Hydrodynamic Limits For Long Range Asymmetric Processes

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Introduction

- Consider a fluid evolving over time. A hydrodynamic equation describes how features of a fluid (e.g. density) evolves over time.
- View the fluid as being made up of many tiny molecules that interact with one another.
- Model the fluid as an interacting particle system. Assume the particles are restricted to a lattice.
- Recover/deduce the hydrodynamic equation in the limit as the number of particles increases to infinity.

Some Hydrodynamic Equations

- Let $\rho(t, u)$ be the density of the fluid at time t at the point u with an initial density $\rho_0(u)$.
- The heat equation:

$$\partial_t \rho = \triangle \rho$$

- A conservation law:

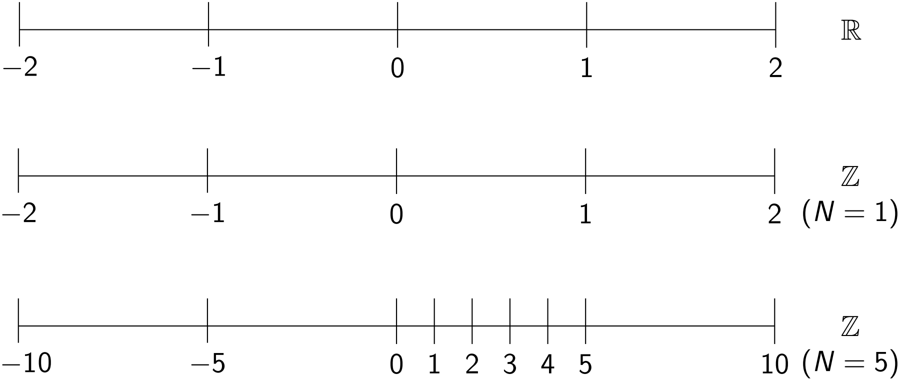
$$\partial_t \rho + \gamma_\alpha \partial_u \Phi(\rho) = 0$$

  where $\gamma_\alpha = \sum_{d=1}^{\infty} \frac{1}{d^\alpha}$ and $\Phi(\rho) = \rho(1-\rho)$.

- With a pseudo-differential operator: $(0 < \alpha < 1)$

$$\partial_t \rho = \int_0^\infty \frac{\rho(u-v)(1-\rho(u)) - \rho(u)(1-\rho(u+v))}{v^{1+\alpha}} dv$$

Space Scaling



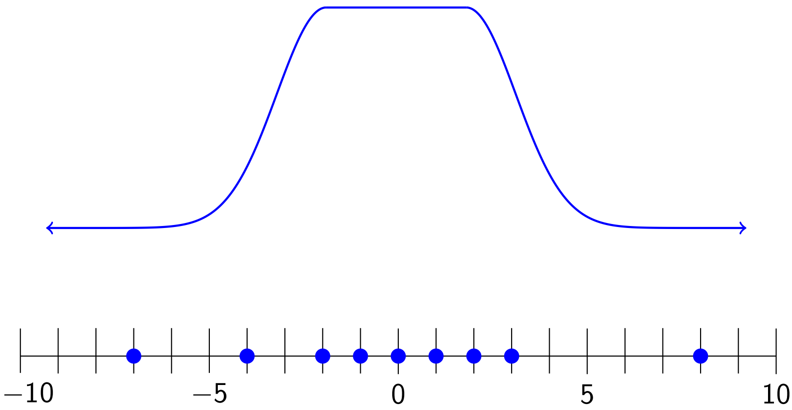
Initial Conditions

- $\eta_t : \mathbb{Z} \to \mathbb{N} \cup \{0\}$ is called a configuration and it gives the number of particles at each location $x \in \mathbb{Z}$ at time t.
- The initial configuration η_0 depends on the initial density $\rho_0(u)$.
- Find invariant measures ν_ρ for the IPS, which are translation invariant product measures. Define probability measures Θ_ρ on $\mathbb{N} \cup \{0\}$ that satisfy $\Theta_\rho(k) = \nu_\rho(\eta_0(x) = k)$. And let the initial probability measure, μ^N, on configurations be the product measure satisfying

$$\mu^N(\eta_0(x) = k) = \Theta_{\rho_0(\frac{x}{N})}(k)$$

- If Θ_ρ is the Bernoulli(p) distribution, a single particle can be placed independently at each site x with probability $\rho_0(\frac{x}{N}) \in [0, 1]$.

