# Modeling web-crawlers on the Internet with random walks on graphs

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- My goal then is to describe properties of random walks on the random graphs that model the growth of the Internet.

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Alternatively, you may view the graph as being built one vertex at a time with each vertex attaching with $d$ edges to the other vertices with the possibility of self-loops.

# Convergence to a Stationary State

The random walk on $G_{d,n}$ is irreducible and aperiodic, because the graph is connected and the first vertex has a selfloop. By the ergodic theorem, $P^t$ $(t \in \mathbb{N})$ converges to the matrix $\Pi$ whose rows are all $\pi$. Here $P$ is the transition matrix, and $\pi$ is the stationary distribution.

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Physically, this means that for large time $t$ the initial state of the system does not influence the final state of the system. In regards to the Internet, this means that eventually the probability that the web crawler starts visits a particular website will be the same regardless of which website it started at. This "eventually" is a time $t$ when $P^t \approx \Pi$. It is related to the time when it is likely that the web-crawler has indexed every website.

# Reversible Markov Chains

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The following matrices will be useful when proving facts about reversible Markov chains.

## Definition

Let $D$ be the diagonal matrix with $D(i,i) = \pi(i)$. Let $Q = DP$, and $A = D^{1/2}PD^{-1/2}$.

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## Theorem

*$Q$ and $A$ are symmetric matrices.*

# Key Bounds

## Definition

The relative pointwise distance is $\Delta(t) = \max_{i,j} |\frac{P^t(i,j)}{\pi(j)} - 1| = \max_{i,j} |\frac{P^t(i,j)}{\Pi(i,j)} - 1|$.

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$\lambda_2$ can be bounded in terms of conductance. Therefore, we will be able to find bounds for $\lambda_{max}$ in terms of conductance when $\lambda_{max} = \lambda_2$.

## Theorem (Cheeger's inequality)

$\lambda_2 < 1 - \frac{(\Phi_G)^2}{2}$

# Random walks with $\lambda_{max} = \lambda_2$

Changes to the random walk can be made to shift all the eigenvalues to be non-negative. Then $1 = \lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_n \geq 0$. So $\lambda'_{max} = \lambda'_2$.

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## Example

A lazy random walk is one in which there is a probability of $1/2$ that you remain at the same vertex. Then new transition matrix is $P' = \frac{I+P}{2}$. All of the eigenvalues of $P'$ are between 1 and 0. $\lambda'_2 = \lambda'_{max}$, and $\lambda'_2 = \frac{1+\lambda_2}{2}$.

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The latter random walk is a continuous-time Markov chain. We still have the same bound as before for $\Delta(t)$. I found a simple proof of this result using the definition of $H_t$ and the result for $P^t$.

# Definitions concerning conductance

## Definition

Let $G$ be a undirected multigraph with self-loops with vertex set $V$ and edge set $E$.

1. $d_G(v)$ is the degree of a vertex, where self loops contribute 2 to the degree.
2. The volume of a set $S \subseteq V$ is $\mathrm{Vol}_G(S) = \sum_{v \in S} d_G(v)$
3. The cutset of $S \subseteq V$ is $C_G(S, S^C) = \{e \in E : e \text{ has one endpoint in } S \text{ and one in } S^C\}$
4. The conductance $\Phi_G$ of $G$ is $\Phi_G = \min\left\{\frac{|C_G(S,S^C)|}{\mathrm{Vol}_G(S)} : \varnothing \neq S \subseteq V, \mathrm{Vol}_G(S) \leq \frac{\mathrm{Vol}_G(V)}{2}\right\}$.

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I have generalized this definition to all graphs.

## Definition

$$\Phi_G = \min\left\{1, \frac{|C_G(S,S^C)|}{\mathrm{Vol}_G(S)} : S \subseteq V, \varnothing \neq S \neq V, 0 \leq \mathrm{Vol}_G(S) \leq \frac{\mathrm{Vol}_G(V)}{2}\right\}.$$

# Results about Conductance

## Theorem

- $0 \leq \Phi_G \leq 1$.
- $\Phi_G = 0$ *iff* $G$ *is disconnected.*
- *Hubs are the only graphs with four or more vertices that have* $\Phi_G = 1$.

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$\Phi_G = \min\{1, \frac{Q(S,S^c)}{P(S)} : S \subseteq V, \varnothing \neq S \neq V, P(S) \leq 1/2\}$, *where* $Q(S, S^c) = \sum_{i \in S, j \in S^c} Q(i,j)$ *and* $P(S) = \sum_{i \in S} \pi(i)$.

