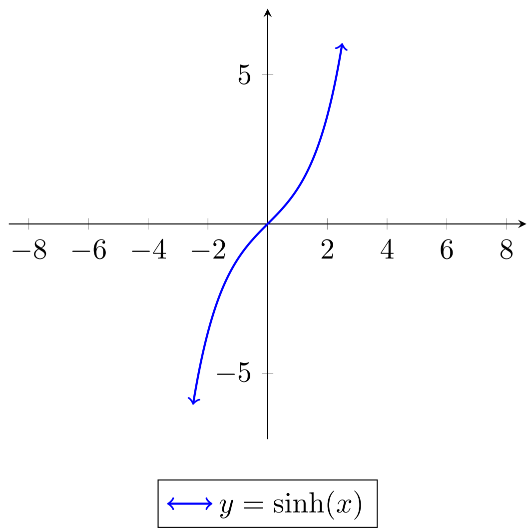
Doron Shahar
Math 122B

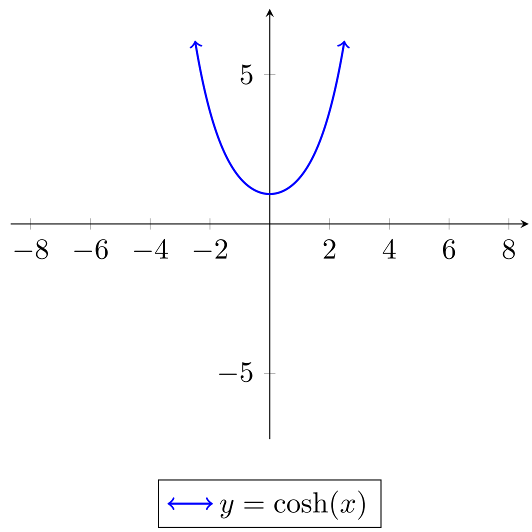
Section 3.8: Hyperbolic trigonometric functions

In this section, we will introduce two new functions commonly used in engineering and for solving partial differential equations. These functions are the hyperbolic sine $\sinh(x)$ (read "sinch of ex") and the hyperbolic cosine $\cosh(x)$ (read "cawsh of ex"). They are defined as follows:

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad\qquad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

Below are there graphs.





Note the following properties of $\sinh(x)$ and $\cosh(x)$.

$\sinh(0) = 0$ $\qquad\qquad$ $\cosh(0) = 1$

$\sinh(x)$ is odd $\qquad\qquad$ $\cosh(x)$ is even

$\sinh(x)$ and $\cosh(x)$ also exhibit the following limit behavior.

As $x \to \infty$, $\qquad \sinh(x) \approx \dfrac{e^x}{2} \qquad\qquad \cosh(x) \approx \dfrac{e^x}{2}$

As $x \to -\infty$, $\qquad \sinh(x) \approx -\dfrac{e^{-x}}{2} \qquad\qquad \cosh(x) \approx \dfrac{e^{-x}}{2}$

$\sinh(x)$ and $\cosh(x)$ also satisfy a common identity:

$$\cosh^2(x) - \sinh^2(x) = 1$$

We can furthermore define other hyperbolic trigonometric functions in terms of $\sinh(x)$ and $\cosh(x)$. For example, the hyperbolic tangent $\tanh(x)$ (read "tansh of ex") can be defined as follows:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

Almost all of the above information does not need to be memorized. Rather, know that such things exist and refer back to them whenever necessary. Below we will discuss what you need to memorize about hyperbolic trigonometric functions.

First, you will need to memorize the derivatives of $\sinh(x)$ and $\cosh(x)$.

$$\frac{d}{dx}\sinh(x) = \frac{d}{dx}\left[\frac{e^x - e^{-x}}{2}\right] = \frac{1}{2}\frac{d}{dx}[e^x - e^{-x}] = \frac{1}{2}[e^x + e^{-x}] = \frac{e^x + e^{-x}}{2} = \cosh(x)$$

$$\frac{d}{dx}\cosh(x) = \frac{d}{dx}\left[\frac{e^x + e^{-x}}{2}\right] = \frac{1}{2}\frac{d}{dx}[e^x + e^{-x}] = \frac{1}{2}[e^x - e^{-x}] = \frac{e^x - e^{-x}}{2} = \sinh(x)$$

The values and derivatives of $\sinh(x)$ and $\cosh(x)$ at $x = 0$ are also important to know by heart.

$$\sinh(0) = 0 \qquad \cosh(0) = 1$$

$$\sinh'(0) = \cosh(0) = 1 \qquad \cosh'(0) = \sinh(0) = 0$$

That's all you need to memorize.

Lastly, we will go over a problem where $\sinh(x)$ and $\cosh(x)$ are actually used. When solving differential equations, you sometimes need to find a function of the form $f(x) = ae^x + be^{-x}$ given the values of $f(0)$ and $f'(0)$. Any function of the form $f(x) = ae^x + be^{-x}$ can also be written in the form $f(x) = c\cosh(x) + d\sinh(x)$, and it easier to find the constants c and d, then it is to find the constants a and b.

Problem: Let $f(x) = c\cosh(x) + d\sinh(x)$. Find constants c and d so that $f(0) = 2$ and $f'(0) = 3$

Answer: $2 = f(0) = c\cosh(0) + d\sinh(0) = c \cdot 1 + d \cdot 0 = c$

$3 = f'(0) = c\sinh(0) + d\cosh(0) = c \cdot 0 + d \cdot 1 = d$

Therefore, $f(x) = 2\cosh(x) + 3\sinh(x)$.