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Fun Math Problem

Problem: The set $A = \{1, 2, 3, \ldots, 2044, 2045\}$ contains 2045 elements. A subset S of A is called triple-free if no element of S equals three times another element of S. For example, $\{1, 2, 4, 5, 10, 2043\}$ is triple-free, but $\{1, 2, 4, 5, 10, 681, 2043\}$ is not triple-free. The triple-free subsets of A that contain the largest number of elements contain exactly 1535 elements. There are n triple-free subsets of A that contain exactly 1535 elements. The integer n can be written in the form $p^a q^b$, where p and q are distinct prime numbers and a and b are positive integers. If $N = p^2 + q^2 + a^2 + b^2$, then the last three digits of N are

(A) 202 (B) 102 (C) 302 (D) 402 (E) 502

Solution: A subset B of A will be called a triple subset if for every $x, y \in B$, $x = 3^m y$ for some integer m. For example, $\{3, 9, 27\}$ is a triple subset. A maximal triple subset of A is a triple subset of A that is not contained in any larger triple subset. For example, $\{1, 3, 9, 27, 81, 243, 729\}$ is a maximal triple subset. A can be written as a disjoint union of its maximal triple sets. That is, every element of A is contained in exactly one of its maximal triple subsets.

It is advantageous to decompose A in that way, because the triple-free subsets of A can be characterized in terms of the maximal triple subsets of A. Specifically, if $B_1, ..., B_k$ are disjoint maximal triple subsets of A, and $S_1, ..., S_k$ are triple-free subsets of $B_1, .., B_k$, respectively, then the union $\cup_{i=1}^{k} S_i$ is a triple-free subset of A. In fact, any triple-free subset of A can be formed in this manner. For example, $B_1 = \{1, 3, 9, 27, 81, 243, 729\}$, $B_2 = \{2, 6, 18, 54, 162, 486, 1458\}$, and $B_3 = \{4, 12, 36, 108, 324, 972\}$ are three disjoint maximal triple subsets of A. $S_1 = \{1, 9, 81\}$, $S_2 = \{6, 162, 1458\}$, and $S_3 = \{4, 108\}$ are triple-free subsets of B_1, B_2, and B_3, respectively. It follows then that $\cup_{i=1}^{3} S_i = \{1, 4, 6, 9, 81, 108, 162, 1458\}$ is a triple-free subset of A.

We will continue by characterizing the maximal triple subsets of A. The length of a maximal triple subset B of A will just be the number of elements in B. For example, the length of B_1 above is 7, the length of B_3 above is 6, and $\{2045\}$ is a maximal triple subset of length 1. If s is the smallest element in a maximal triple subset B of A, then the length of B is the largest natural number L such that $s \cdot 3^{L-1} \leq 2045$. Since s must be at least 1, L can be at most 7. Thus, all maximal triple subsets of A have length between 1 and 7.

We can then proceed to find the number of maximal triple subsets of A having a given length. The maximal triple subsets that have length 7 must have a smallest element s such that $s \cdot 3^{7-1} \leq 2045$. So $s \leq 2045/729 \approx 2.8$. Therefore, the only maximal triple subsets of A with length 7 are those whose smallest elements are 1 and 2. These are the sets $B_1 = \{1, 3, 9, 27, 81, 243, 729\}$ and $B_2 = \{2, 6, 18, 54, 162, 486, 1458\}$. Thus, there are 2 maximal triple subsets of length 7.

The maximal triple subsets that have length 6 must have a smallest element s such that $s \cdot 3^{6-1} \leq 2045$, but $s \cdot 3^{7-1} > 2045$. Therefore, $2.8 \approx 2045/729 < s \leq 2045/243 \approx 8.4$. So $s = 3, 4, 5, 6, 7, 8$.

s, however, cannot be 3 or 6 because these are elements of the maximal triple subsets B_1 and B_2, which have length 7. In fact, s cannot be a multiple of 3, for then it would not be the smallest element of the maximal triple subset to which it belongs. Therefore, the maximal triple subsets of length 6 are those whose smallest elements are 4, 5, 7, and 8. There are 4 such subsets.

The maximal triple subsets that have length 5 must have a smallest element s such that $s \cdot 3^{5-1} \leq 2045$, but $s \cdot 3^{6-1} > 2045$. Therefore, $8.4 \approx 2045/243 < s \leq 2045/81 \approx 25.2$. So s is a integer between 9 and 25 that is not a multiple of 3. There are 11 such s. Therefore, there are 11 the maximal triple subsets of length 5.

We can repeat the above procedure to find the number of maximal triple subsets of any length. The results are presented in the table below.

Length of maximal triple subsets	Number of maximal triple subsets
7	2
6	4
5	11
4	33
3	102
2	302
1	910

Earlier, we mentioned that if $B_1, ..., B_k$ are disjoint maximal triple subsets of A, and $S_1, ..., S_k$ are triple-free subsets of $B_1, .., B_k$, respectively, then $\cup_{i=1}^{k} S_i$ is a triple-free subset of A. To form the largest triple-free subsets of A, the subsets S_i must be the largest possible triple-free subsets of the B_i.

