Recitations Week 5

1. Let X be a normal random variable with mean μ and variance σ^2. Show that $Z = (X - \mu)/\sigma$ is a normal random variable with mean 0 and variance 1. Hint: Find the CDF and/or PDF of Z. (A normal random variable with mean 0 and variance 1 is called a standard normal random variable.)

2. Let X be a normal random variable with mean $\mu = 10$ and standard deviation $\sigma = 5$. Write each of the following probabilities in terms of Φ, the CDF of a standard normal random variable. Hint: Use the result from the previous problem.

   (a) $P(X \leq 15)$

   (b) $P(-5 < X < 25)$

   (c) $P((X - 10)^2 < 4)$

3. Let $X_1, X_2, ..., X_n$ be independent random variables with mean μ and variance σ^2. Let $X = \frac{1}{n}\sum_{i=1}^{n} X_i$.

   (a) Find $E[X]$.

   (b) Find $\text{Var}(X)$ and $\sigma(X)$

4. Let $Y_1, Y_2, ..., Y_{1000}$ be independent random variables with mean 17 and variance 250. Let $Y = \frac{1}{1000}\sum_{i=1}^{1000} X_i$.

   (a) By the central limit theorem, Y is approximately what type of random variable.

   (b) Approximate $P(15 < Y < 18)$. Hint: Use your answer from number 5.

5. A coin is tossed 10,000 times for an experiment. The probability of it landing heads on any flip is 0.51. Let Z be the number of heads obtained throughout the experiment.

   (a) What type of random variable is Z?

   (b) What is the expected value and standard deviation of Z?

   (c) Approximate the probability that there are more heads than tails using the function Φ. (Hint: Z can be written as a sum of independent random variables.)

6. Let X be a random variable with mean μ and standard deviation σ. Find a constant k such that $P(|X - \mu| > k\sigma) \leq 0.1$. (Chebyshev's inequality will help.)

7. Let $X_1, X_2, ..., X_n$ be independent random variables with mean μ and variance σ^2. Let $X = \frac{1}{n}\sum_{i=1}^{n} X_i$. Find the smallest n could be so that $P(|X - \mu| \geq \sigma) \leq 0.05$.

8. A machine can be either working or broken down on a given day. If it is working, it will break down in the next day with probability b, and will continue working with probability $1-b$. If it breaks down on a given day, it will be repaired and be working the next day with probability r, and it will continue to be broken down with probability $1-r$.

   (a) Construct a Markov chain model for this process.

   (b) Write the transition matrix for the Markov chain.

   (c) Suppose when you bought the machine there was an 80% chance it was working and a 20% chance it was broken. Find the probability it will be working in two days.

9. Suppose in the previous question the machine can be broken down for at most 3 days, after which it is replaced by a new, working machine.

   (a) Construct a Markov chain model for this process. *Remark: You should have four total states.*

   (b) Write down the transition matrix.

10. A professor gives tests that are hard, medium, or easy. If he gives a hard test, his next test will be either medium or easy, with equal probability. However, if he gives a medium or easy test, there is a 0.5 probability that the next test will be of the same difficulty, and a 0.25 probability for each of the two other difficulty levels.

    (a) Construct an appropriate Markov chain.

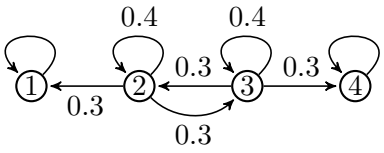
    (b) Suppose the are 3 tests in the course. If the first test was easy, what is the probability that the last test will be hard?

11. Consider a Markov chain with state space $S = \{1, 2\}$ with transition matrix

$$P = \begin{bmatrix} 1/3 & 2/3 \\ 3/4 & 1/4 \end{bmatrix}$$

    Assuming that the chain starts in state 1 at time $n = 0$, what is the probability that it is in state 1 at time $n = 3$?

12. Consider the Markov chain



    (a) Write down the transition matrix.

    (b) What is $\mathbf{P}(X_1 = 2, X_2 = 3, X_3 = 4 | X_0 = 2)$?

    (c) What is $\mathbf{P}(X_2 = 3 | X_0 = 2)$?

    (d) Suppose the initial state (i.e. value of X_0) is chosen at random. What is $\mathbf{P}(X_1 = 2)$?

    (e) What are the communication classes?

    (f) Which are recurrent and which are transient?

(g) Are there any absorbing states? If so, what are they?

13. Consider a Markov chain with state space $S = \{1, 2, 3, 4, 5, 6\}$ and transition matrix

$$P = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0 & 0.9 & 0 \\ 0.25 & 0.25 & 0 & 0 & 0.25 & 0.25 \\ 0 & 0 & 0.7 & 0 & 0.3 & 0 \\ 0 & 0.2 & 0 & 0.2 & 0.2 & 0.4 \end{bmatrix}$$

(a) Draw the Markov chain.

(b) What are the communication classes?

(c) Which are recurrent and which are transient?

REVIEW PROBLEMS

14. Let X and Y be independent discrete random variables taking values $0, 1, 2, ...$

$$P(X + Y = n) = \sum_{k=0}^{n} P(X = k)P(Y = n - k)$$

15. Let X and Y be independent Poisson random variables with parameters λ_1 and λ_2, respectively. Show that $Z = X + Y$ is a Poisson random variable with parameter $\lambda_1 + \lambda_2$. You may use the fact that

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

16. A random variable has the CDF $F(x) = a + b \arctan(x)$. Find a and b.

17. Let X_1, X_2, X_3 be independent random variables taking values $1, 2, 3, ..., 10$ with equal probability.

(a) Find the PMF of $Y = \max\{X_1, X_2, X_3\}$.

(b) Find the PMF of $Z = \min\{X_1, X_2, X_3\}$.

18. Let X and Y be independent continuous random variables with CDF F and PDF f.

(a) Show that $V = \max(X, Y)$ has CDF $F_V(x) = F(x)^2$ and PDF $f_V(x) = 2f(x)F(X)$.

(b) Find the density function of $U = \min(X, Y)$.

19. Let X, Y be independent exponential random variables with parameters λ, μ. Compute

$$\mathbf{P}(X < Y).$$