Initial Conditions (continued)



The Simple Exclusion Process

Each particle holds an exponential alarm clock with rate 1. When a particle's alarm clock goes off, a displacement d is picked with probability proportional to $p(d)$. If the new site $x+d$ is unoccupied, then the particle relocates to $x+d$.

- The generator for the process is given by

$$Lf(\eta) = \sum_{x,d} p(d)\eta(x)(1-\eta(x+d))(f(\eta^{x,x+d}) - f(\eta))$$

- Some invariant measures:

$$\nu_\rho(\eta_0(x) = 1) = \rho, \quad \nu_\rho(\eta_0(x) = 0) = 1 - \rho$$

The Zero Range Process

Each site x holds an exponential alarm clock with rate $g(\eta(x))$. When a site's alarm clock goes off, a displacement d is picked with probability $p(d)$. Then a single particle moves from x to $x + d$. Note: If there is no particle at a location x, it doesn't make sense to have a particle move from x. Hence, we require that $g(0) = 0$.

- The generator for the process is given by

$$Lf(\eta) = \sum_{x,d} p(d)g(\eta(x))(f(\eta^{x,x+d}) - f(\eta))$$

- Some invariant measures:

$$\nu_\rho(\eta_0(x) = k) \propto \frac{\lambda^k}{\prod_{j=1}^k g(j)}$$

The Misanthrope Process

A more general interacting particle system that encompasses both the simple exclusion and many zero range processes.

- The generator for the process is given by

$$Lf(\eta) = \sum_{x,d} p(d) b(\eta(x), \eta(x+d))(f(\eta^{x,x+d}) - f(\eta))$$

- $b(\cdot,\cdot)$ is increasing in the first coordinate and decreasing in the second coordinate.

Since $b(\eta(x), \eta(x+d))$ increases with $\eta(x)$, the number of particles at the initial site, and decreases with $\eta(x+d)$, the number of particles at the destination site, particles tend to avoid crowded sites; hence the name "misanthrope".

Properties of $p(d)$

There are the following properties concerning the range of $p(d)$.

Definition

- Nearest neighbor: $p(1) + p(-1) = 1$. ($p(d) = 0$ if $d \neq \pm 1$.)
- Finite range: For some R, $p(d) = 0$ whenever $|d| > R$.
- Long range: Not finite range.

And there are the following properties concerning the symmetry of $p(d)$.

Definition

- Symmetric: $p(d) = p(-d)$
- Asymmetric: $E_p(d) = \sum_d dp(d) \neq 0$

Previous Results and Current Work

- The symmetric simple exclusion process with nearest neighbor jumps is one of the simplest processes. It gives the heat equation.
- Rezakhanlou (1990) proved results for finite range asymmetric simple exclusion and zero range processes.
- Jara (2009) proved results for long range symmetric simple exclusion and zero range processes.
- I have been working on the long range asymmetric simple exclusion process with

$$p(d) = 1(d > 0)\frac{1}{d^{1+\alpha}}$$

where $\alpha > 0$. This splits in to three cases: $\alpha < 1, \alpha = 1$, and $\alpha > 1$.

Time Scaling

The generator L gives jumps in $\mathbb{Z}$. Displacement in $\mathbb{Z}$ will not correspond to the same displacements in $\mathbb{R}$ after scaling space. To compensate, time is scaled by a factor of γ_N giving the generator $\gamma_N L$ for the time scaled process.

- Diffusive scaling: $\gamma_N = N^2$ used to describe diffusion
- Euler scaling: $\gamma_N = N$ describes drift (used for $\alpha > 1$ case)
- Logarithmic correction: $\gamma_N = \frac{N}{\ln(N)}$ (used for $\alpha = 1$ case)
- Other scaling: $\gamma_N = N^\alpha$ (used for $\alpha < 1$ case)

