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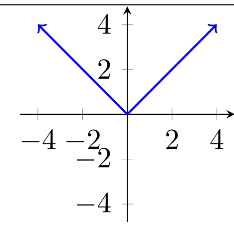
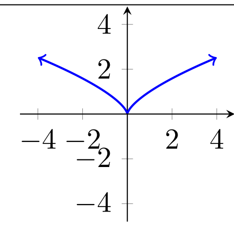
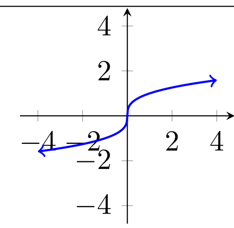
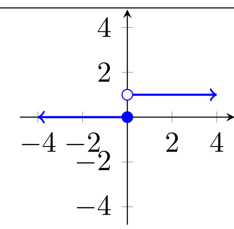
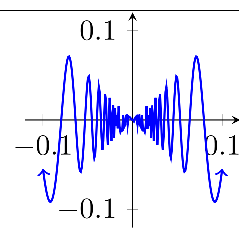
Math 122B

Section 2.6: Differentiablity and piecewise functions

Definition: A functions f is differentiable at $x = a$ if $\lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists.

Graphically, f is differentiable at $x = a$ if f has a non-vertical tangent line at $x = a$. (The derivative, which is the value of the above limit, is the slope of the tangent line.)

Below is a table of all the reasons why a function may fail to be differentiable at a point.

Reason	Example	Graph
Corner	$f(x) = \|x\|$ at $x = 0$	
Cusp	$f(x) = x^{2/3}$ at $x = 0$	
Vertical tangent line	$f(x) = \sqrt[3]{x}$ at $x = 0$	
Discontinuity	$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$ at $x = 0$	
Other weirdness	$f(x) = \begin{cases} x\sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ at $x = 0$	

Next, we will go through and explain and prove why the above functions are not differentiable at $x = 0$.

Corners: The key characteristic of a corner is that slope of the secant line (i.e., the difference quotient) approaches one value as h approaches zero from the right and a different value as h approaches zero from the left.

Before proving that $f(x) = |x|$ is not differentiable at $x = 0$, we need to understand the function $|x|$. So let me ask you is the following ever true?

$$|x| = -x$$

The answer is yes. We often read "$-x$" as "negative ex", but if we are being careful we should say "the opposite of ex". x itself could be negative in which case $-x$ is positive. Consider the following example in which $x = -3$

$$|x| = |-3| = 3 = -(-3) = -x$$

In general, if $x \leq 0$ then $|x| = -x$, and if $x \geq 0$ then $|x| = x$. Now we are ready to prove that $f(x) = |x|$ is not differentiable at $x = 0$.

Proof: We will show that $f(x) = |x|$ is not differentiable at $x = 0$ by trying to take the derivative at $x = 0$ and finding that the limit doesn't exist.

If h approaches zero from the right, then $h > 0$. Therefore,

$$\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$$

If h approaches zero from the left, then $h < 0$. Therefore,

$$\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$$

Therefore, $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$ does not exist, which is exactly what we mean when we say that $f(x) = |x|$ is not differentiable at $x = 0$. QED

Cusps: Cusps are very similar to corners except that slope of the secant line (i.e., the difference quotient) approaches ∞ from one side and $-\infty$ from the other. The proof that $f(x) = x^{2/3}$ is not differentiable at $x = 0$ is left as a homework problem. You can prove by the same steps as the proof for $|x|$.

Vertical tangent lines: Vertical tangent lines are exist when the slope of the secant (i.e., the difference quotient) approaches $\pm\infty$. $f(x) = \sqrt[3]{x}$ has a verticale tangent line at $x = 0$; this is one of the standard examples of vertical tangent lines. Next, we shall prove that $f(x) = \sqrt[3]{x}$ is not differentiable at $x = 0$. Give this one a try on your own first.

