Recitations Week 1

1. Consider rolling a six-sided die. Let A be the set of outcomes where the roll is an even number. Let B be the set of outcomes where the roll is greater than 3.

   (a) Write A and B using the appropriate set notation.

   (b) Calculate $(A \cup B)^c$.

   (c) Calculate $A^c \cap B^c$.

   (d) Calculate $(A \cap B)^c$.

   (e) Calculate $A^c \cup B^c$.

   Note: The fact that $(A \cup B)^c = A^c \cap B^c$ is an instance of DeMorgan's law. Similarly for $(A \cap B)^c = A^c \cup B^c$.

2. Let A, B, C be sets. For each of the sets below, express it in terms of A, B, C by using unions, intersections, and complements.

   (a) $\{x | x \text{ is an element of exactly one of } A, B, C\}$

   (b) $\{x | x \text{ is an element of exactly two of } A, B, C\}$

   (c) $\{x | x \text{ is an element of at least two of } A, B, C\}$

   As an example, the set $\{x | x \text{ is an element of at least one of } A, B, C\}$ may be written as $A \cup B \cup C$.

3. Let Ω be a set of all outcomes of interest, and consider sets A, B, C that are subsets of Ω. Suppose that A and B are disjoint.

   (a) Are A^c and B^c disjoint? Justify your answer.

   (b) Are $A \cap C$ and $B \cap C$ disjoint?

   (c) Are $A \cup C$ and $B \cup C$ disjoint?

4. List all the subsets of $\{1, 2, 3\}$. There should be 8.

5. Consider two sets A, B such that neither is a subset of the other and the two are not disjoint.

   (a) Draw a diagram depicting the sets A and B.

   (b) Write the set A as a union of two disjoint sets.

   (c) Write $A \cup B$ as a union of three disjoint sets.

6. Consider three sets A, B, C such that $A \subseteq B$ and $B \subseteq C$. Also assume that $A \neq B$ and $B \neq C$.

   (a) Draw a diagram depicting the sets A, B, C.

   (b) Write the set C as a union of three disjoint sets.

7. Prove that
$$P(\cap_{i=1}^n A_i) = 1 - P(\cup_{i=1}^n A_i^c)$$

8. Draw a diagram to see why the identity
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
is true. Now use the axioms of probability to prove the identity.

9. The formula above may be generalized to allow for more than two events. Let A, B, C be events. Prove
$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$
[Drawing a diagram may be helpful.] This is a special case of the more general inclusion-exclusion formula.

10. Of the students in a class, 60% are over 21, 70% like chocolate, and 40% fall into both categories. What is the probability that a randomly selected student is not over 21 nor a chocolate lover? You should be able to do this intuitively. But also do this formally using the equations from problems 7 and 8.

11. The following are two "continuity properties" of probability laws.

    (a) Let $A_1, A_2, \ldots$ be an infinite sequence of events, which is "monotonically increasing", meaning that $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$. Let $A = \cup_{n=1}^\infty A_n$. Show that
    $$\mathbf{P}(A) = \lim_{n\to\infty} \mathbf{P}(A_n).$$
    Hint: Express the event A as a union of countably many disjoint sets.

    (b) Suppose now that the events are "monotonically decreasing", i.e. $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$. Let $A = \cap_{n=1}^\infty A_n$. Show that
    $$\mathbf{P}(A) = \lim_{n\to\infty} \mathbf{P}(A_n).$$
    Hint: Apply the result of part (a) to the complements of the events.

    (c) Consider a probabilistic model whose sample space is the real line. Show that
    $$\mathbf{P}([0,\infty)) = \lim_{n\to\infty} \mathbf{P}([0,n]), \qquad \text{and} \qquad \lim_{n\to\infty} \mathbf{P}([n,\infty)) = 0.$$

12. Ten books are placed in random order on a bookshelf. Find the probability of three given books being side by side.

13. One of the numbers 2, 4, 6, 7, 8, 11, 12, 13 is chosen at random as the numerator of a fraction, and then one of the remaining numbers is chosen at random as the denominator of the fraction. What is the probability of the fraction being in lowest terms?

14. In throwing $6n$ dice, what is the probability of getting each face n times? Approximate for large n.

15. Suppose we draw the top 7 cards from a well-shuffled standard 52-card deck. Find the probability that

    (a) The 7 cards include exactly 3 aces.

    (b) The 7 cards include exactly 2 kings.

    (c) The 7 cards include exactly 3 aces, or exactly 2 kings, or both.

16. Eight rooks are placed in distinct squares of an 8 by 8 chessboard, with all possible placements being equally likely. Find the probability that all the rooks are safe from another, i.e. that there is no row or column with more than one rook.