Recitations Week 7

1. Suppose that you get email according to a Poisson process at a rate of 15 messages per day.
   (a) What is the expected time until you receive another email?
   (b) You have received 80 emails so far this week. What is the expected amount of time until you receive another 40 emails?
   (c) You are expecting an email that you want to reply to as soon as it arrives. What is the probability you won't receive an email in the next hour?
   (d) You go on vacation for 3 days. What is the probability you received fewer than 30 emails in those 3 days?
   (e) You are away on a five day conference. What is the probability that you receive no emails in the first two days, but 20 by the fourth day?
2. Suppose that the number of calls per hour arriving at an answering machine service follows a Poisson process with $\lambda = 4$ calls per hour.
   (a) What is the probability that fewer than two calls come in the first hour?
   (b) Suppose that six calls arrive in the first hour. What is the probability that at least two calls will arrive in the second hour?
   (c) The person answering the phones waits until fifteen phone calls have arrived before going to lunch. What is the expected amount of time that the person will wait?
   (d) Suppose it is known that exactly eight calls arrived in the first two hours. What is the probability that exactly five of them arrived in the first hour?
   (e) Suppose it is known that exactly k calls arrive in the first four hours. What is the probability that exactly j of them arrived in the first hour?
3. You are practicing shooting a basketball. The number of times you score is a Poisson process with $\lambda = 30$ baskets an hour.
   (a) What is the probability you score 100 baskets in under 3 hours?
   (b) You decide to stay until you score 10 baskets. What is the probability that it takes you less than 15 minutes to score 10 baskets?
4. A father and son are going fishing. Let F_t denote the number of fish the father has caught by time t, and let S_t denote the number of fish the son has caught by time t. Suppose F_t and S_t are two independent Poisson processes with parameters λ_1 and λ_2, respectively. Let $Z_t = F_t + S_t$ denote the total number of fish caught by time t.
   (a) Let T denote the first time a fish is caught. Find the PDF of T.
   (b) Use your answer from part (a) to show that Z_t is a Poisson process. What is the parameter of this Poisson process?
   (c) What is the probability that the son catches the first fish?
5. Let X_t and Y_t be two independent Poisson processes with parameters λ_1 and λ_2, respectively, measuring the number of customers arriving in stores 1 and 2 respectively.

(a) What is the probability that in the first hour, a total of exactly four customers arrived at the two stores?

(b) Given that exactly four customers have arrived at the two stores by time t. What is the probability that all four went to store 1?

(c) What is the probability that a customer arrives in store 1 before any arrive in store 2.

6. Consider a Poisson process with rate λ. Find the probability that there are m events by time s given that there are n events by time t, where $m \leq n$ and $s \leq t$.

7. Suppose that we have three light bulbs. Each bulb's lifetime is random, given by an exponential random variable with parameter λ. What is the expected time until the last light bulb dies out?

8. Let X_t be a Poisson process with rate λ. Let T_m be the time of the mth event. (That is, the time when $X_t = m$.)

   (a) Find the CDF of T_m. (Hint: $\{T_m \leq t\} = \{X_t \geq m\}$)

   (b) Find the PDF of T_m.

9. One observation X is normally distributed with mean 0 and variance σ^2.

   (a) Find the likelihood function for σ.

   (b) What is the MLE for σ?

   (c) Show that X^2 is an unbiased estimator of σ^2.

10. Let X have a binomial distribution with parameters n and p.

    (a) Find the log likelihood function for p.

    (b) Find the MLE of p.

    (c) Is the MLE unbiased?

11. Let $X_1, ..., X_n$ be independent Bernoulli random variables with parameter p

    (a) Find the likelihood function for p

    (b) What is the MLE of p?

12. Let $X_1, \ldots, X_n$ be a random sample from a Gamma distribution with parameters 2 and β. The PDF of the Gamma distribution is

$$f(x|\beta) = \frac{1}{\Gamma(2)\beta^2} x e^{-x/\beta} 1(x > 0)$$

    Find the MLE of β.

13. Let $X_1, \ldots, X_n$ be i.i.d. random variables with pdf

$$f(x|\theta) = \begin{cases} \theta x^{\theta-1} & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

    where $\theta > 0$. Find the MLE of θ.

14. Let T be the current temperature outside. Suppose X and Y are two unbiased measurements of the temperature outside. Suppose X and Y have standard deviations σ_X and σ_Y, respectively.
    (a) For what value of a and b is $aX + bY$ unbiased?
    (b) Given the restrictions you found in part (a), what values of a and b minimize the variance of $aX + bY$. (The result is called the best linear unbiased estimator of T.)
15. Let $X_1, X_2, ..., X_n$ be normally distributed with mean 0 and variance σ^2.
    (a) Find the MLE of σ^2.
    (b) Find the expected value of $\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$
    (c) Is the MLE unbiased?
    (d) Find the MSE of the MLE and of the expression in part (b). Which is smaller?
16. Let $X_1, \ldots, X_n$ be i.i.d. from a uniform distribution on $[0, \theta]$ where $\theta > 0$.
    (a) Find a constant c so that cX_1 will be an unbiased estimator.
    (b) Find the MLE of θ and its MSE.
    (c) Find the MME of θ and its MSE.
17. Consider a random sample of size n from the PDF

$$f(x|\theta) = \frac{1}{\theta}xe^{-x^2/(2\theta)}1(x > 0)$$

    where $\theta > 0$. Find the MME of θ using
    (a) the first moment.
    (b) the second moment.
18. Consider a random sample of size m from a binomial distribution with parameters n and p. Find the MME of n and P.
19. A source emits a random number of photons K each time that it is triggered. We assume that the PMF of K is

$$p_K(k;\theta) = c(\theta)e^{-\theta k}, \quad k = 0, 1, 2, \ldots,$$

    where θ is the inverse of the temperature of the source and $c(\theta)$ is a normalization factor. We also assume that the photon emissions each time that the source is triggered are independent. We want to estimate the temperature of the source by triggering it repeatedly and counting the number of emitted photons.
    (a) Determine the normalization factor $c(\theta)$.
    (b) Find the expected value and the variance of the number K of photons emitted if the source is triggered once.
    (c) Derive the MLE for the temperature $\psi = 1/\theta$, based on $K_1, \ldots, K_n$, the number of photons emitted when the source is triggered n times.
    (d) Show that the ML estimator is consistent.