Proof: We will show that $f(x) = \sqrt[3]{x}$ is not differentiable at $x = 0$ by trying to take the derivative at $x = 0$ and finding that the limit doesn't exist.

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{\sqrt[3]{h}}{h} = \lim_{h\to 0}\frac{1}{h^{2/3}} = \infty \quad \text{(DNE)}$$

Now while we could say that the limit is infinity, we hold to the convention that if the limit is infinity, then it does not exist. Therefore, $f(x) = \sqrt[3]{x}$ is not differentiable at $x = 0$. QED

Discontinuities: When a function has a discontinuity at a point, the slope of the secant line (i.e., the difference quotient) approaches $\pm\infty$ as h approaches zero from at least one side. A discontinuity is not classified as any of the other reasons simply because it is a discontinuity. Corners, cusps, and vertical tangent lines can only exist where a function is continuous. Next we will prove that the function

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

is not differentiable at $x = 0$. Give this one a try on your own first.

Proof: Clearly, $\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h} = 0$, but

$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+}\frac{1}{h} = \infty \quad \text{(DNE)}$$

Therefore, $f(x)$ is not differentiable at $x = 0$. QED

Other weirdness: Functions can fail to be differentiable for other reasons than those listed above. The inherent characteristic of all these other functions is that the limit of the difference quotient fails to exist for more "bizarre" reasons (e.g., it oscillates endlessly between ± 1).

$f(x) = \begin{cases} x\sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is one of the standard examples of a function that is not differentiable at $x = 0$ for "bizarre" reasons. We prove that next.

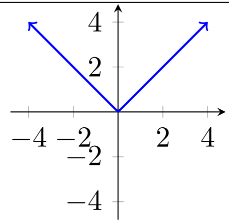
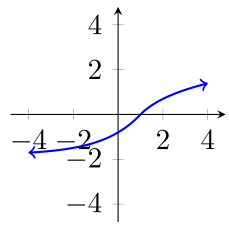
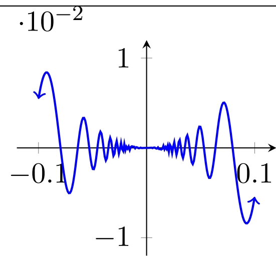
Proof:

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h\sin(\frac{1}{h})}{h} = \lim_{h\to 0}\sin\left(\frac{1}{h}\right)$$

$\sin\left(\frac{1}{h}\right)$ oscillates between ± 1 indefinitely as h approaches zero. Therefore, the limit does not exist, and $f(x)$ is not differentiable at $x = 0$. QED

A word of caution: some functions may look like they are differentiable but may not be differentiable or vice versa. It is important to actually check. Don't judge a book by its cover. For this reason, questions involving graphs often ask whether a function appears to be differentiable, and not whether it is actually differentiable.

Consider the functions in the table below. (We will learn how to take the derivatives of these functions later in the course. For now, you should be able to show that the third of these functions is differentiable at $x = 0$.)

Function	Graph	Explanation
$f(x) = \sqrt{x^2 + 0.001}$		The function appears to have a corner at $x = 0$. If you zoom in, however, you will see that the function rounds out and is actually differentiable at $x = 0$.
$f(x) = \begin{cases} \ln(x) & \text{if } x \geq 1 \\ 1.8^x - 1.8 & \text{if } x \leq 1 \end{cases}$		This function appears to be differentiable near $x = 1$. In reality, this function has a slight corner at $x = 1$. From one direction the slope of the secant line approaches 1; from the other it approaches $\ln(1.8)1.8 \approx 1.058$.
$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$		While this function appears to be just as crazy as $x\sin(1/x)$, it is actually differentiable at $x = 0$. The slope of the secant line oscillates endlessly but still approaches zero.

Before moving on to piecewise functions, we prove one theorem about differentiability. We mentioned earlier that if a function is not continuous at a point, then it won't be differentiable there. Therefore, differentiability implies continuity as stated in the following theorem.

