Probabilistic Opinion Dynamics

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Introduction

- Consider a group of n people who interact with each other.
- As they interact, their opinions may change over time. Time can be both discrete or continuous.
- $p_i(t)$ will denote person i's opinion at time t time. We will consider $p_i(t) \in \mathbb{R}$, but one can also have $p_i(t) \in \mathbb{R}^n$ or other spaces.
- $\vec{p}(t) = (p_1(t), ..., p_n(t))^T$ will denote the arrangement of opinions.

Behaviors

- Repulsion: Upon interaction, individuals disagree with each other more than they did initially.
- No interaction/Stubbornness: Not altering an initial opinion.
- Attraction: Individuals attempt to agree with one another.
- Agreement: Adopting the other person's opinion right away.
- Overshooting: Liking another opinion so strongly, so as to hold a more extreme version of it.
- External influences
- Internal changes

DeGroot's Model

- Each person's opinion is simply a weighted average of everyone's opinions at the previous time step.
- Let $A = (a_{ij})$ be a stochastic matrix. ($a_{ij} \geq 0$ and $\sum_j a_{ij} = 1$)

$$p_i(t+1) = \sum_j a_{ij} p_j(t)$$

- Alternative formulation: $p_i(t+1) = p_i(t) + \sum_j a_{ij}(p_j(t) - p_i(t))$
- a_{ij} quantifies the extent to which person i values person j's opinion.
  1. Attraction: $a_{ij} \in (0, 1)$; Agreement: $a_{ij} = 1$ for some $j \neq i$.
  2. No interaction/Stubborn: $a_{ij} = 0 / a_{ii} = 1$.
  3. Repulsion: $a_{ij} < 0$; Overshooting: $a_{ij} > 1$

Extensions of DeGroot's Model

- Extension: Let $a_{ij} = a_{ij}(\vec{p}(t))$
  1. Bounded confidence model: Each person only interacts with those who have similar enough opinions. Let $I_i(t) = \{j : |p_j(t) - p_i(t)| \le r_i\}$.

$$a_{ij} = \begin{cases} 0 & \text{if } j \notin I_i(t) \\ 1/|I_i(t)| & \text{if } j \in I_i(t) \end{cases}$$

  2. $a_{ij} \propto \phi(p_j(t) - p_i(t))$ (with or without normalization)
- Continuous time models:

$$\frac{d}{dt} p_i(t) = \sum_j a_{ij}(p_j(t) - p_i(t))$$

related to the alternative formulation

$$p_i(t+1) = p_i(t) + \sum_j a_{ij}(p_j(t) - p_i(t))$$

Probabilistic Models

- We consider two models
  1. One-at-a-time Model: At each time step, one randomly selected person's opinion is updated.
  2. Simultaneous Motion Model: Everyone's opinion is updated at every time step.
- Person i interacts with person j with probability a_{ij}. $A = (a_{ij})$ is a stochastic $n \times n$ matrix, which we allow to depend on the $\{p_i(t)\}$.
- Adjust person i's opinion as follows:

$$p_i(t+1) = p_i(t) + v_i(t)(p_j(t) - p_i(t))$$

  The $v_i(t) \in [-\epsilon, 1+\delta]$ are i.i.d. random variables for all i and t.
- v quantifies the degree to which person i tries to agree with person j.
  1. Stubborn: $v = 0$; Attraction: $v \in (0,1)$; Agreement: $v = 1$
  2. Repulsion: $v < 0$; Overshooting: $v > 1$

Generalized Probabilistic Model

- Generalized model:

$$p_i(t+1) = p_i(t) + \sum_j \delta_{ij}(t) v_{ij}(t)(p_j(t) - p_i(t))$$

- Interaction frequency: $\delta_{ij}(t)$ are Bernoulli random variables describing whether person i considers person j's opinion.
- Influence factors: $v_{ij}(t)$ are random variables. They quantify the strength of the relationship between person i and j.

