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Math 122B

Section 3.1: The power rule

Power rule: For all numbers $r \neq 0, 1$,

$$\frac{d}{dx}[x^r] = rx^{r-1}$$

The power rule also holds, more or less, when $r = 0, 1$ if we ignore the fact that 0^0 is undefined. It is easier, however, to just remember the $r = 0, 1$ cases on their own without this rule.

The $r = 0$ case states

$$\frac{d}{dx}[1] = 0$$

The $r = 1$ case states

$$\frac{d}{dx}[x] = 1$$

The proof of the power rule involves four separate cases: One when r is a positive integer, one when r is a negative integer, one when r is a rational number, and then one when r is an irrational number. We shall only illustrate the proof in the first case when r is positive integer.

The crux of the proof involves finding a general expression for $(x+h)^r$ when r is a positive integer. Let's first see what happens when $r = 2, 3$ before considering the general case.

$$(x+h)^2 = (x+h)(x+h) = x^2 + 2xh + h^2$$
$$(x+h)^3 = (x+h)(x+h)(x+h) = x^3 + 3x^2h + 2xh^2 + h^3$$

And in general for a positive integer r,

$$(x+h)^r = (x+h)\cdots(x+h) = x^r + rx^{r-1}h + \frac{r(r-1)}{2}x^{n-2}h^2 + \cdots + h^n$$

The key point to note is that $(x+h)^r = x^r + rx^{r-1}h+$ (terms involving higher powers of h).

Finally, we get to the proof.

Proof:

$$\begin{aligned}
\frac{d}{dx}[x^r] &= \lim_{h\to 0} \frac{(x+h)^r - x^r}{h} \\
&= \lim_{h\to 0} \frac{(x^r + rx^{r-1}h + \frac{r(r-1)}{2}x^{n-2}h^2 + \cdots + h^n) - x^r}{h} \\
&= \lim_{h\to 0} \frac{rx^{r-1}h + \frac{r(r-1)}{2}x^{n-2}h^2 + \cdots + h^n}{h} \\
&= \lim_{h\to 0} rx^{r-1} + \frac{r(r-1)}{2}x^{n-2}h + \cdots + h^{n-1} \\
&= rx^{r-1}
\end{aligned}$$

QED

Let's get some practice in using the power rule.

In simple cases, you apply the power by bringing down a copy of the exponent in front and then subtracting 1 from the exponent.

Example: $\frac{d}{dx}[x^2] = 2x^{2-1} = 2x^1 = 2x$

Once you feel comfortable using the power rule, you may take such derivatives in one step. Find the derivatives of the following functions.

First some simple ones.

Problems: $\frac{d}{dx}[x^3]$ $\quad$ $\frac{d}{dx}[x^4]$ $\quad$ $\frac{d}{dx}[x^5]$

Answers: $\frac{d}{dx}[x^3] = 3x^2$ $\quad$ $\frac{d}{dx}[x^4] = 4x^3$ $\quad$ $\frac{d}{dx}[x^5] = 5x^4$

Some with weird exponents.

Problems: $\frac{d}{dx}[x^\pi]$ $\quad$ $\frac{d}{dx}[x^e]$ $\quad$ $\frac{d}{dx}[x^{\sqrt{2}}]$

Answers: $\frac{d}{dx}[x^\pi] = \pi x^{\pi-1}$ $\quad$ $\frac{d}{dx}[x^e] = ex^{e-1}$ $\quad$ $\frac{d}{dx}[x^{\sqrt{2}}] = \sqrt{2}x^{\sqrt{2}-1}$

Sometimes the exponent is not clear when we have a fraction or a root function. In that case, we write the function in the form x^r and then take derivatives. At the end, we can simplify the derivative.

Example: $\frac{d}{dx}[\sqrt{x}] = \frac{d}{dx}[x^{1/2}] = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$ after simplifying.

Try some with fractions and root functions.

Problems: $\frac{d}{dx}[\sqrt[3]{x}]$ $\quad$ $\frac{d}{dx}[\frac{1}{x}]$ $\quad$ $\frac{d}{dx}[\frac{1}{\sqrt{x}}]$

Answers: $\frac{d}{dx}[\sqrt[3]{x}] = \frac{d}{dx}[x^{1/3}] = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$

$\frac{d}{dx}[\frac{1}{x}] = \frac{d}{dx}[x^{-1}] = -1x^{-2} = -\frac{1}{x^2}$

$\frac{d}{dx}[\frac{1}{\sqrt{x}}] = \frac{d}{dx}[x^{-1/2}] = -\frac{1}{2}x^{-3/2} = -\frac{1}{2x^{3/2}}$

As we saw earlier, if our function is multiplied by a constant we simply carry the constant along in the derivative.