Theorem 1: *If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.*

Proof: Suppose $f(x)$ is differentiable at $x = a$. We want to show that $f(x)$ is continuous at $x = a$. That is, we want to show one of the following equivalent conditions:

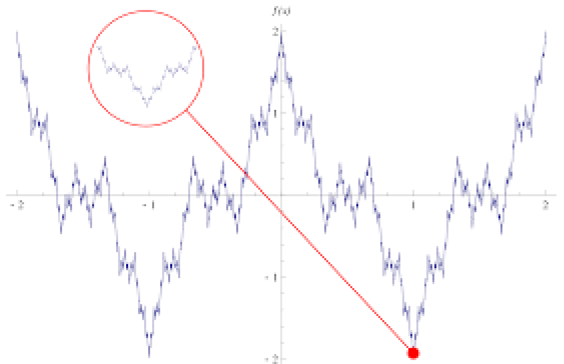
1. $\lim_{x \to a} f(x) = f(a)$
2. $\lim_{h \to 0} f(a + h) = f(a)$
3. $\lim_{h \to 0} f(a + h) - f(a) = 0$

We will show that the third one holds. Specifically, we want to relate the third limit to the limit we already know exists, namely $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$. We also know that $\lim_{h \to 0} h = 0$. Therefore,

$$\lim_{h \to 0} f(a + h) - f(a) = \lim_{h \to 0} h \cdot \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} h \cdot \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = 0 \cdot f'(a) = 0$$

QED

A function is said to be differentiable if it is differentiable at all points. By the above theorem, if f is differentiable, then f is continuous. For a long time, it was thought that the converse almost holds. Specifically, people thought that if a function is continuous it must be differentiable almost everywhere. This, however, is not the case. There exist continuous functions which are nowhere differentiable. Notably the Weierstrass function shown below. It is defined by $f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$ with some constraints on the constants a and b.



(The picture is from the Wikipedia page about the Weierstrass function.)

Finally, we discuss taking the derivative of piecewise functions. In particular, we will try to determine whether a function is differentiable at the point where the function switches from being defined one way to begin defined another.

Example: Find the derivative of

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$$

At the point $x = 0$, $f(x)$ switches from being defined in terms of $-x^2$ to being defined in terms of x^2. Away from $x = 0$, we can simply take the derivative of each piece of the function $f(x)$. The derivative of x^2 is $2x$ and the derivative of $-x^2$ is $-2x$. Therefore,

$$f'(x) = \begin{cases} 2x & \text{if } x > 0 \\ -2x & \text{if } x < 0 \end{cases}$$

Except that we still have to determine $f'(0)$ if it exists. To determine, if $f'(0)$ exists we need to verify two things: 1) whether plugging $x = 0$ into the rules for f (x^2 and $-x^2$) on both sides of $x = 0$ gives the same value, 2) whether plugging $x = 0$ into the rules for f' ($2x$ and $-2x$) on both sides of $x = 0$ gives the same value. This will be more clear once we finish the example.

1) To verify condition one we need to check whether x^2 and $-x^2$ are equal after we plug in $x = 0$. This is indeed the case:

$$x_2|_{x=0} = 0^2 = -(0)^2 = -x^2|_{x=0}$$

2) To verify condition two we need to check whether $2x$ and $-2x$ are equal after we plug in $x = 0$. This is indeed the case:

$$2x|_{x=0} = 2 \cdot 0 = -2 \cdot 0 = -2x|_{x=0}$$

Therefore, f is differentiable at $x = 0$. The value of $f'(0)$ is the value we got in 2) after plugging in $x = 0$. Therefore,

$$f'(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases}$$

We could also write $\leq$ instead of $<$ in the second line.

Note: Condition 1) by itself is equivalent to showing that the function is continuous at $x = 0$. But both conditions 1) and 2) are needed to show that the function is differentiable at $x = 0$. Lastly, this method is not valid unless both rules defined in f and their derivatives can be evaluated at the point of interest. That is, it cannot be used to check whether weird functions are differentiable.