Convergence

- Goal: Show that $\pi_t^N = \frac{1}{N}\sum_{x\in\mathbb{Z}} \eta_t(x)\delta_{\frac{x}{N}}$ converges to $\rho(t,u)du$ in some sense where $\rho(t,u)$ is the solution of a hydrodynamic equation.
- Define probability measures $\{P^N\}_{N=1}^{\infty}$ that correspond to the process $\{\pi_t^N\}_{N=1}^{\infty}$ with the appropriate time scaling and initial distributions.
- $\{P^N\}_{N=1}^{\infty}$ will be probability measures on the Skorokhod space $D([0,T],M_+)$, the space of càdlàg functions of t taking values in M_+, the set of positive Radon measures on $\mathbb{R}$.
- Show weak convergence to a deterministic limit:

$$P^N \Rightarrow \delta_{\rho(t,u)du}$$

Method

- Define a metric on the space $D([0, T], M_+)$ to have a notion of weak convergence. (The Skorokhod metric often works.)
- It is difficult to show weak convergence directly, because it is difficult to verify that P^N converges at all. Instead we use the following method:
  1. Show that $\{P^N\}_{N=1}^{\infty}$ is tight (equivalent to weakly relatively compact).
  2. Show every convergent subsequence converges to the same limit. (Abusing notation P^N will denote a convergent subsequence.)

Two Useful Martingales

Let X_t be a continuous time Markov process with generator L. The following are mean zero martingales.

1. $M^F(t) = F(t, X_t) - F(0, X_0) - \int_0^t \frac{\partial F}{\partial s}(s, X_s) + LF(s, X_s)ds$
   - used to resemble the weak formulation of a hydrodynamic equation/condition.
   - Choose F well: $F(t, \pi_t^N) = \langle \pi_t^N, g_t \rangle = \frac{1}{N} \sum_{x \in \mathbb{Z}} \eta_t(x) g(t, \frac{x}{N})$.
2. $(M^F(t))^2 - \langle M^F \rangle_t$ where $\langle M^F \rangle_t$ is the quadratic variation.

$$\langle M^F \rangle_t = \int_0^t LF^2(s, X_s) - 2F(s, X_s) LF(s, X_s) ds$$

   - Used to show the first martingale goes to zero.

$$P^N \left(\left| M_t^F \right| > \epsilon_0 \right) \leq \frac{1}{\epsilon_0^2} E^N \left(\left(M_t^F \right)^2 \right) = \frac{1}{\epsilon_0^2} E^N \left(\langle M^F \rangle_t \right)$$

Tightness

Theorem

$\{P^N\}_{N=1}^{\infty}$ is tight if for all test functions g the following hold:

1. *For all $t \in [0, T]$ and every $\epsilon > 0$, there is a compact $K \subseteq \mathbb{R}$ such that*

$$\sup_N P^N[\langle \pi_t^N, g \rangle \notin K] \leq \epsilon$$

2. *Aldous's criterion is satisfied:*

$$\limsup_{\gamma \to 0^+} \limsup_{N \to \infty} \sup_{\theta \leq \gamma} \sup_{\tau \leq T} P^N(|\langle \pi_{\tau+\theta}^N, g \rangle - \langle \pi_\tau^N, g \rangle| > \epsilon) = 0$$

where τ is a stopping time.

Substitution Lemma

The following lemma that is used to replace an old expression O_k by a new expression N_k in certain types of probability statements. It is used repeatedly in later proofs to make $M_t^{N,g}$ look more like the weak formulation of a hydrodynamic equation.

Substitution Lemma

Suppose $\lim_{k\to\infty} E[|D_k|] = 0$ where $D_k = O_k - N_k$.
For all $\epsilon_0 > 0$, if $\lim_{k\to\infty} P(|O_k| > \epsilon_0) = 0$, then $\lim_{k\to\infty} P(|N_k| > \epsilon_0) = 0$

Proof:

$P(|N_k| > \epsilon_0) \le P(|D_k| + |O_K| > \epsilon_0) \le P(|D_k| > \epsilon_0/2) + P(|O_K| > \epsilon_0/2) \le \frac{2}{\epsilon_0} E[|D_k|] + P(|O_K| > \epsilon_0/2)$ which goes to zero as k goes to infinity. QED

The substitution lemma has many variations that are used in practice but all are similarly easy to prove.

Hydrodynamic Limit ($\alpha < 1$)

Theorem

When $\alpha < 1$, every limit point P^* of $\{P^N\}_{N=1}^{\infty}$ is supported on absolutely continuous measures $\pi_t = \rho(t,u)du$ whose densities are weak solutions of the hydrodynamic equation

$$\partial_t \rho = \int_0^{\infty} \frac{\rho(u-v)(1-\rho(u)) - \rho(u)(1-\rho(u+v))}{v^{1+\alpha}} dv$$

- Additional conditions may be necessary guarantee a unique weak (or possibly strong) solution of the hydrodynamic equation.
- Details showing that π_t is absolutely continuous will be skipped.

