Geometry Qualifying Exam Notes

Definition: The Jacobian matrix of a map $f : N \to M$ is $\begin{pmatrix} \frac{\partial F^1}{\partial x^1} & \cdots & \frac{\partial F^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial F^m}{\partial x^1} & \cdots & \frac{\partial F^m}{\partial x^n} \end{pmatrix}$. When this is a square matrix, its determinant is called the Jacobian determinant.

Definition: The rank of a smooth map F is the rank of the Jacobian matrix, which is the largest number of linearly independent columns of the matrix.

Constant Rank Theorem: Let N and M be manifolds of dimension n and m respectively. If $f : N \to M$ has contant rank k is a neighborhood of a point $p \in N$, then there are charts (U, ϕ) centered at p and (V, ψ) centered at $f(p)$ such that $\psi \circ f \circ \phi^{-1}(r^1, ...r^n) = (r^1, ...r^k, 0, ..., 0)$

Constant-rank level set theorem: If $f : N \to M$ is smooth, and if f has constant rank k in a neighborhood of $f^{-1}(c)$, then $f^{-1}(c)$ is a regular submanifold of N of codimension k.

Proposition 1: *Let $f : N \to M$ be a smooth function of manifolds where N and M have dimension n and m respectively.*

1. *f is an immersion at p iff the rank of the Jacobian matrix of F equals $n \leq m$. f is an immersion if it is an immersion at all points.*
2. *f is a submersion at p iff the rank of the Jacobian matrix of F equals $m \leq n$. f is a submersion if it is a submersion at all points.*

Immersion theorem: If $f : N \to M$ is an immersion at p, then there are charts (U, ϕ) centered at p and (V, ψ) cnetered at $f(p)$ sucht that $\psi \circ f \circ \phi^{-1}(r^1, ..., r^n) = (r^1, ..., r^n, 0, ..., 0)$ in a neighborhood of $\phi(p)$.

Submersion theorem: If $f : N \to M$ is a submersion at p, then there are charts (U, ϕ) centered at p and (V, ψ) cnetered at $f(p)$ sucht that $\psi \circ f \circ \phi^{-1}(r^1, ..., r^m, r^{m+1}, ..., r^n) = (r^1, ..., r^m)$ in a neighborhood of $\phi(p)$.

Inverse function theorem: Let $F : N \to M$ be a smooth map of manifolds with same dimension. Then F is locally invertible at $p \in N$ iff its Jacobian determinant at p is nonzero.

Implicit function theorem: If $F : U \to \mathbb{R}^m$ is smooth where U is an open subset of $\mathbb{R}^n \times R^m$, $f(a, b) = 0$, and the Jacobian determinant of f at (a, b) is not zero, then there is a smooth function $h : A \to B$ such that $f(x, y) = 0 \Leftrightarrow y = h(x)$ in $A \times B$ where $A \times B$ is an open subset of U containing (a, b).

Whitneys theorem: Any smooth n-dimensional manifold can be smoothly embedded into $\mathbb{R}^{2n}$

Stoke's Theorem: Let ω be a smooth $(n-1)$-form with compact support on an oriented n dimensional manifold M. Then $\int_M d\omega = \int_{\partial M} \omega$.

Proposition 2: 1. $df = \sum_i \frac{\partial f}{\partial x^i} dx^i$

2. *If* $\omega = \sum_I a_I dx^I$*, then* $d\omega = \sum_I da_I \wedge dx^I$
3. $d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^{\deg \omega} w \wedge d\tau$

Proposition 3: *Let* ω *be an* n*-form on* S^n*.* $\int_{S^n} \omega = 0 \Leftrightarrow \omega$ *is exact*

Proof: If ω is exact, then $\omega = d\tau$. So $\int_{S_n} \omega = \int_{S_n} d\tau = \int_{\partial S_n} \tau = \int_{\varnothing} \tau = 0$. For the converse, note that $H^n(S^n) = \mathbb{R}$ and that the volume ν form on S^n does not integrate to zero. Therefore, the cohomology classes of the n-forms are $[r\nu]$ where $r \in \mathbb{R}$. So every n-form ω of on S_n can be written in the form $\omega = r\nu + \tau$ where τ is exact. $\int_{S_n} \omega = r \int_{S_n} \nu$. If $\int_{S_n} \omega = 0$, then $r = 0$. So $\omega = \tau$ is exact. QED