Generalized Probabilistic Model (Continued)

$$p_i(t+1) = p_i(t) + \sum_j \delta_{ij}(t) v_{ij}(t)(p_j(t) - p_i(t))$$

- Interaction frequency:
  1. One-at-a-time: The $\delta_{ij}(t)$ are dependent Bernoulli($\frac{1}{n}a_{ij}$). Exactly one of them equals 1 at each time step.
  2. Simultaneous motion: The $\delta_{ij}(t)$ are dependent Bernoulli(a_{ij}). Exactly one of the $\delta_{ij}(t)$ equals 1 for each i at each time step.
  3. Patterson & Bamieh: The $\delta_{ij}(t)$ are independent Bernoulli(a_{ij}) random variables for all i and j that are allowed to interact.
- Influence factors:
  1. Our models: $v_{ij}(t)$ are i.i.d.
  2. Patterson & Bamieh: v_{ij} are degenerate RV with $\sum_j v_{ij} = 1$.

Phenomena

- Consensus: In the long run, everyone agrees with each other.
- Clustering: Several distinct non-interacting groups form.
- Shifting opinions: Opinions change in a distinct manner over time.
- Fluctuations: Opinions vary over time in no particular direction. (e.g. Brownian motion)
- Extremism: Opinions achieve at more extreme values than initially present.

Key difficulty: Finding objective as opposed to subjective definitions.

Consensus (One-at-a-time Model)

- Person i and j interact with probability a_{ij}/n.

$$p_i(t+1) = p_i(t) + v(p_j(t) - p_i(t))$$

- Let $v \in [0,1]$ be chosen from a fixed distribution with pdf $f(v)$ on $[0,1]$. We allow for $f(v)$ to have singularities.
- The convex hull of opinions $C_t = [\min_i p_i(t), \max_i p_i(t)]$ is non-increasing with time.
- Consensus: $\cap_{t=0}^{\infty} C_t$ is a singleton.

Theorem

If $\int_{(0,1)} f(v)dv > 0$ (i.e. f is not a linear combination of point masses at 0 and 1) and if there exists an $a_0 > 0$ such that $\min_{i \neq j} a_{ij}(t) > a_0$ for all t, then consensus is reached with probability 1.

Outline of Proof

- **With probability 1, the length of C_t eventually decreases by a factor less than 1 regardless of the initial arrangement of opinions.**
- By the strong Markov property, the length of C_t will continue to decrease by this factor again and again.
- Therefore, $\cap_{t=0}^{\infty} C_t$ is a singleton, and consensus is reached, with probability 1.

Outline of Proof (Decreasing Convex Hull)

- Suppose $C_t = [a, b]$ with $a < b$ at some time.
- Let $\epsilon \in (0, \frac{b-a}{2})$.
- Leftists: Those with opinions in $[a, a+\epsilon]$
  Rightists: Those with opinions in $[b-\epsilon, b]$
- Let $\phi = \frac{b-a-2\epsilon}{b-a} \in (0,1)$. We may choose ϵ small enough so that

$$\int_{\frac{1-\phi}{2\phi}}^{\phi} f(v)dv = c > 0$$

  This will be important later.
- **There is a positive probability bounded away from zero, that there will be no more leftist or rightists after n time steps.**
- C_t decreases by a factor of at least $\frac{b-a-\epsilon}{b-a} = \frac{1+\phi}{2} < 1$

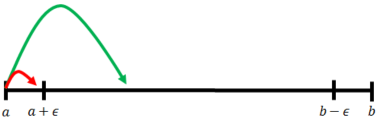
Outline of Proof (Disappearing Leftists and Rightists)

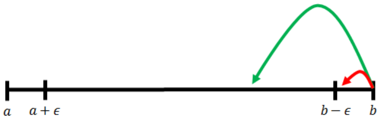
We proceed to bound below the probability that a particular leftist's (or rightist's) opinion is within the interval $(a+\epsilon, b-\epsilon)$ in the next time step.

- By assumption, at time t there are both leftists and rightists.
- There is a probability of $1/n$ that a particular leftist (or rightist), person i, is picked.
- And a probability of at least a_0 of interaction with a rightist (or leftist), person j.
- Person i's opinion changes by $v|p_j(t) - p_i(t)|$. It must change by at least ϵ but not more than $b - a - 2\epsilon$ to be within the interval.

