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Math 122B

Section 3.5: Trigonometric functions

Henceforth, all angles are in radians, and all trigonometric functions are defined as functions of angles written in radians. This is important as the derivatives of trigonometric functions depend on the angles being in radians.

The derivatives of $\sin(x)$ and $\cos(x)$ can be found using the limit definition of the derivative:

$$\frac{d}{dx}\sin(x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \qquad \frac{d}{dx}\cos(x) = \lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h}$$

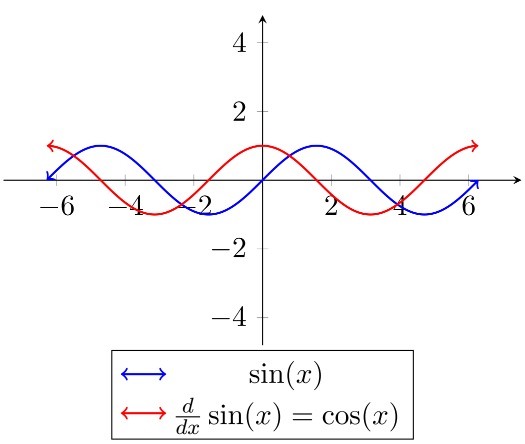
The following trigonometric identities can then be used to help evaluate these limits.

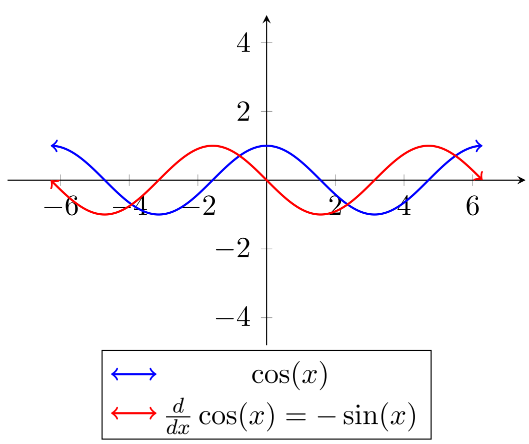
$$\sin(x+h) = \sin(x)\cos(h)+\sin(h)\cos(x) \qquad \cos(x+h) = \cos(x)\cos(h)-\sin(x)\sin(h)$$

The derivatives of $\sin(x)$ and $\cos(x)$ are as follows:

$$\frac{d}{dx}\sin(x) = \cos(x) \qquad \frac{d}{dx}\cos(x) = -\sin(x)$$

The only difficulty in memorizing these derivatives is remembering which one has a minus sign. By considering the graphs of $\sin(x)$ and $\cos(x)$, we can easily figure out which derivative has a minus sign if ever we forget.





$\sin(x)$ is increasing at $x=0$. Therefore, its derivative at $x=0$ must be positive, which tell us that the derivative of $\sin(x)$ must be $\cos(x)$ and not $-\cos(x)$. $\cos(x)$, on the other hand, is decreasing for small positive x. Therefore, its derivative for small positive x must be negative, which tells us that the derivative of $\cos(x)$ must be $-\sin(x)$ and not $\sin(x)$.

The derivatives for the other trigonometric functions can now be found using the quotient rule and the derivatives for $\sin(x)$ and $\cos(x)$.

Example: $\frac{d}{dx}\tan(x) = \frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right] = \frac{\cos(x)\cos(x)-\sin(x)(-\sin(x))}{\cos^2(x)} = \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$

We used the trigonometric identity $\cos^2(x)+\sin^2(x)=1$ and the definition of $\sec(x)$ to simplify the derivative of $\tan(x)$.

Try finding the derivatives of $\sec(x), \cot(x)$, and $\csc(x)$ using the same procedure. That is, use the quotient and then simplify the derivative using trigonometric identities.

Problems: $\frac{d}{dx}\sec x$ $\qquad\qquad$ $\frac{d}{dx}\cot x$ $\qquad\qquad$ $\frac{d}{dx}\csc(x)$

Answers: $\frac{d}{dx}\sec(x) = \frac{d}{dx}\left[\frac{1}{\cos(x)}\right] = \frac{\cos(x)\cdot 0 - 1(-\sin(x))}{\cos^2(x)} = \frac{\sin(x)}{\cos^2(x)} = \frac{\sin(x)}{\cos(x)}\frac{1}{\cos(x)} = \tan(x)\sec(x)$

$$\frac{d}{dx}\cot(x) = \frac{d}{dx}\left[\frac{\cos(x)}{\sin(x)}\right] = \frac{\sin(x)(-\sin(x)) - \cos(x)\cos(x)}{\sin^2(x)} = -\frac{\cos^2(x)+\sin^2(x)}{\sin^2(x)} = \frac{-1}{\sin^2(x)} = -\csc^2(x)$$

$$\frac{d}{dx}\csc(x) = \frac{d}{dx}\left[\frac{1}{\sin(x)}\right] = \frac{\sin(x)\cdot 0 - 1\cdot\cos(x))}{\sin^2(x)} = \frac{-\cos(x)}{\sin^2(x)} = -\frac{\cos(x)}{\sin(x)}\frac{1}{\sin(x)} = -\cot(x)\csc(x)$$

You should memorize the derivative for $\sin(x), \cos(x), \tan(x)$, and $\sec(x)$. The derivatives for $\cot(x)$ and $\csc(x)$ aren't used as often, but you should be able to derive them if necessary.

Lastly, try the following practice problems.

Problems: $\frac{d}{dx}[e^{-x}\cos(x)]$ $\qquad\qquad$ $\frac{d}{dy}\left[\frac{\sin(y)}{y}\right]$ $\qquad\qquad$ $\frac{d^2}{dt^2}\sec(t)$

Answers: $\frac{d}{dx}[e^{-x}\cos(x)] = -e^{-x}\cos(x) + e^{-x}(-\sin(x)) = -e^{-x}(\cos(x)+\sin(x))$

$$\frac{d}{dy}\left[\frac{\sin(y)}{y^2}\right] = \frac{y^2\cos(y) - \sin(y)2y}{y^4} = \frac{y\cos(y) - 2\sin(y)}{y^3}$$

$$\frac{d^2}{dt^2}\sec(t) = \frac{d}{dt}[\tan(t)\sec(t)] = \sec^2(t)\sec(t) + \tan(t)\tan(t)\sec(t) = \sec^3(t) + \tan^2(t)\sec(t)$$