Hydrodynamic Limit ($\alpha < 1$, continued)

First we verify

$$\limsup_{N\to\infty} P^N \left(\left| M_t^{N,g} \right| > \epsilon_0 \right) = 0$$

which gives us

$$\limsup_{N\to\infty} P^N \left(\left| \langle \pi_0^N, g_0 \rangle + \int_0^t \langle \pi_s^N, \partial_s g_s \rangle ds + \int_0^t N^\alpha L \langle \pi_s^N, g_s \rangle ds \right| > \epsilon_0 \right) = 0$$

for large t, where

$$N^\alpha L \langle \pi_s^N, g_s \rangle = \frac{N^\alpha}{N} \sum_{x\in\mathbb{Z}} \sum_{d=1}^{\infty} \frac{1}{d^{1+\alpha}} \eta_s^d(x) \left[g_s\left(\frac{x+d}{N} \right) - g_s\left(\frac{x}{N} \right) \right]$$

and $\eta_s^d(x) = \eta_s(x)(1 - \eta_s(x+d))$.

Hydrodynamic Limit ($\alpha < 1$, continued)

The substitution lemma is now used repeatedly to replace $N^\alpha L \langle \pi_s^N, g_s \rangle$ by several similar expressions:

- Technical replacements: Limit the sum over d to be at least $\epsilon N + 1$ and at most DN, where $\epsilon \to 0$ and $D \to \infty$.
- Common replacement: Introduce an operator on g. (Not needed here.)

The Replacement Lemma

The replacement lemma is used to **close the equation**. Every term in the martingale $M_t^{N,g}$ should be written in terms of π_s^N.

1. Replace $\eta_s^d(x)$ with it's average over l-blocks: $\frac{1}{2l+1}\sum_{|y|\leq l}\eta_s^d(x+y)$.
   - Relies on DIBP, and smoothness and compact support of g
2. 1-block replacement: Replace the average of the function with the function of the average. Replace

$$\frac{1}{2l+1}\sum_{|y|\leq l}\eta_s^d(x+y) \quad \text{by} \quad (\eta_s^l(x))^d = \eta_s^l(x)(1-\eta_s^l(x+d))$$

3. 2-blocks replacement: Replace $\eta_s^l(x)$ by $\eta_s^{\epsilon' N}(x)$.

Hydrodynamic Limit ($\alpha < 1$, continued)

The equation is essentially closed due to the relation

$$\eta_s^{\epsilon' N}(x) = \frac{2\epsilon' N}{2\epsilon' N + 1} \left(\pi_s^N * \iota_{\epsilon'}\right)\left(\frac{x}{N}\right)$$

where $\iota_{\epsilon'} = \frac{1}{2\epsilon'} 1_{[-\epsilon',\epsilon']}$. We are then left with

$$\frac{1}{N}\sum_{x\in\mathbb{Z}} \frac{1}{N} \sum_{d=\epsilon N+1}^{DN} \left(\left(\pi_s^N * \iota_{\epsilon'}\right)\left(\frac{x}{N}\right)\right)^d \left[\frac{g_s\left(\frac{x}{N}+\frac{d}{N}\right) - g_s\left(\frac{x}{N}\right)}{\left(\frac{d}{N}\right)^{1+\alpha}}\right]$$

- Replace the Riemann sums by integrals.
- Remove residuals from technical replacements, if any.
- Let N go to infinity to obtain a limit point P^*.
- Replace $(\pi_s * \iota_{\epsilon'})\left(\frac{x}{N}\right)$ by $\rho(s,u)$, where $\pi_s = \rho(s,u)du$.

Hydrodynamic Limit ($\alpha > 1$)

- It has not been possible to prove a 2-blocks estimate when $\alpha > 1$.
- To **close the equation**, we use Young measures:

$$\pi_t^{N,l}(du, d\lambda) = \frac{1}{N} \sum_{x \in \mathbb{Z}} \delta_{\frac{x}{N}}(du) \delta_{\eta_t^l(x)}(d\lambda)$$

- Probability measures $Q^{N,l}$ that correspond to the process are introduced. We prove a weak convergence result for $Q^{N,l}$, which is used to prove weak convergence for P^N.