Example: $\frac{d}{dx}[3x^6] = 3\frac{d}{dx}[x^6] = 3 \cdot 6x^5 = 18x^5$

Again once you feel comfortable taking derivatives of functions multiplied by constants, you may take such derivatives in one step.

Try the following.

Problems: $\frac{d}{dx}[5x]$ $\quad$ $\frac{d}{dx}[0.2x^3]$ $\quad$ $\frac{d}{dx}[\sqrt{5}x^9]$

Answers: $\frac{d}{dx}[5x] = 5$ $\quad$ $\frac{d}{dx}[0.2x^3] = 0.6x^2$ $\quad$ $\frac{d}{dx}[\sqrt{5}x^9] = 9\sqrt{5}x^8$

And a few more.

Problems: $\frac{d}{dx}[\frac{x^4}{4}]$ $\frac{d}{dx}[-\frac{5}{x^2}]$ $\frac{d}{dx}[\frac{1}{3\sqrt{x}}]$

Answers: $\frac{d}{dx}[\frac{x^4}{4}] = \frac{d}{dx}[\frac{1}{4}x^4] = x^3$ $\frac{d}{dx}[-\frac{5}{x^2}] = \frac{d}{dx}[-5x^{-2}] = \frac{10}{x^3}$ $\frac{d}{dx}[\frac{1}{3\sqrt{x}}] = \frac{d}{dx}[\frac{1}{3}x^{-1/2}] = \frac{-1}{6x^{3/2}}$

If multiple terms are summed together, the derivative is simply taken term by term as follows.

Example: $\frac{d}{dx}[5x^3+3x^2+2x+1] = \frac{d}{dx}[5x^3]+\frac{d}{dx}[3x^2]+\frac{d}{dx}[2x]+\frac{d}{dx}[1] = 15x^2+6x+2+0 = 15x^2+6x+2$

Again you can do this in one step after you understand the principle.

Try the following two problems.

Problems: $\frac{d}{dx}[\frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + x + 1]$ $\frac{d}{dx}[\sqrt{2}x^7 + \frac{x^\pi}{3} + x^{-4} + \frac{1}{x} - 12\frac{1}{\sqrt[3]{x}}]$

Answers: $\frac{d}{dx}[\frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + x + 1] = x^3 + x^2 + x + 1$

$\frac{d}{dx}[\sqrt{2}x^7 + \frac{x^\pi}{3} + x^{-4} + \frac{1}{x} - 12\frac{1}{\sqrt[3]{x}}] = 7\sqrt{2}x^6 + \frac{\pi}{3}x^{\pi-1} - \frac{4}{x^5} - \frac{1}{x^2} + \frac{4}{x^{4/3}}$

Lastly, it's sometimes necessary to simplify an expression before we use the power rule.

Example: $\frac{d}{dx}[\frac{x^2+1}{x}] = \frac{d}{dx}[x + \frac{1}{x}] = 1 - \frac{1}{x^2}$

Try the following problems.

Problems: $\frac{d}{dx}[\frac{\sqrt{x}(1+x)}{x^2}]$ Find $\frac{dA}{dy}$ where $A = (\frac{100-y-\pi y}{2})y + \frac{\pi}{2}(\frac{y}{2})^2$

Answers: $\frac{d}{dx}[\frac{\sqrt{x}(1+x)}{x^2}] = \frac{d}{dx}[\frac{x^{1/2}+x^{3/2}}{x^2}] = \frac{d}{dx}[x^{-3/2} + x^{-1/2}] = \frac{-3}{2}x^{-5/2} - \frac{1}{2}x^{-3/2}$

$A = (\frac{100-y-\pi y}{2})y + \frac{\pi}{2}(\frac{y}{2})^2 = 50y - \frac{1}{2}y^2 - \frac{\pi}{2}y^2 + \frac{\pi}{8}y^2$ $\frac{dA}{dy} = 50 - y - \pi y + \frac{\pi}{4}y$

Both of the problems above can be done without simplifying once we learn the product and quotient rules. It is easier and faster, however, to just simplify and take derivatives with the power rule. Remember this for when you take harder derivatives. It is worthwhile to simplify to make derivatives easier.

Recall that we sometimes needed to find the second derivative of a function. By applying the power rule twice, we can find the second derivatives of certain functions.

Example: $f(x) = x^2,\ f'(x) = 2x,\ f''(x) = 2$

Find the second derivatives of the following functions.

Problems: $g(x) = x^3$ $h(x) = \frac{1}{3}x^3 + x^2 - 15x + 2$

Answers: $g'(x) = 3x^2,\ g''(x) = 6x$ $h'(x) = x^2 + 2x - 15,\ h''(x) = 2x + 2$