Recitations Week 6

1. A machine can be either working or broken down on a given day. If it is working, it will break down in the next day with probability $b$, and will continue working with probability $1-b$. If it breaks down on a given day, it will be repaired and be working the next day with probability $r$, and it will continue to be broken down with probability $1-r$.

   (a) Construct a Markov chain model for this process.

   (b) Write the transition matrix for the Markov chain.

   (c) Suppose when you bought the machine there was an 80% chance it was working and a 20% chance it was broken. Find the probability it will be working in two days.

2. Suppose in the previous question the machine can be broken down for at most 3 days, after which it is replaced by a new, working machine.

   (a) Construct a Markov chain model for this process. *Remark: You should have four total states.*

   (b) Write down the transition matrix.

3. A professor gives tests that are hard, medium, or easy. If he gives a hard test, his next test will be either medium or easy, with equal probability. However, if he gives a medium or easy test, there is a 0.5 probability that the next test will be of the same difficulty, and a 0.25 probability for each of the two other difficulty levels.

   (a) Construct an appropriate Markov chain.

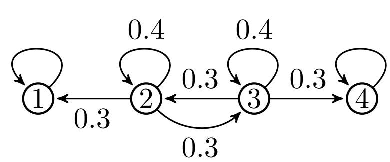
   (b) Suppose the are 3 tests in the course. If the first test was easy, what is the probability that the last test will be hard?

4. Consider a Markov chain with state space $S = \{1, 2\}$ with transition matrix

$$P = \begin{bmatrix} 1/3 & 2/3 \\ 3/4 & 1/4 \end{bmatrix}$$

   Assuming that the chain starts in state 1 at time $n = 0$, what is the probability that it is in state 1 at time $n = 3$?

5. Consider the Markov chain




   (a) Write down the transition matrix.

   (b) What is $\mathbf{P}(X_1 = 2, X_2 = 3, X_3 = 4 | X_0 = 2)$?

   (c) What is $\mathbf{P}(X_2 = 3 | X_0 = 2)$?

   (d) Suppose the initial state (i.e. value of $X_0$) is chosen at random. What is $\mathbf{P}(X_1 = 2)$?

   (e) What are the communication classes?

(f) Which are recurrent and which are transient?

(g) Are there any absorbing states? If so, what are they?

6. Consider a Markov chain with state space $S = \{1, 2, 3, 4, 5, 6\}$ and transition matrix

$$P = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0 & 0.9 & 0 \\ 0.25 & 0.25 & 0 & 0 & 0.25 & 0.25 \\ 0 & 0 & 0.7 & 0 & 0.3 & 0 \\ 0 & 0.2 & 0 & 0.2 & 0.2 & 0.4 \end{bmatrix}$$

(a) Draw the Markov chain.

(b) What are the communication classes?

(c) Which are recurrent and which are transient?

(d) Find the period of the recurrent chains.

7. Consider a Markov chain with the following transition matrix.

$$P = \begin{bmatrix} 0 & 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.4 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

(a) Draw the Markov chain.

(b) What are the communication classes?

(c) Which are recurrent and which are transient?

(d) Find the period of the recurrent chains.

8. Consider the Markov chain with the following transition probabilities. $p_{12} = p_{23} = p_{31} = p_{24} = p_{45} = p_{56} = p_{61} = 1$.

(a) Draw this Markov chain.

(b) Convince yourself it has only one communication class. Is this class recurrent or transient?

(c) Find its period.

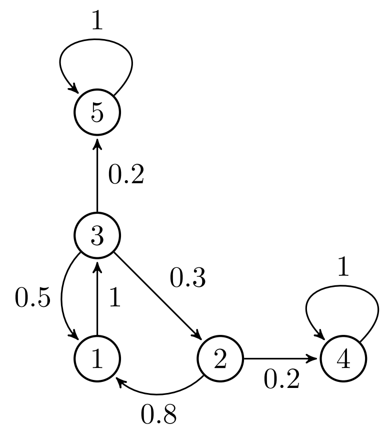
9. Consider a Markov chain with states $1, 2, \ldots, 9$ and transition probabilities $p_{12} = p_{17} = \frac{1}{2}$, $p_{i(i+1)} = 1$ for $i = 2, 3, 4, 5, 7, 8$, and $p_{61} = p_{91} = 1$.

(a) Draw this chain.

(b) Identify the recurrent class.

(c) Find the period of the recurrent class. Is the recurrent class periodic?

10. Consider the Markov chain shown below:




(a) Which states are absorbing states?

(b) Let $a_i$ denote the probability of being absorbed by state 5 if we start from state $i$. Write the formulas for $a_1, \ldots, a_5$.

(c) Solve for $a_2$.

11. Using the same Markov chain in the previous problem, let $b_i$ denote the expected time until absorption (into either state 4 or state 5), if we start from state $i$.

(a) Write equations for $b_1, \ldots, b_5$.

(b) Solve for $b_2$.

12. Suppose you have a biased coin, with probability of heads 2/3 and probability of tails 1/3. You start with $i$ dollars ($i = 1, 2, 3$), and play the following game: at each turn, you flip the coin. If it is heads, you gain a dollar. If it is tails, you lose a dollar. Whenever you reach \$0 or \$4, you quit playing.

(a) Draw a Markov chain diagram for this game.

(b) What is the probability that, if you start with \$2, when the game ends you have \$4?

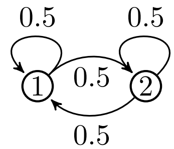
(c) Suppose you start with \$2. What is the expected number of coin tosses until the game is over?

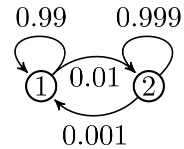
13. Suppose you are bored and are flipping a coin until you get 4 heads in a row. Find the expected number of flips until you get 4 heads.

14. For problem number 5 above, find all of the absorption probabilities and the expected number of time steps until you are absorbed into one of the absorbing states.

15. For number 6 above, find the probability that you are absorbed by each recurrent class given that you start in state 4.

16. Consider the chains



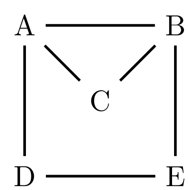

For each chain, compute the steady-state probabilities $\pi_1, \pi_2$.

17. Consider the Markov chains with transition matrices

$$P = \begin{bmatrix} 0.4 & 0.2 & 0.4 \\ 0.6 & 0 & 0.4 \\ 0.2 & 0.5 & 0.3 \end{bmatrix} \qquad P = \begin{bmatrix} 0.2 & 0.4 & 0.4 \\ 0.1 & 0.5 & 0.4 \\ 0.6 & 0.3 & 0.1 \end{bmatrix}$$

Find the probability in the long run that the chain is in state 1.

18. Consider a random walk on the following graph. That is, from each point $A, B, C, D$, or $E$ you randomly pick an adjacent edge and move along it to the next state. Suppose that many people are walking along this graph.




(a) After a long time, what fractions of people are present at each site? What fraction of time

(b) After a long time, how much traffic is there on the path between $A$ and $C$? That is to say, what fraction of people are traveling along the path between $A$ and $C$ after a long time?

(c) Assume a walker starts in state $D$. What is the expected amount of time until they reach state $C$?

19. Referring to problem 2, what is the expected long run frequency with which the machine is replaced? (That is, the frequency we move from the 3 day old state to the new state.) What is the expected long run frequency with which the machine is fixed one day after it breaks?

20. Referring to problem 2. In the long run, what fraction of the time is the machine broken?

21. Referring to problem 3. Suppose the professor has given a 400 tests over the years. Estimate how many of them were hard.