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Math 122B

Section 3.2: Exponential functions

Derivatives of exponential functions:

$$\frac{d}{dx}[e^x] = e^x$$

For any constant constant β,

$$\frac{d}{dx}[e^{\beta x}] = \beta e^{\beta x}$$

For any constant constant $a > 0$,

$$\frac{d}{dx}[a^x] = \ln(a)a^x$$

Note the difference between the power functions covered in the last section and exponential functions covered in this section. Power functions are functions of the form x^r whereas exponential functions are of the form a^x. For a power function x is the base, and the exponent is constant. For an exponential function x is the exponent, and the base is constant.

The proof of the of these derivatives essentially involves finding the derivative of e^x. We will skip the key technical step of the proof. The other two cases follow easily, and can be done more formally once we have the chain rule.

Proof:

$$\frac{d}{dx}[e^x] = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x e^h - e^x}{h} = \lim_{h\to 0} e^x\frac{e^h-1}{h} = e^x\lim_{h\to 0}\frac{e^h-1}{h} = e^x$$

We have use the fact that $\lim\limits_{h\to 0}\frac{e^h-1}{h} = 1$. We will not prove this as it takes some time.

The proof for $e^{\beta x}$ is similar, so we will skip some steps.

$$\frac{d}{dx}[e^{\beta x}] = \lim_{h\to 0}\frac{e^{\beta(x+h)}-e^{\beta x}}{h} = e^{\beta x}\lim_{h\to 0}\frac{e^{\beta h}-1}{h} = \beta e^{\beta x}\lim_{h\to 0}\frac{e^{\beta h}-1}{\beta h} = \beta e^{\beta x}\lim_{s\to 0}\frac{e^s-1}{s} = \beta e^{\beta x}$$

In the second to last equality, we let $s = \beta h$ and used the fact that $\lim\limits_{h\to 0}\frac{e^{\beta h}-1}{\beta h} = \lim\limits_{s\to 0}\frac{e^s-1}{s} = 1$

The proof for a^x follows from the result for $e^{\beta x}$. Note that $a^x = e^{\ln(a^x)} = e^{x\ln(a)} = e^{\ln(a)x}$. Therefore,

$$\frac{d}{dx}[a^x] = \frac{d}{dx}[e^{\ln(a)x}] = \ln(a)e^{\ln(a)x} = \ln(a)a^x$$

QED

We could also remember the result for $e^{\beta x}$ using the rule for a^x:

$$\frac{d}{dx}[e^{\beta x}] = \frac{d}{dx}[(e^{\beta})^x] = \ln(e^{\beta})(e^{\beta})^x = \beta e^{\beta x}$$

Let's get some practice finding such derivatives.

When taking the derivative of a function of the form $e^{\beta x}$, you simply bring down the number multiplied by the x.

Example: $\frac{d}{dx}[e^{2x}] = 2e^{2x}$

Find the derivatives of the following functions.

First some simple ones.

Problems: $\frac{d}{dx}[e^{3x}]$ $\frac{d}{dx}[e^{5x}]$ $\frac{d}{dx}[e^{x}]$

Answers: $\frac{d}{dx}[e^{3x}] = 3e^{3x}$ $\frac{d}{dx}[e^{5x}] = 5e^{5x}$ $\frac{d}{dx}[e^{x}] = e^{x}$

Some with weird exponents.

Problems: $\frac{d}{dx}[e^{\sqrt{2}x}]$ $\frac{d}{dx}[e^{0.2x}]$ $\frac{d}{dx}[e^{-x}]$

Answers: $\frac{d}{dx}[e^{\sqrt{2}x}] = \sqrt{2}e^{\sqrt{2}x}$ $\frac{d}{dx}[e^{0.2x}] = 0.2e^{0.2x}$ $\frac{d}{dx}[e^{-x}] = -e^{-x}$

When taking a derivative of a function of the form a^x, you multiply a^x by $\ln(a)$.

Example: $\frac{d}{dx}[2^x] = \ln(2) \cdot 2^x$

Practice with the following functions.

First some simple ones.

Problems: $\frac{d}{dx}[3^x]$ $\frac{d}{dx}[4^x]$ $\frac{d}{dx}[7^x]$

Answers: $\frac{d}{dx}[3^x] = \ln(3) \cdot 3^x$ $\frac{d}{dx}[4^x] = \ln(4) \cdot 4^x$ $\frac{d}{dx}[7^x] = \ln(7) \cdot 7^x$

Some with weird bases.

Problems: $\frac{d}{dx}[\ln(2)^x]$ $\frac{d}{dt}[\ln(7)^t]$ $\frac{d}{dy}[\pi^y]$

Answers: $\frac{d}{dx}[\ln(2)^x] = \ln(\ln(2)) \cdot \ln(2)^x$ $\frac{d}{dt}[\ln(7)^t] = \ln(\ln(7)) \cdot \ln(7)^t$ $\frac{d}{dy}[\pi^y] = \ln(\pi) \cdot \pi^y$

As we saw earlier, if our function is multiplied by a constant we simply carry the constant along in the derivative.