Theorem

$\{Q^{N,l}\}$ converges weakly to the probability measure Q supported on the measure $\pi_s(du, d\lambda) = \rho(t, u, d\lambda)du$, where $\rho(t, u, d\lambda) = \delta_{\rho(t,u)}(d\lambda)$ is the Dirac measure valued entropy solution of the hydrodynamic equation

$$\partial_t \rho + \gamma_\alpha \partial_u \Phi(\rho) = 0$$

Hydrodynamic Limit ($\alpha > 1$, continued)

Theorem (Using DiPerna's Uniqueness Theorem)

Every limit point Q^* of $\{Q^{N,l}\}$ is supported on measures $\pi_s(du, d\lambda)$ absolutely continuous in the first coordinate whose densities $\rho(t, u, d\lambda)$ are

1. *Measure valued solutions of $\partial_t \rho + \gamma_\alpha \partial_u \Phi(\rho) = 0$*
2. *Entropy condition holds measure weakly for any $c \in \mathbb{R}$:*

$$\partial_t |\rho - c| + \gamma_\alpha \partial_u [sgn(\rho - c)(\Phi(\rho) - \Phi(c))] \leq 0$$

3. *Initial condition:*

$$\lim_{t \to 0^+} \frac{1}{t} \int_0^t \int_{\mathbb{R}} \int_0^\infty |\lambda - \rho_0(u)| \rho(s, u, d\lambda) duds = 0$$

4. *Technical condition:*

$$\sup_t \int_{\mathbb{R}} \int_0^\infty |\lambda| \rho(t, u, d\lambda) du < \infty$$

Hydrodynamic Limit ($\alpha > 1$, continued)

Condition 1 can be proven by starting as in the $\alpha < 1$ case. After the 1-block replacement, $NL\langle \pi_s^N, g_s \rangle$ will look like

$$\gamma_\alpha \frac{1}{N} \sum_{x \in \mathbb{Z}} \Phi(\eta_s^l(x)) g_s' \left(\frac{x}{N} \right) = \gamma_\alpha \langle \pi_s^{N,l}, g_s'(u)\, \Phi(\lambda) \rangle$$

The equation will can thus be closed with respect to $\pi_s^{N,l}$:

$$Q^{N,l} \left(\left| \langle \pi_0^{N,l}, g_0(u)\lambda \rangle + \int_0^t \langle \pi_s^{N,l}, \partial_s g_s(u)\lambda \rangle ds \right.\right.$$
$$\left.\left. + \gamma_\alpha \int_0^t \langle \pi_s^{N,l}, g_s'(u)\Phi(\lambda) \rangle ds \right| > \epsilon_0 \right)$$

goes to zero as N and then l go to infinity.

Coupling

Conditions 2 and 3 are proven using coupling. Two processes η_t and ξ_t are coupled so that each marginal process is the long range asymmetric simple exclusion process, but whenever possible the two processes exhibit the same jumps.

- New measures $\tilde{P}^N$ on the space $D([0,T],\{0,1\}^{\mathbb{Z}} \times \{0,1\}^{\mathbb{Z}})$ corresponding to the coupled process (η_t, ξ_t).
- Define new empirical measures $\tilde{\pi}_t^N = \frac{1}{N}\sum_{x\in\mathbb{Z}} |\eta_t(x) - \xi_t(x)|\delta_{\frac{x}{N}}$
- The martingale $\tilde{M}_t^{N,g} = M^F(t)$, where $F(t,\pi_t^N) = \langle\tilde{\pi}_t^N, g_t\rangle$, is used because it resembles the entropy condition.
- The distribution ξ_0 is chosen to be ν_c so that $\xi_t^l(x)$ approximates c.
- Young measures are then used to close the equation.

Hydrodynamic Limit ($\alpha > 1$, continued)

- $\{Q^{N,l}\}$ converges weakly to the measure Q supported on $\delta_{\rho(t,u)}(d\lambda)du$ where $\rho(t,u)$ is the unique entropy solution.
- By relating P^N to $Q^{N,l}$, $\{P^N\}$ can be shown to converge weakly to the measure P supported on $\rho(t,u)du$:

$$Q^{N,l}\left(\left|\int_0^t \langle \pi_s^{N,l}, g_s\lambda\rangle ds - \int_0^t \int \rho(s,u)g(s,u)duds\right| > \epsilon_0\right)$$

goes to zero, and $\langle \pi_s^{N,l}, g_s\lambda\rangle \approx \langle \pi_s^N, g_s\rangle$ so

$$P^N\left(\left|\int_0^t \langle \pi_s^N, g_s\rangle ds - \int_0^t \int \rho(s,u)g(s,u)duds\right| > \epsilon_0\right)$$

converges to zero as N goes to infinity.