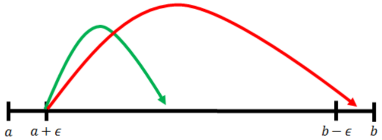
Outline of Proof (Disappearing Leftists and Rightists)

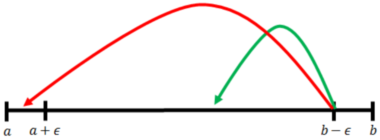
If person i's opinion is at the extremes (a or b), it must change at least ϵ to be in the interval $(a+\epsilon, b-\epsilon)$.





If person i's opinion is as moderate as it can be ($a+\epsilon$ or $b-\epsilon$) while still being a leftist or a rightist, then it must change by less than $b-a-2\epsilon = \phi(b-a)$.





Outline of Proof (Disappearing Leftists and Rightists)

The probability that person i's opinion is within the interval $(a+\epsilon, b-\epsilon)$ in the next time step is bounded below by

$$\frac{a_0}{n} P(\epsilon < v|p_j(t) - p_i(t)| < b - a - 2\epsilon)$$

$$= \frac{a_0}{n} P\left(\frac{\epsilon}{|p_j(t)-p_i(t)|} < v < \frac{b-a-2\epsilon}{|p_j(t)-p_i(t)|}\right)$$

$$\geq \frac{a_0}{n} P\left(\frac{1-\phi}{2\phi} < v < \phi\right) = \frac{a_0}{n} \int_{\frac{1-\phi}{2\phi}}^{\phi} f(v)dv = \frac{a_0 c}{n} > 0$$

Note:

$$\frac{\epsilon}{|p_j(t) - p_i(t)|} \leq \frac{\epsilon}{b - a - 2\epsilon} = \frac{1-\phi}{2\phi}$$

$$\frac{b - a - 2\epsilon}{|p_j(t) - p_i(t)|} \geq \frac{b - a - 2\epsilon}{b - a} = \phi$$

Outline of Proof (Disappearing Leftists and Rightists)

The probability that there will be...

- No more leftists or rightists after n time steps is at least $(a_0 c/n)^n$.
- Both leftists and rightists after n time steps is bounded by $1 - (a_0 c/n)^n$
- Both leftists and rightists after kn time steps is bounded by $(1 - (a_0 c/n)^n)^k$.
- Both leftists and rightists forever is bounded by $\lim_{k \to \infty} (1 - (a_0 c/n)^n)^k = 0$.

Consensus (One-at-a-time Model)

- Agreement without attraction: $v = 0$ or 1. At each time step, a person either holds the same opinion or adopts another's opinion.
- The set of opinions $P_t = \{p_1(t), ..., p_n(t)\}$ is non-increasing in time.
- Consensus: $\cap_{t=0}^{\infty} P_t$ is a singleton

Theorem

If $f(v) = w_0\delta_0(v) + w_1\delta_1(v)$ with $w_0 + w_1 = 1, w_0 \geq 0$, and $w_1 > 0$ and if there exists an $a_0 > 0$ such that $\min_{i\neq j} a_{ij}(t) > a_0$ for all t, then consensus is reached with probability 1.

- Similar proof, but we consider individual opinions being abandoned.

Consensus (Simultaneous Motion Model)

Theorem

Consensus is reached with probability 1 provided there exists an $a_0 > 0$ *such that* $\min_{i \neq j} a_{ij}(t) > a_0$ *for all* t *and one of the following three holds.*

1. $\int_{(0,1)} f(v) dv > 0$
2. $f(v) = w_0 \delta_0(v) + w_1 \delta_1(v)$ *with* $w_0 + w_1 = 1$*, and* $w_0, w_1 > 0$
3. $f(v) = \delta_1(v)$ *and there are at least three people* $(n \geq 3)$

- Proofs split into three cases.
- For the first, the earlier proof nicely extend to the simultaneous motion model, but one needs to worry about moderates becoming leftists or rightists at each time step.
- For the third case, slightly different estimates are used. One must rely on interaction between those with the same opinion.