Proposition 4: *A form* $\omega = \sum_I a_I dx^I$ *is smooth iff all the* a_I *are smooth functions on each chart.*

Proposition 5:

1. *A manifold is orientable iff it has a nowhere-vanishing smooth top form.*
2. *A manifold has a trivial tangent bundle iff it has a smooth global frame.*
3. *If a manifold has a smooth global frame, then it is orientable.*

Definition: A smooth partition of unity on a manifold M is a collection of nonegative function such that their sum is 1 and every point in M has a neighborhood that intersects only finitely many of the supports of the functions.

Proposition 6:

1. *Every manifold has a smooth partition of unity with each function having* **compact support** *and they may be chosen so that each support lies inside a set of an open cover.*
2. *Every manifold has a smooth partition of unity that is subordinate to any chosen open cover.*

Proposition 7:

1. *For a function* $f : V \to W$ *between vector spaces the pullback of the differential* n*-form* ω *is* $f^*(\omega)(v_1, .., v_n) = \omega(f_* v_1, ..., f_* v_n)$
2. *The pullback is linear* $f^*(a\omega + b\tau) = af^*(\omega) + bf^*(\tau)$
3. *The pullback commutes with the differential* $f^* d\omega = df^* \omega$
4. *The pullback distributes with the wedge produce* $f^*(\omega \wedge \tau) = f^*(\omega) \wedge f^*(\tau)$

Definition: The push forward of a map $f : M \to N$ is defined by $f_*(\frac{\partial}{\partial x^j}) = \sum_i \frac{\partial f^i}{\partial x^j}\frac{\partial}{\partial y^i}$

Hairy ball theorem: If n is even, then any continuous tangent vector field on S^n must vanish.

Proposition 8: *S^0, S^1, S^3, S^7 only spheres with trivial tangent bundle.*

Borsuk-ulam theorem: For every continuous map $f : S^2 \to \mathbb{R}^2$ there is a pair of antipodal points x and $-x$ in S^2 such that $f(x) = f(-x)$.

Definition: The degree of a map $f : S^n \to S^n$ is the integer d such that $f_*(\alpha) = d\alpha$, where $f_* : \tilde{H}_n(S^n) \to \tilde{H}_n(S^n)$.

Proposition 9:

1. *deg$(f) = 0$ if f is not surjective.*
2. *deg$(f \circ g) =$ deg$(f)\cdot$ deg(g)*
3. *deg$(f) = \pm 1$ if f is a homotopy equivalence*
4. *deg$(f) = (-1)^{n+1}$ if f is the antipodal map*
5. *deg(f) =deg(g) if $f \simeq g$. The converse is also true for $n > 0$.*

Definition: The connected sum $M\#N$ of two surfaces is the space constructed by removing a disc from each and identfying the boundary circles of the removed discs.

Van Kampen's theorem: If X is the union of path-connected open sets A_α each containing the same base point, and each triple intersection of the A_α is path-connected, then $\Pi_1(X) \cong (*_\alpha \Pi_1(A_\alpha))/N$ where N is the normal subgroup generated by all elements of the form $i_{\alpha\beta}(\omega)i^{-1}_{\beta\alpha}(\omega)$. $i_{\alpha\beta}$ maps into $\Pi_1(A_\alpha)$ and $i_{\beta\alpha}$ maps into $\Pi_1(A_\beta)$.

Proposition 10: *If X is path-connected, $H_1(X) = \Pi_1(X)^{ab}$*

Mayer-Vietoris: Let U, V be open sets whose union is the entire space X. Then the Mayer-Vietoris sequences are exact.