Comments

- The proof when $\alpha > 1$ mimicks Rezakhanlou's proof in many respects.
- Rezankhanlou relies heavily on attractiveness, a property of an IPS when coupled. Our proof seems to remove the need for attractiveness.
- Rezakhanlou's initial conditions are more general. We need $\rho_0(u)$ to be a positive constant outside a compact set.
- We don't have a uniqueness criterion when $\alpha < 1$.
- Our results should extend to higher dimensions and the misanthrope process with $b(n, m) = g(n)h(m)$

Entropy and the Dirichlet Form

We define the entropy of μ_t^N with respect to ν_λ as follows:

$$H(\mu_t^N \mid \nu_\lambda) = \int f_t^N \ln(f_t^N) d\nu_\lambda$$

where $f_t^N = d\mu_t^N / d\nu_\lambda$. If μ_t^N is not absolutely continuous with respect to ν_λ and f_t^N cannot be defined, then the entropy is taken to be infinite.

- Entropy is positive and decreasing in time.
- $H(\mu_0^N \mid \nu_\lambda) \leq CN$ if ρ_0 is a positive constant outside a compact set.

We define the Dirichlet form by $D(f) = -\int \sqrt{f} L^{sym} \sqrt{f} d\nu_\lambda$

We may show

$$D\left(\frac{1}{t}\int_0^t f_s^N ds\right) \leq \frac{C_1 N}{\gamma_N}$$

Entropy and the Dirichlet Form (continued)

It is useful to have a bound on the Dirichlet form of $\bar{f}_t^N = \frac{1}{t}\int_0^t f_s^N ds$ because expectations can be written in terms of it. Specifically,

$$E^N\left[\int_0^t h(\eta_s)ds\right] = t\int h(\eta)\bar{f}_t^N(\eta)\nu_\lambda(d\eta)$$

The 1-block and 2-blocks estimates involve showing that such expectations go to zero. It is enough to show

$$\sup_{D(f)\leq\frac{C_1 N}{\gamma_N}} \int h(\eta)f(\eta)\nu_\lambda(d\eta)$$

goes to zero.

1-Block and 2-Blocks Estimates

Below are the functions $h(\eta)$ that appear in the 1-block and 2-blocks estimates. For the 1-block estimate, $h(\eta)$ is

$$\sum_{d=\epsilon N+1}^{\infty} \frac{N^{\alpha}}{d^{1+\alpha}} \frac{1}{N} \sum_{|x|\leq RN} \left| \frac{1}{2l+1} \sum_{|y|\leq l} \eta^{d}(x+y) - \left(\eta^{l}(x)\right)^{d} \right|$$

where $\left(\eta^{l}(x)\right)^{d} = \eta^{l}(x)(1-\eta^{l}(x+d))$. For the 2-blocks estimate, $h(\eta)$ is

$$\frac{1}{N} \sum_{|x|\leq (R+D)N} \left| \eta^{l}(x) - \eta^{\epsilon' N}(x) \right|$$

The function $f(\eta)$ is now manipulated using the translation invariance of ν_{λ} to produce a new function $\bar{f}_N$. $\bar{f}_N = \frac{1}{2RN+1} \sum_{|x|\leq RN} \tau_x f(\eta)$ is commonly obtained, where τ_x is the shift operator.

New Dirichlet Form

A new Dirichlet form D^* is introduced such that $D^*(\bar{f}_N)$ will go to zero.

$D(f) = \sum_{x,y} D^{x,y}(f)$ where $D^{x,y}$ is the piece of the Dirichlet form corresponding to jumps from x to y.

$$D^{x,y}(f) = -\int \sqrt{f} L_{x,y}^{sym} \sqrt{f} \, d\nu_\lambda$$

$$D^{x,y}(f) = \frac{1}{2} \int p(y-x)\eta(x)(1-\eta(y))(\sqrt{f(\eta^{x,y})} - \sqrt{f(\eta)})^2 d\nu_\lambda$$

$D^{x,y}(f)$ measures how much $f(\eta)$ can vary as a particle is moved from x to y. If $D^{x,y}(f) = 0$, then $f(\eta) = f(\eta^{x,y})$.

The new Dirichlet form will be a sum of only some of the $D^{x,y}(f)$. If it is zero, f will be constant on hyperplanes.