Recitations Week 2

1. We are given two urns, each containing a collection of colored balls. Urn 1 contains two white and three blue balls. Urn 2 contains three white and four blue balls. A ball is drawn at random from the urn 1 and put into urn 2, and then a ball is picked at random from urn 3 and examined. What is the probability that it is blue?

2. A test for jambes lourdes ("heavy legs") is assumed to be correct 95% of the time: if a person has the illness, the test results are positive with probability 0.95, and if the person does not have the disease, the test results are negative with probability 0.95. A random person drawn from a certain population has probability 0.001 of having the illness. Given that the person just tested positive, what is the probability of having the disease?

3. Prove that if $P(A) = P(B) = 2/3$, then $P(A|B) \geq 1/2$. Hint: Find the smallest possible value for $P(A \cap B)$.

4. A radar, which is attached to an alarm, is used to detect planes flying overhead. Let $F$ be the event that a plane will fly overhead tomorrow, and let $A$ be the event that the radar generates an alarm tomorrow. The radar is not perfect so it may not generate an alarm when a plane is present or set off the alarm when there isn't one.

   (a) Interpret the meaning of the following: $P(F) = 0.05$, $P(A|F) = 0.99$, and $P(A|F^c) = 0.1$.

   (b) Find the probability that a plane will fly overhead tomorrow, but the alarm doesn't go off. Hint: Write the event in terms of $A$ and $F$.

   (c) Find the probability that the alarm goes off tomorrow, but no plane will be flying overhead. Hint: Write the event in terms of $A$ and $F$.

   (d) Find $P(A)$

   (e) Find $P(F|A)$ and $P(F|A^c)$. How worried should you be if the alarm goes off? How safe should you feel if the alarm doesn't go off?

5. This is a much discussed puzzle, based on an old American game show. You, the contestant, must choose one of three doors. One of these doors conceals a new car, whereas there is no prize behind the other two. When you choose a door, it is not opened immediately. Instead, the host randomly opens a different door (one without the car behind it). The host then offers you the opportunity to change your choice to the third door (the door not chosen by you and not opened by the host). Consider two possible strategies:

   (a) Stay with your initial choice.

   (b) Switch to the third door.

   Compute the probability of winning the car for each of these. Which is better? Hint: Set up a tree diagram.

6. Suppose $A$ and $B$ are mutually exclusive (i.e., $A$ and $B$ are disjoint), and both $A$ and $B$ have positive probabilities. Are $A$ and $B$ independent? Prove your answer.

7. Prove that $A$ and $B$ are independent if $P(B|A) = P(B|A^c)$.

8. Suppose $A$ and $B$ are independent. Show $A^c$ and $B$ are independent.

9. Alice wants to go to the opera, but Bob wants to go to the movies. They agree to settle this by flipping a coin. Unfortunately, the only available coin is biased (though the bias is not known exactly). How can they use the biased coin to make a decision that either option (opera or the movies) is equally likely to be chosen?

10. A power utility can supply electricity to a city from $n$ different power plants. Power plant $i$ fails with probability $p_i$, independent of the others.

    (a) Suppose that any one plant can produce enough electricity to supply the entire city. What is the probability that the city will experience a black-out?

    (b) Suppose that two power plants are necessary to keep the city from a black-out. Find the probability that the city will experience a black-out.

11. Suppose we have a class with $n$ people. We assume that every person has an equal probability of being born on any day during the year, independent of everybody else (and we ignore leap years).

    (a) Write an expression for the probability that each person has a distinct birthday, in terms of $n$.

    (b) What is this probability, for $n = 23$? Hint: You may want to use the approximation $e^{-k/365} = 1 - k/365$ to help compute the product from part (a).