If B_i has length 7, then the largest triple-free subsets of B_i contain 4 elements. For example, $B_1 = \{1, 3, 9, 27, 81, 243, 729\}$ is a maximal triple subset of length 7. Its largest triple-free subset is $S_1 = \{1, 9, 81, 729\}$. S_1 contains the first, third, fifth, and seventh elements of B_1. Note that these elements are at least two apart B_1 in the sense that there is another element of B_1 between any two of the elements of S_1. If a set contained any of these other elements it would either contain fewer than 4 elements or cease to triple-free.

If B_i has length 6, then the largest triple-free subsets of B_i contain 3 elements. For example, $B_3 = \{4, 12, 36, 108, 324, 972\}$ is a maximal triple subset of length 6. One of its largest triple-free subsets is $S_3 = \{4, 36, 324\}$. S_3 contains the first, third, and fifths elements of B_3. Again note that these elements are at least two apart in B_3 in the sense that there is another element of B_1 between any two of the elements of S_1. Any other triple-free subset of B_3 with 3 elements must contain elements that are also at least two apart. The other triple-free subsets of B_3 containing 3 elements are $\{4, 36, 972\}$, which contains the first, third, and sixth elements of B_3; $\{4, 108, 972\}$, which contains the first, fourth, and sixth elements; and $\{12, 108, 972\}$, which contains the second, fourth, and sixth elements. Therefore, there are 4 triple-free subsets of B_3 that have length 3.

In general, if B_i has an odd length equal to $2n+1$ for some integer n, then its largest triple-free subsets have $n+1$ elements, and there is only 1 such subset. If B_i has an even length equal to $2n$ for some integer n, then its largest triple-free subsets have n elements, and there will be $n+1$ such subsets. This is displayed in the table below.

Length of maximal triple subsets	Number of maximal triple subsets	Number of elements in the largest triple-free subsets	Number of largest triple-free subsets
7	2	4	1
6	4	3	4
5	11	3	1
4	33	2	3
3	102	2	1
2	302	1	2
1	910	1	1

We can quickly check that these numbers are correct. We were told that the largest triple-free subsets of A contain 1535 elements. The largest triple-free subsets of A must contain

(a) 4 elements from every maximal triple subset of length 7.
(There are 2 such subsets giving a total $2 \cdot 4 = 8$ elements.)

(b) 3 elements from every maximal triple subset of length 6.
(There are 4 such subsets giving a total $4 \cdot 3 = 12$ elements.)

(c) 3 elements from every maximal triple subset of length 5.
(There are 11 such subsets giving a total $11 \cdot 3 = 33$ elements.)

(d) 2 elements from every maximal triple subset of length 4.
(There are 33 such subsets giving a total $33 \cdot 2 = 66$ elements.)

(e) 2 elements from every maximal triple subset of length 3.
(There are 102 such subsets giving a total $102 \cdot 2 = 204$ elements.)

(f) 1 element from every maximal triple subset of length 2.
(There are 302 such subsets giving a total $302 \cdot 1 = 302$ elements.)

(g) 1 element from every maximal triple subset of length 1.
(There are 910 such subsets giving a total $910 \cdot 1 = 910$ elements.)

Therefore, the largest triple-free subsets of A contain $8 + 12 + 33 + 66 + 204 + 302 + 910 = 1535$ elements as we were told.

Finally, we can find the number of triple-free subsets with 1535 elements.

(a) There is only 1 way to choose the 4 elements from a maximal triple subset of length 7.
(There are 2 such subsets giving a total of 1^2 ways of choosing the 8 elements.)

(b) There are 4 ways to choose the 3 elements from a maximal triple subset of length 6. (There are 4 such subsets giving a total of 4^4 ways of choosing the 12 elements.)

(c) There is only 1 way to choose the 3 elements from a maximal triple subset of length 5. (There are 11 such subsets giving a total of 1^{11} ways of choosing the 33 elements.)

(d) There are 3 ways to choose the 2 elements from every maximal triple subset of length 4. (There are 33 such subsets giving a total of 3^{33} ways of choosing the 66 elements.)

(e) There is only 1 way to choose the 2 elements from a maximal triple subset of length 3. (There are 102 such subsets giving a total of 1^{102} ways of choosing the 204 elements.)

(f) There are 2 ways to choose the 1 element from every maximal triple subset of length 2. (There are 302 such subsets giving a total of 2^{302} ways of choosing the 302 elements.)

(g) There is only 1 way to choose the 1 element from a maximal triple subset of length 1. (There are 910 such subsets giving a total 1^{910} ways of choosing the 910 elements.)

Therefore, there are $n = 1^2 \cdot 4^4 \cdot 1^{11} \cdot 3^{33} \cdot 1^{102} \cdot 2^{302} \cdot 1^{910} = 2^{310} \cdot 3^{33}$ different triple-free subsets of A that have 1535 elements.

Referring back to the problem, it is clear that $p = 2$, $q = 3$, $a = 310$, and $b = 33$. Or we may interchange the values of p and q as well as those of a and b. In any case, $N = p^2 + q^2 + a^2 + b^2 = 2^2 + 3^2 + 310^2 + 33^2 = 97,202$. So the last three digits of N are 202, and the answer is (A).