Multiple Opinions

- Suppose there the people have opinions on M different issues. p_i^m is person i's opinion on the mth issue, and $\vec{p_i} \in \mathbb{R}^M$ is a vector of person i's opinions.
- Opinions are updated as in the earlier model, but the v_m can have different distributions.

$$p_i^m(t+1) = p_i^m(t) + v_m(p_j^m(t) - p_i^m(t))$$

With vector notation,

$$\vec{p_i}(t+1) = \vec{p_i}(t) + \vec{v}(\vec{p_j}(t) - \vec{p_i}(t))$$

where the vector multiplication is coordinate-wise, and $\vec{v} \in [0,1]^M$ is chosen from the joint distribution of the v_m's.

- Consensus is reached (on all issues) if the the marginal distribution for each v_m satisfies the conditions of the earlier theorems.

Repulsion and Overshooting

- Repulsion: $v \in [-\epsilon, 1]$ where $\epsilon > 0$.
- Overshooting: $v \in [0, 1 + \delta]$ where $\delta > 0$.
- Studying consensus becomes more difficult.
- The convex hull of the set of opinions is no longer non-increasing. Possible expansion, oscillation, or shifting.

Scaling opinions

- Consider an external influence on people's opinions
- This can be modeled by scaling everyone's opinions at each time step:

$$\frac{p_i(t)}{(1+\tau)^t}$$

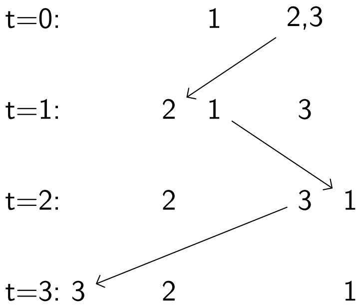
- For large enough, τ consensus is guaranteed as the scaling drives opinion to a common value.

  If $\int_{[0,1]} f(v)dv = P(v \in [0,1]) > 0$

  1. Repulsion: $\tau \geq \epsilon$ (One-at-a-time); $\tau \geq 2\epsilon$ (Simultaneous motion)
  2. Overshooting: $\tau \geq 2\delta$ (Simultaneous motion)
  3. Overshooting: $\tau \geq \delta$ ($\tau \geq \delta - 1$ when $n = 2$) (One-at-a-time, $\delta \geq 1$);

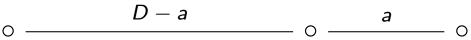
Worst Case Motions (One-at-a-time; $\delta < 1$)

- Diameter of opinions: $D_t = \max p_i(t) - \min p_i(t)$
- Worst Case: Largest possible diameter D_t^w.
- Maximizing the diameter overall comes from maximizing the diameter at each time step.
- When $n = 3$, the worst case motion is a leap frog motion.



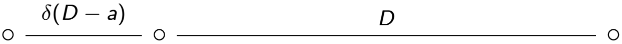
Worst Case Motions (One-at-a-time; $\delta < 1$)

Consider the following diagram where $a \leq D/2$.



- Either the second person or the third person should interact with the first person to maximize the diameter at the next step.
- If the second person moves, the diameter will be $D + \delta(D - a)$.
  If the third person moves, the diameter will be $D + \delta D - a$.
  The diameter is greater when the second person moves.

At the next time step, the diagram is



The second person is closer to the first person, so the second person now swings to the right, and so on, guaranteeing the leap frog motion.

Worst Case Motions (One-at-a-time; $\delta < 1$)

- Two possible worst case motions:
  1. Eventual leap frog motion
  2. The person who jumps is always the one second from the left (or always the one second from the right)
- For $n \geq 4$, if the same three people are outer-most two times in a row according to the worst case motion, then they will remain so for the rest of time.
- If $n = 4$, the worst case motion is eventually the leap frog motion.
- For $n \geq 5$, both worst case motions are possible.

Comments

- Theorems about Consensus: If everyone tries to agree with one another, then they will eventually agree.
- Finding an all encompassing proof for consensus in the general model.
- Numerical Results: Consensus is more likely if people try to agree with those of different opinions more than those with similar opinions.
- Clustering: Difficult to get “useful” results.
- Subjective notions of clusters. “Distinct” groups that interact.
- The space of opinions.