Recitations Week 3

1. Let X be a random variable with PMF

$$p_X(k) = \begin{cases} x^2/a, & \text{if x = } -3,-2,-1,0,1,2,3, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Find a.

   (b) Calculate $\mathbf{P}(X^2 \leq 1)$.

2. The random variable X has the PDF

$$f_X(x) = \begin{cases} cx^{-4}, & \text{if } x \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Determine the value of c.

   (b) Calculate $\mathbf{P}(X > 2|X > 1)$.

3. An archer takes 10 shots at a target and has probability 0.2 of hitting the target with each shot, independently of all other shots. Let X be the number of hits.

   (a) Calculate the PMF of X.

   (b) What is the probability of scoring no hits?

   (c) What is the probability of scoring more hits than misses?

4. A world chess champion plays against 100 amateurs in a large simultaneous exhibition. It has been estimated from past experience that the champion wins each game with probability 0.99, independent of any other games. Use the Poisson approximation to the probabilities that the champion wins (a) 100 games (b) 98 games (c) 90 games.
   Using the binomial PMF gives $0.366, 0.185$*, and* $7.007 \cdot 10^{-8}$*.*

5. Let X be a random variable that takes values from 0 to 9 with equal probability 1/10. Find the pmf of the random variable $Y = X \mod 3$. *Note:* $Y = X \mod 3$ *is the remainder when dividing* X *by 3. For example,* $7 \mod 3 = 1$*, whereas* $6 \mod 3 = 0$*.*

6. Let X be uniformly distributed in the unit interval $[0, 1]$. Consider the random variable $Y = g(X)$.

   (a) Suppose $g(x) = x^2$. Find the CDF and PDF of Y.

   (b) Suppose instead $g(x) = e^x$. Find the CDF and PDF of Y.

7. Suppose Dave fails quizzes with probability 1/4, independent of other quizzes.

   (a) What is the probability that Dave fails exactly two of the next six quizzes?

   (b) What is the probability that the second and third time that Dave fails a quiz will occur when he takes his eighth and ninth quizzes, respectively?

8. Four buses carrying 148 students arrive at a convention. The buses carry 40, 33, 25, and 50 students, respectively.

   (a) One of the bus drivers is randomly selected. Let X denote the number of students on the selected driver's bus. Compute $E[X]$.

   (b) One of the students is randomly selected. Let Y denote the number of students that were on the selected student's bus. Compute $E[Y]$.

9. You toss independently a fair coin and you count the number of tosses until the first tail appears. If this number is n, you receive 2^n dollars. What is the expected amount of money you will receive? *Consider also how much money you would be willing to pay in order to play this game. This is known as the St. Petersburg Paradox.*

10. Calculate the mean and variance of the random variables in problems 2 and 3.

11. Consider problem 4.

    (a) Find the expectation and the variance of X.

    (b) Suppose the archer has to pay \$3 to enter the archery range and he gets \$2 for each hit. Let Y be his profit. Find the expectation and variance of Y.

12. Let X be a geometric random variable with parameter p. Calculate the following (a) $E[X]$ (b) $E[(X+1)X]$ (c) Var(x) Hint: Use parts (a) and (b).

13. Let X be an exponential random variable with parameter λ. Calculate the following:

    (a) $\mathbf{E}[X]$ (b) $\mathbf{E}[X^2]$ (c) Var(X)

14. Let T be an exponential random variable with parameter λ. An exponential distribution has the nice property of being memoryless meaning that

$$P(T > t + s | T > s) = P(T > t)$$

    where $t, s > 0$. Prove that this true.

15. A city's temperature is modeled as a random variable with mean and standard deviation both equal to 10 degrees Celsius. A day is described as "normal" if the temperature during that day ranges within one standard deviation from the mean. What would the temperature range for a normal day be if temperature were expressed in Fahrenheit? *Note:* $F = \frac{9}{5}C + 32$.