Recitations Week 4

1. The random variables X and Y have joint PMF

$$p_{X,Y}(k_1, k_2) = \frac{c\lambda^{k_1+k_2}}{k_1!}$$

where $k_1, k_2 = 0, 1, 2, ...$ and $\lambda \in (0, 1)$ is a parameter.

   (a) Find the constant c.

   (b) Find the PMF of X. What type of random variable is X?

   (c) Find the PMF of $Z = Y + 1$. What type of random variable is Z?

2. The random variables X and Y have joint density function

$$f_{X,Y}(x, y) = \begin{cases} e^{-x-y} & \text{if } x, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find $P(X > Y)$

   (b) Find $P(X + Y \leq 1)$

3. The random variables X and Y have joint density function

$$f_{X,Y}(x, y) = \begin{cases} c(x^2 + 0.5xy) & \text{if } 0 < x < 1, 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the constant c.

   (b) Find the marginal PDF, $f_X(x)$, of X.

   (c) Find the conditional PDF, $f_{X|Y}(x|y)$ of X.

   (d) Find the conditional expectation $E[X|Y = y]$ where $y \in (0, 2)$

4. Suppose two random variables X, Y have joint pdf

$$f_{X,Y}(x, y) = \begin{cases} \lambda^2 e^{-\lambda y} & \text{if } 0 \leq x \leq y \\ 0 & \text{otherwise} \end{cases}$$

   (a) Show that the above is indeed a PDF.

   (b) Find $P(X > Y^2)$

   (c) Find the conditional pdf $f_{Y|X}(y|x)$.

   (d) Find the conditional expectation $\mathbf{E}[Y|X = x]$.

5. Let X be a normal random variable with mean μ and variance σ^2. Show that $Z = (X - \mu)/\sigma$ is a normal random variable with mean 0 and variance 1. (A normal random variable with mean 0 and variance 1 is called a standard normal random variable.)

6. Let X be a normal random variable with mean $\mu = 10$ and standard deviation $\sigma = 5$. Write each of the following probabilities in terms of Φ, the CDF of a standard normal random variable. Hint: Use the result from the previous problem.

   (a) $P(X \leq 15)$

   (b) $P(-5 < X < 25)$

   (c) $P((X - 10)^2 < 4)$

7. Let $X_1, X_2, ..., X_n$ be independent random variables with mean μ and variance σ^2. Let $X = \frac{1}{n}\sum_{i=1}^{n} X_i$.

   (a) Find $E[X]$.

   (b) Find $\text{Var}(X)$ and $\sigma(X)$

8. Let $Y_1, Y_2, ..., Y_{1000}$ be independent random variables with mean 17 and variance 250. Let $Y = \frac{1}{1000}\sum_{i=1}^{1000} X_i$.

   (a) By the central limit theorem, Y is approximately what type of random variable.

   (b) Approximate $P(15 < Y < 18)$. Hint: Use your answer from number 5.

9. A coin is tossed 10,000 times for an experiment. The probability of it landing heads on any flip is 0.51. Let Z be the number of heads obtained throughout the experiment.

   (a) What type of random variable is Z?

   (b) What is the expected value and standard deviation of Z?

   (c) Approximate the probability that there are more heads than tails using the function Φ. (Hint: Z can be written as a sum of independent random variables.)

10. Let X be a random variable with mean μ and standard deviation σ. Find a constant k such that $P(|X - \mu| > k\sigma) \leq 0.1$. (Chebyshev's inequality will help.)

11. Let X and Y be independent discrete random variables taking values $0, 1, 2, ...$

$$P(X+Y=n) = \sum_{k=0}^{n} P(X=k)P(Y=n-k)$$

12. Let X and Y be independent Poisson random variables with parameters λ_1 and λ_2, respectively. Show that $Z = X + Y$ is a Poisson random variable with parameter $\lambda_1 + \lambda_2$. You may use the fact that

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

13. A random variable has the CDF $F(x) = a + b \arctan(x)$. Find a and b.

14. Let X_1, X_2, X_3 be independent random variables taking values $1, 2, 3, ..., 10$ with equal probability.

    (a) Find the PMF of $Y = \max\{X_1, X_2, X_3\}$.

    (b) Find the PMF of $Z = \min\{X_1, X_2, X_3\}$.

15. Let X and Y be independent continuous random variables with CDF F and PDF f.

    (a) Show that $V = \max(X, Y)$ has CDF $F_V(x) = F(x)^2$ and PDF $f_V(x) = 2f(x)F(X)$.

    (b) Find the density function of $U = \min(X, Y)$.

16. Let X, Y be independent exponential random variables with parameters λ, μ. Compute

$$\mathbf{P}(X < Y).$$

17. Jane goes to the bank to make a withdrawal, and is equally likely to find 0 or 1 customers ahead of her. The service time of the customer ahead, if present, is exponentially distributed with parameter λ. What is the CDF of Jane's waiting time?

18. Let $Y = X_1 + \ldots + X_N$, where the random variables X_i are geometric with parameter p and N is geometric with parameter q. Assume that $N, X_1, X_2, \ldots$ are independent. Show that Y is geometric with parameter pq.

19. Suppose we repeatedly flip a coin with probability $2/3$ of heads. Find the PMF, mean, and variance of the following random variables:

    (a) $T =$ the time index of the first tail.

    (b) $B =$ the length (number of flips) of the first run of heads.

    (c) $I =$ the length of the first run of tails

    (d) $Z =$ the number of flips after the first head up to and including the first tail.

    For example, if the result of the coin flips was $HHTHTTT...$ then we would have $T = 1$, $B = 2$, $I = 1$, $Z = 2$.