1. For homology H_n, you go from $U \cap V$ to $U \amalg V$ to $X = U \cup V$ and then to a lower homology, decreasing n
2. For cohomology H^n, you go from $X = U \cup V$ to $U \amalg V$ to $U \cap V$ and to a higher cohomology, increasing
3. For compact support cohomology H^n_c, go up as in cohomology but reverse the direction. n

Proposition 11: *The alternating sum of the degrees in the Mayer-Vietoris sequence is zero*

Proof: This follows from exactness. QED

Proposition 12: $H^n(M \times \mathbb{R}) \cong H^n(M)$

Kunneth formula: For manifolds M and F, $H^n(M \times F) = \oplus_{p+q=n} H^p(M) \otimes_{\mathbb{R}} H^q(F)$

Proposition 13: $\tilde{H}_n(\vee_\alpha X_\alpha) \cong \oplus_\alpha \tilde{H}_n(X_\alpha)$ *provided that* (X_α, x_α) *are good pairs, where* x_α *are the base points.*

Proposition 14: $H_0(x) \cong \mathbb{Z}^k$, *where* k *is the number of path-components.*

Definition: The reduced homology $\tilde{H}_n$ is simply H_n for $n \neq 0$ and has one less $\mathbb{Z}$ summand for $n = 0$.

Definition: A singular n-simplex is a continuous map $\sigma : \Delta^n \to X$. $\partial\sigma = \sum_i (-1)^i \sigma | face_i$. An n-chain is a formal finite $\mathbb{Z}$-linear combination of n-simplices. The singular homology is those chains without boundary mod those that are the boundary of other chains.

Singluar homology:

Definition: The cellular homology of a space X is found by considering the sequence $\cdots \to H_{n+1}(X^{n+1}, X^n) \to H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2}) \to \cdots$, where $H_n(X^n, X^{n-1}) \cong \mathbb{Z}^k$ where k is the number of n-cells in the complex. The maps between the H_n are the boundary maps, and the cellular homology $H_n(X) = \ker d_n / \text{im } d_{n-1}$ where d_n maps from $H_n(X^n, X^{n-1})$. That is, it maps from the n-cells

Homology and Cohomology of common spaces.

1. S^n The de Rham cohomology is $\mathbb{R}$ in dimension 0 and n and is zero otherwise. For homology replace $\mathbb{R}$ by $\mathbb{Z}$.
2. $\mathbb{R}^n$ The de Rham cohomology is $\mathbb{R}$ in dimension 0 and is zero otherwise (Poincare lemma). For homology replace $\mathbb{R}$ by $\mathbb{Z}$.
3. T^2. The homology is $\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}$ for $n = 0, 1, 2$ respectively, otherwise it's zero.

Definition: Two maps $f_0 : X \to Y$ and $f_1 : X \to Y$ are homotopic if there exists a continuous function $F : X \times [0,1] \to Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. For two paths to be homotopic the end points must be fixed.

Definition: $X \simeq Y := X$ is homotopy equivalent to Y. This means there exist map $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity.

Definition: $G \cong H := G$ and H are isomorphic

.

Proposition 15: *Homotopy equivalent spaces have the same homology and cohomology.*

Definition: A covering space of a space X is a space $\tilde{X}$ together with a map $p : \tilde{X} \to X$ for which there exists an open conver $\{U_\alpha\}$ of X such that $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\tilde{X}$ each of which is homeomorphic to U_α via p. Note p need not be surjective.

Definition: Two covering spaces $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ are isomorphic if there exists a homeomorphism $f : \tilde{X}_1 \to \tilde{X}_2$ such that $p_1 = p_2 \circ f$.

Definition: The deck transformations $G(\tilde{X})$ of a covering space is the group of isomorphisms of the covering space with itself.

Definition: A covering space is normal if for each pair of lifts $\tilde{x}, \tilde{x}'$ of x there is a deck transformation sending one to the other.

Prop 1.33 in Hatcher: Let $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a covering space and Y be a path-connected and locally path-connected space. Then a lift $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x_0})$ of $f : (Y, y_0) \to (X, x_0)$ exists iff $f_*(\Pi_1(Y, y_0)) \subseteq p_*(\Pi_1(\tilde{X}, \tilde{x_0}))$. The lift is unique once a based point is fixed.

Proposition 16: *If X is path-connected, locally path-connected, and semi-locally simply-connected, then there is a one to one correspondence between subgroups $H \leq \Pi_1(x)$ and covering spaces X_H of X. $p_*(\Pi_1(X_H)) = H$.*

Proposition 17: *Let $p : \tilde{X} \to X$ be a path-connected covering space of the path-connected locally path-connected space X. Let $H = p_*(\Pi_1(\tilde{X}))$. Then*

1. *The convering space is normal iff H is a normal subgroup of $\Pi_1(X)$*
2. *$G(\tilde{X}) \cong N(H)/H$, where $N(H)$ is the normalizer of H in $\Pi_1(X)$*
3. *It follows that for the universal cover, $G(\tilde{X}) \cong \Pi_1(X)$*

Definition: The action of a group G on a space Y is properly discontinuous if each $y \in Y$ has a neighborhood U such that the $g(U)$ are disjoint for different $g \in G$.