# Theorem Bounding Conductance

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*For all integers* $d \geq 2$, $\lim_{n\to\infty} P(\Phi_{G_{d,n}} < \frac{\alpha}{2d+\alpha}) = 0$ *for all* $\alpha < \min\{\frac{2d-3}{4}, \alpha_0\}$ *where* $\alpha_0$ *is the unique positive solution less than* $d$ *of the equation* $\frac{3}{2}\alpha(1 + \ln(d) - \ln(\alpha)) = (d - 1 - \frac{5}{2}\alpha)\ln 2$.

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Bound $\Phi_{G_{d,n}}$ below in terms of edge expansion $\rho_{G_{d,n}} = \min\{\frac{|C_G(S,S^c)|}{|S|}\}$

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$P(\rho_{G_{d,n}} < \alpha) \leq \sum_{k=2}^{n/2} \binom{n}{k}\alpha k\binom{dn}{\alpha k} f(k)$ where $f(k) \geq P(A = Good(S))$ for all $S$ with $|S| = k \leq n/2$ and for all $A$ with $|A| = j$ for any $j \leq \alpha k - 1$.

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Lastly, we want to find a suitable candidate for $f(k)$.

$P(A = Good(S)) \leq \binom{dn}{|A|/2} \times \binom{dn-|A|/2}{dk-|A|/2}^{-1}$. This is a bound that I found, which is slightly better than the one given in the paper.

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$P(\rho_{G_{d,n}} < \alpha) \leq \sum_{k=2}^{n/2} \alpha k(2)^{\frac{\alpha}{2}k}(\frac{ed}{\alpha})^{\frac{3}{2}\alpha k}(\frac{k}{n})^{(d-1-2\alpha)k}$ after some trickery. The first or last term in the sum is the largest. So $n/2$ times the first or last term bounds $P(\rho_{G_{d,n}} < \alpha)$. The result goes to zero for specified $\alpha$.

# Good and Bad Mini-vertices

## Definition

A mini-vertex is said to be associated with a set if its corresponding vertex is in the set. A mini-vertex is good with respect to a set $S \subseteq V$ if it is associated with $S$ and its father is associated with $S^c$ or if it is associated with $S^c$ and its father it associated with $S$. A mini-vertex is said to be bad with respect to $S$ if it is not good with respect to $S$. By convention, mini-vertex 1 is bad. $Good(S)$ is the set of all mini-vertices that are good with respect to $S$.

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Note that $|Good(S)| = |C_G(S, S^c)|$. Also note that every good mini-vertex contributes 1 to the volume of $S$; every bad mini-vertex associated with $S$ contributes 2 to the volume of $S$; and every bad mini-vertex associated with $S^c$ contributes 0 to the volume of $S$. Using these facts and some simple but neat computations, I found the bound $P(A = Good(S)) \leq \binom{dn}{|A|/2} \times \binom{dn-|A|/2}{dk-|A|/2}^{-1}$ in the same way a different bound was found in the paper.

# Results

In the following table are the values that conductance is bounded above.

| $\Phi_G \geq$ | d=2 | d=3 | d=4 | d=5 |
|---|---|---|---|---|
| Their bound | 0.00257956 | 0.02168342 | 0.02439024 | 0.01960784 |
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Using Cheeger's inequality, I found the values that $\lambda_2$ is bounded below.