Now try the following problems.

Problem: Find the derivative of

$$g(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Solution: At the point $x = 0$, $g(x)$ switches from being defined in terms of $-x$ to being defined in terms of x. Away from $x = 0$, we can simply take the derivative of each piece of the function $g(x)$ to obtain

$$g'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

We still have to determine $g'(0)$ if it exists.

1) To verify condition one we need to check whether x and $-x$ are equal after we plug in $x = 0$. This is indeed the case:

$$x|_{x=0} = 0 = -(0) = -x|_{x=0}$$

2) To verify condition two we need to check whether 1 and -1 are equal after we plug in $x = 0$. This is not true:

$$1|_{x=0} = 1 \neq -1 = -1|_{x=0}$$

Therefore, g is continuous at $x = 0$, but not differentiable at $x = 0$. Therefore,

$$g'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Note $g(x)$ is formally how we define the absolute value of x.

Problem: Consider the function

$$h(x) = \begin{cases} x^2 & \text{if } x \leq 2 \\ ax + b & \text{if } x > 2 \end{cases}$$

Find values of a and b such that $g(x)$ is a) continuous but not differentiable at $x = 2$, and b) differentiable at $x = 2$.

Solution: At the point $x = 2$, $h(x)$ switches from being defined in terms of x^2 to being defined in terms of $ax + b$. Away from $x = 2$, we can simply take the derivative of each piece of the function $h(x)$ to obtain

$$h'(x) = \begin{cases} 2x & \text{if } x > 2 \\ a & \text{if } x < 2 \end{cases}$$

We still have to determine $h'(0)$ in terms of a and b.

1) To verify condition one we need to check whether x^2 and $ax+b$ are equal after we plug in $x = 2$. This gives the condition $4 = 2a + b$.

$$x^2|_{x=2} = (2)^2 = 4 == 2a + b = (ax + b)|_{x=2}$$

This condition must be satisfied if h is to be continuous at $x = 2$

2) To verify condition two we need to check whether $2x$ and a are equal after we plug in $x = 2$. This gives the condition $a = 4$.

$$2x|_{x=2} = 4 = a = a|_{x=2}$$

a) If $a \neq 4$, then $h(x)$ is not differentiable at $x = 2$. Take $a = 3$ (or any value other than 4). By condition 1), $b = -2$. Therefore, when $a = 3$ and $b = -2$, $h(x)$ is continuous at $x = 2$, but not differentiable there.

b) For $h(x)$ to be differentiable at $x = 2$, both conditions 1) and 2) must hold in which case $a = 4$ and $b = -4$.

Lastly, some of the problems in this section involve functions that we have yet to learn how to differentiate. Often times we can still verify that the function will not be differentiable by noting that one derivative must be positive and the other negative.

Problem: Determine whether

$$i(x) = \begin{cases} kx & \text{if } x \leq 1 \\ \frac{k}{x} & \text{if } x > 1 \end{cases}$$

is a) continuous, and b) differentiable. ($k > 0$ is a constant.)

Solution: a) $i(x)$ is clearly continuous everywhere except possibly at $x = 1$. To determine whether $i(x)$ is continuous at $x = 1$, we need to attempt to verify condition one. Plugging $x = 1$ into both kx and k/x we have,

$$kx|_{x=1} = k = \frac{k}{x}|_{x=1}$$

Therefore, $i(x)$ is indeed continuous.

b) $i(x)$ is clearly differentiable everywhere except possibly at $x = 1$. To determine whether $i(x)$ is differentiable at $x = 1$, we need to attempt to verify condition two. While we could take the derivative of $\frac{k}{x}$ by hand (or wait until we learn derivative shortcuts), it is easier just to obtain qualitative estimates.

kx is an increasing function, and k/x is an decreasing function. Therefore, there derivatives could match at $x = 1$ only if both derivatives were zero, which is not the case. Therefore, $i(x)$ is not differentiable.