Prop 1.40 in hatcher If an action of a group G on a path-connected and locally path-connected space Y is properly discontinuous, then $G \cong \Pi_1(Y/G)/p_*(\Pi_1(Y))$

Proposition 18: *If $G \leq Y$ is a finite group acting on Y via multiplication and mutliplication on*

Y is continuous and every element in Y has an inverse, then the action is properly discontinuous.

Proof: Let $y \in Y$. Then the gy are all distinct, because every element in Y has an inverse. So for each $g \in G$, there is an open neighborhood V_g of gy such that the V_g are all disjoint. This is possible because G is finite. Let $U_g = g^{-1}(V_g)$, and let $U = \cap_{g \in G} U_g$. Each U_g contains y because $gy \in V_g$. So U contains y, and it is open, because it is a finite intersection of open sets. Finally, note that $g(U) \subseteq g(U_g) \subseteq V_g$. So that the different $g(U)$ are disjoint. Therefore, the action is properly discontinuous. QED

Proposition 19: *The two dimensional fractal tree is the universal cover of $S^1 \vee S^1$. The three dimensional fractal tree is the universal cover of $S^1 \vee S^1 \vee S^1$.*

Definition: For a finite CW complex, the Euler characteristic $\chi(X) = \sum_n (-1)^n c_n$ where c_n is the number of n-cells in X

Proposition 20: *$\chi(X) = \sum_n (-1)^n rank H_n(X)$. Hence, $\chi(X)$ depends only on homotopy type, and is independent of the CW structure on X*

Cauhcy Riemann equations: If $f(x+iy) = u(x,y) + iv(x,y)$ is differentiable and u, v are real, then $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Picardi's great theorem: If f has an essential singularity at $z_0 \in U$ where U is a neighborhood of z_0, then $f(U) = \mathbb{C}$ or $\mathbb{C} \setminus \{\text{point}\}$.

Residue theorem: If f is meromorphic on D and γ is a simple closed curve not passing through a pole of f, then $\int_\gamma f(z)dz = 2\pi i \sum_{\text{poles } z_0 \text{ enclosed by } \gamma} \text{Res}_{z_0} f$, where $\text{Res}_{z_0} f(z)$ is the coeffient of the term $\frac{1}{z - z_0}$ in the Laurent series.

Cauchy integral formula: If f is analytic, $f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(s)}{s-z} ds$, where γ is a path around z.

Open mapping theorem: Any non-constant analytic function is an open mapping.

Maximum modulus principle: If $f(z)$ is a nonconstant function on an open set U, f does not attain a maximum modulus, $|f(z)|$.

Louiville's theorem: Any bounded entire function is constant.

Picardi's little theorem: If f is entire and $f(\mathbb{C})$ omits at least two values, then f is constant.

Riemann's mapping theorem: If $U \subseteq \mathbb{C}$ is open and simply connected and $U \neq \mathbb{C}$, then there exists an analytic isomorphism $U \to D$, where D is the unit disk.

Conformal maps are all diffeomorphisms.

conformal mapping ingredients: Translation $z \mapsto z + z_0$, Rotation $z \mapsto e^{i\theta} z$, Wrap around origin $z \mapsto z^k$ (k not necessarily an integer), $\sin(z)$ half strip to upper half plane, $\log(z)$ upper half of unit disc to half horizontal strip in second quadrant, $z \mapsto (z - i)/(z + i)$ takes upper half plane to disc, exponential takes horizontal strip to upper half plane.

Example: quarter disc in first quadrant to vertical half strip in first quadrant.

1. $z \mapsto z^2$ Quarter to half disk

2. $z \mapsto \log z$ half disk to half horizontal strip in second qaudrant

3. $z \mapsto e^{-i\pi/2} z$ rotates clockwise 90 degress.