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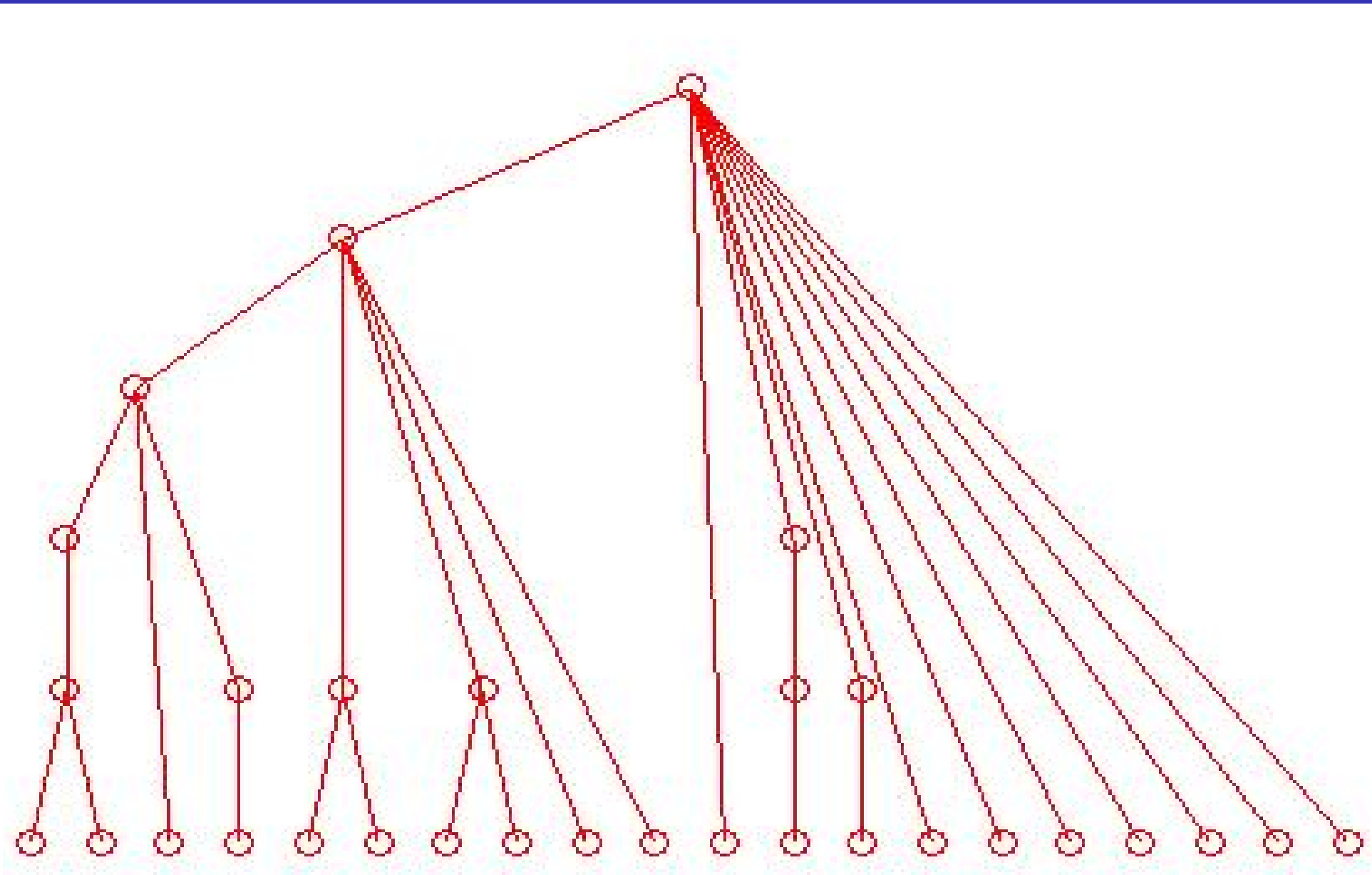
$\lambda_2$ seems to approach the following constants based on a program I wrote.

| $\lambda_2$ | d=2 | d=3 | d=4 | d=5 |
|---|---|---|---|---|
| | 0.8354 | 0.7212 | 0.6428 | 0.5852 |

I conjecture that the constant $\lambda_2$ decreases with $d$ and approaches 0.5.

# A randomly generated graph with 30 vertices (d=1)



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## Idea behind Proof:

Let $S_n$ be the set of the second vertex added to the graph and all of its descendants at time $n$. Then $S_n^c$ is the set of the first vertex and all of its descendants except for the second vertex and its descendants.
$|C_G(S_n, S_n^c)| = 1$ for all $n$ because the only edge in $C_G(S_n, S_n^c)$ is the edge connecting the first and second vertices.

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Both $\text{Vol}_{G_{d,n}}(S_n)$ and $\text{Vol}_{G_{d,n}}(S_n^c)$ approach infinity. Using these sets to estimate conductance gives us the desired result, since
$\Phi_{G_{d,n}} \leq \frac{|C_G(S_n, S_n^c)|}{\min\{\text{Vol}_{G_{d,n}}(S_n), \text{Vol}_{G_{d,n}}(S_n^c)\}}$.

# Summary

For $d \geq 2$, $\lambda_2$ is very likely to be less than $1 - c$ for a constant $c$. So $P^t$ most likely approaches $\Pi$ faster than $(1 - c)^t$ when the graph is large. When $d = 1$, this is not the case.

If this is a good model of the Internet with $d \geq 2$, then a web-crawler will be able to index new websites well even as the Internet continues to grow. If this is a good model with $d = 1$, then the web-crawler will preform more and more poorly as the Internet grows.