Example: $\frac{d}{dx}[7e^{2x}] = 7\frac{d}{dx}[e^{2x}] = 7 \cdot 2e^{2x} = 14e^{2x}$

Once you feel comfortable taking derivatives of functions multiplied by constants, you may take such derivatives in one step.

Try the following.

Problems: $\frac{d}{dx}[3e^{0.5x}]$ $\frac{d}{dx}[-e^{-x}]$ $\frac{d}{dx}[-2\cdot 5^x]$

Answers: $\frac{d}{dx}[3e^{0.5x}] = 1.5e^{0.5x}$ $\frac{d}{dx}[-e^{-x}] = e^{-x}$ $\frac{d}{dx}[-2\cdot 5^x] = -2\ln(5)5^x$

And a few more.

Problems: $\frac{d}{dx}[\ln(5)\cdot 5^x]$ $\frac{d}{dx}[\ln(\frac{1}{3})\cdot 3^x]$ $\frac{d}{dx}[\frac{4^x}{4}]$

Answers: $\frac{d}{dx}[\ln(5)\cdot 5^x] = (\ln(5))^2 5^x$ $\frac{d}{dx}[\ln(\frac{1}{3})\cdot 3^x] = -\ln(3)^2\cdot 3^x$ $\frac{d}{dx}[\frac{4^x}{4}] = \frac{1}{4}\frac{d}{dx}[4^x] = \frac{\ln(4)}{4}4^x$

Sometimes before taking derivatives we need to use properties of exponents to simplify exponential functions into the forms $e^{\beta x}$ and a^x.

Example: $\frac{d}{dx}[2^{3x}] = \frac{d}{dx}[(2^3)^x] = \frac{d}{dx}[8^x] = \ln(8)8^x$

Try the following problems

Problems: $\frac{d}{dx}[5^{2x}]$ $\frac{d}{dt}[e^{-t+2}]$ $\frac{d}{dy}[e^{y+3}]$

Answers: $\frac{d}{dx}[5^{2x}] = \frac{d}{dx}[25^x] = \ln(25)25^x$

$\frac{d}{dt}[e^{-t+2}] = \frac{d}{dt}[e^2e^{-t}] = -e^2e^{-t} = -e^{-t+2}$

$\frac{d}{dy}[e^{y+3}] = \frac{d}{dy}[e^3e^y] = e^3e^y = e^{y+3}$

And some other ones

Problems: $\frac{d}{dx}[b^{-x}]$ $\frac{d}{dx}[\frac{e^{3x}}{e^{2x}}]$ $\frac{d}{dx}[\frac{1}{e^x}]$

Answers: $\frac{d}{dx}[b^{-x}] = \frac{d}{dx}[(\frac{1}{b})^x] = \ln(\frac{1}{b})(\frac{1}{b})^x = -\ln(b)b^{-x}$

$\frac{d}{dx}[\frac{e^{3x}}{e^{2x}}] = \frac{d}{dx}[e^x] = e^x$

$\frac{d}{dx}[\frac{1}{e^x}] = \frac{d}{dx}[e^{-x}] = -e^{-x}$

It is quite easy to get mixed up when taking derivatives. Below are a few problems to help you differentiate between the power rule and the rules for exponential functions, and between functions and constants. Recall the following facts: 1) The derivative of a constant function is zero, and 2) if multiple terms are summed together, the derivative is simply taken term by term.

Example: $\frac{d}{dx}[e^x + x^e] = \frac{d}{dx}[e^x] + \frac{d}{dx}[x^e] = e^x + ex^{e-1}$

Practice with the following problems.

Problems: $\frac{d}{dx}[\pi^x + x^\pi]$ $\frac{d}{dx}[e^\pi + \pi^e]$ $\frac{d}{dy}[3e^{2y} + y^5 + e^2]$

Answers: $\frac{d}{dx}[\pi^x + x^\pi] = \ln(\pi)\pi^x + \pi x^{\pi-1}$

$\frac{d}{dx}[e^{\pi} + \pi^{e}] = 0$ (Constants!)

$\frac{d}{dy}[3e^{2y} + y^5 + e^2] = 6e^{2y} + 5y^4 + 0 = 6e^2y + 5y^4$

Next, we have a super problem that combines most of the techniques we have learned so far.

Super Problem: $\frac{d}{dx}[3^{2x} + \ln(5)^x + e^{x-1} + \sqrt{7}e^{-x} + e^4]$

Answer: $\ln(9)9^x + \ln(\ln(5))\ln(5)^x + e^{x-1} - \sqrt{7}e^{-x}(+0)$

Lastly, we sometimes need to find the second derivative of a function. Since the derivative of an exponential function is also an exponential functions, we can apply the rules in this section twice to obtain the second derivative of exponential functions.

Example: $f(x) = e^{3x}$, $f'(x) = 3e^{3x}$, $f''(x) = 9e^{3x}$

Find the second derivatives of the following functions.

Problems: $g(x) = e^{\sqrt{2}x}$ $\qquad$ $h(x) = 3^x$

Answers: $g'(x) = \sqrt{2}e^{\sqrt{2}x}$, $g''(x) = 2e^{\sqrt{2}x}$ $\qquad$ $h'(x) = \ln(3)3^x$, $h''(x) = (\ln(3))^2 3^x$