Math 129: Test 1 Review

How do you approach evaluating an integral that you see? We have five methods that we have learned so far for integrating functions:

- u-substitution
- Integration by parts
- Partial fractions
- Trigonometric substitutions
- Table

Look at the integral and see which method could be used to evaluate the integral.

- u-substitution is used when there is a clear inside and outside function or else when we see a function and its derivative in the integral
- Integration by parts is used when there is a product of two functions or when integrating inverse functions such as $\ln(x)$, $\arctan(x)$, and $\arcsin(x)$.
- Partial fractions are used when we have a rational function and the denominator can be factored.
- Trigonometric substitutions are used when we see something of the from $a \pm bx^2$ for some constants a and b.
- You should be familiar enough with the table to recognize which problems are on the table.

1. u-substitution: Good guesses for u.
   - The quantity in parentheses
   - The quantity in the square root
   - The quantity in the exponent
   - The quantity in the denominator
   - $\ln(x)$ if there is an x on the denominator.
   - Anything whose derivative also appears in the integral.

2. Integration by parts: Good guesses for u and v'.
   - v' must be something whose antiderivative you already know or can find easily.
   - If we know the antiderivative of both functions in the product, then u should be the one with an easier derivative.
   - If there doesn't seem to be a product, we let $v' = 1$ and u be the function we are integrating.
   - Occasionally, multiple integration by parts may be needed.

- If we have a polynomial times $e^x, \sin(x)$, or $\cos(x)$, then multiple integration by parts will work. The shortcut in this case is to use tabular integration. These types of integrals are also found on the table.

3. Partial fractions:

   - If the degree of the numerator is greater than or equal to the degree of the denominator, first use long division.
   - Next factor the denominator.
   - If the denominator factors into distinct roots (e.g. $(x-1)(x+2)(x-5)$), use a partial fraction decomposition of the form $\frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x-5}$.
   - If there is a repeated root in the factorization (e.g. $(x-1)^2(2x-3)$), use a partial fraction decomposition of the form $\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{2x-3}$.
   - If there is an unfactorable quadratic in the factorization (e.g. $x(x^2+5)$), use a partial fraction decomposition of the form $\frac{A}{x} + \frac{Bx+C}{x^2+5}$.
   - Multiply by the original denominator to solve for the constants in the partial fraction decomposition.
   - The resulting decomposition can then be integrated using logarithms and/or the table.
   - Note: if there is an unfactorable quadratic in the partial fraction decomposition, you may need to complete the square before being able to use the table.

4. Trigonometric substitutions:

   - Trigonometric substitutions are used when we see something of the from $a \pm bx^2$ for some constants a and b. First check whether a simple u-substitution could work.
   - If you see something of the form $a^2 - b^2x^2$, then use the substitution $x = \frac{a}{b}\sin(\theta)$. You will then need to use the trigonometric identity $1 - \sin^2(\theta) = \cos^2(\theta)$.
   - If you see something of the form $a^2 + b^2x^2$, then use the substitution $x = \frac{a}{b}\tan(\theta)$. You will then need to use the trigonometric identity $1 + \tan^2(\theta) = \sec^2(\theta)$.
   - Depending on how difficult the resulting integral is you may need to use the table.
   - Don't forget to write your answer in terms of x. This may simplifying trigonometric functions of inverse trigonometric functions. For example, $\cos(\arcsin(x)) = \sqrt{1-x^2}$.

5. Table:

   - If none of the above methods seem to work, see if the integral is on the table. Even when some of the above methods work the answer may already be on the table. You should know the table well enough to recognize what might be on it.
   - Find which number on the table looks most like your integral.

- You may need to perform a u-substitution, factor out a constant, perform long division, or complete the square to be able to use the table.

There are quite a few things two watch for when integrating rational functions (e.g. $\frac{x^3+2x^2+5x-1}{x^2+3x-1}$) and similar functions (e.g. $\frac{1}{\sqrt{x^2+4x-5}}$). We go over the approach to integrating such functions below:

- If the numerator and denominator are both polynomials and the degree of the numerator is greater than or equal to the denominator, first use long division.
- Next factor the denominator. If the denominator does not factor or if the denominator is the square root of a polynomial, you will need to complete the square.
- If you needed to complete the square, then a simple u-substitution should give you an integral that can be found on the table; a trigonometric substitution could also work.
- If the denominator factored into something of the form $(x-a)^n$ a simple u-substitution will work.
- If the denominator factored into anything more complicated, partial fractions or the table will be helpful.

Estimating integrals

- The left rule is an overestimate of $\int_a^b f(x)dx$ if $f(x)$ is decreasing, and it is an underestimate if $f(x)$ is increasing.
- The right rule is an underestimate of $\int_a^b f(x)dx$ if $f(x)$ is decreasing, and it is an overestimate if $f(x)$ is increasing.
- The trapezoid rule is an underestimate of $\int_a^b f(x)dx$ if $f(x)$ is concave down, and it is an overestimate if $f(x)$ is concave up.
- The midpoint rule is an overestimate of $\int_a^b f(x)dx$ if $f(x)$ is concave down, and it is an underestimate if $f(x)$ is concave up.
- Trapezoid rule is just one half of the left rule plus the right rule.
- You should be able to estimate using any of the above rules using a table, a function, or a graph.