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Hydrodynamic Limits

1 Introduction

A hydrodynamic equation describes how a fluid evolves over time. Specifically, a solution of a hydrodynamic equation is a function $\rho(t, u)$ that gives the density of the fluid at time t at the point u subject to an initial density $\rho_0(u)$. For example, the heat equation is a common hydrodynamic equation:

$$\partial_t \rho = \triangle \rho$$

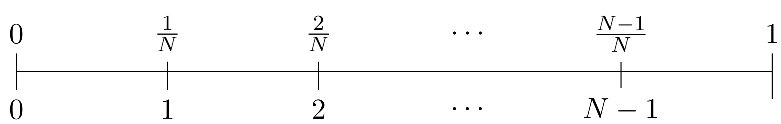
One way to derive a hydrodynamic equation relies on physical assumptions at the molecular level. That is, the fluid is viewed as being made up of many tiny molecules that interact with one another. By making assumptions about the molecular interactions, the hydrodynamic equation can be recovered by taking a limit as the number of particles increases to infinity.

The physical interactions in a fluid are often chaotic enough that we may model the interactions as probabilistic, or we may assume that the underlying interactions between molecules are, in fact, probabilistic. In any case, we will model the hydrodynamic process as a Markov process for an interacting particle system. Furthermore, we will make the simplification that the particles are restricted to a lattice. This approach allows for approximations as the gap between successive lattice points decreases. In the limit as the gap shrinks to zero, the lattice approximation should converge to the real life phenomenon.

2 Microscopic and Macroscopic Locations

To take such a limit we shall first need to embed our lattice into a continuum of positions. A point u in the continuum of position will be called a macroscopic point, and a point x in the lattice will be called a microscopic point.

We first consider the case where a particle can have any location in $[0, 1)$. In this case, we shall use the lattice $\{0, 1, ..., N-1\}$. The N lattice points $0, 1, ..., N-1$ are equally spaced on the interval $[0, 1)$ starting at zero as shown in the figure below.



A macroscopic point in $[0, 1)$ corresponds to the lattice point to its left. That is, a macroscopic point $u \in [0, 1)$ corresponds to the microscopic point $x = \lfloor Nu \rfloor$. Conversely, a lattice point in $\{0, 1, ..., N-1\}$ corresponds to the half-open interval to its right. That is, a microscopic point $x \in \{0, 1, ..., N-1\}$ corresponds to the macroscopic interval $[\frac{x}{N}, \frac{x+1}{N})$. We can also view the microscopic point x as corresponding to the macroscopic point $\frac{x}{N}$. See [3].

This embedding easily generalizes for other continua of positions and lattices. If the continuum of positions is $\mathbb{T} = S^1$, then the lattice $T_N = \mathbb{Z}/N\mathbb{Z}$ is used and the lattice points are equally spaced along the circle starting at 0. If the continuum of positions is $\mathbb{R}$, then the lattice $\mathbb{Z}$ is used and N lattice points are equally spaced as in the figure above along each interval $[m, m+1)$ for all integers m. If we consider the n-dimensional continua $[0, 1)^n, \mathbb{T}^n$, and $\mathbb{R}^n$, then along each cardinal direction we equally space N lattice points across any unit interval as in the one dimensional case. We then fill in the remaining lattice points so as to make a grid. Thus, there are N^n lattice points in every unit box.

In this report, we will restrict our attention to the continuum $\mathbb{R}$ and the lattice $\mathbb{Z}$.

3 Initial conditions

Let Z_t^i be the position of the ith particle at time $t \geq 0$. Then for each $x \in \mathbb{Z}$, $\eta_t(x) = \sum_i 1(Z_t^i = x)$ is the number of particles at site x at time t. The function $\eta_t : \mathbb{Z} \to \mathbb{N} \cup \{0\}$ is called a configuration and it gives the number of particles at each location $x \in \mathbb{Z}$ at time t. The set of all configurations is called the configuration space and is denoted by $E = \mathbb{N}^{\mathbb{Z}}$.

η_t evolves over time according to a Markov process. So for each $t > 0$, η_t depends on the initial configuration η_0. η_0, however, is determined by the initial density $\rho_0(u)$. We will assume that $\rho_0(u)$ is piece-wise continuous. In the simplest case, when $\rho_0(u) \in [0, 1]$, a single particle can be placed independently at each site x with probability $\rho_0(\frac{x}{N})$. [Note that the distribution of η_0 depends on N, and we should technically write $\eta_0^{(N)}$ instead of η_0.]

More generally, we look for invariant measures ν_ρ on E for the interacting particle system in question. Specifically, we look for measures ν_ρ which are translation invariant product measures. That is, ones for which $\{\eta_0(x)\}_{x \in \mathbb{Z}}$ are i.i.d. random variables. We then introduce probability measures Θ_ρ on $\mathbb{N} \cup \{0\}$ that satisfy

$$\Theta_\rho(k) = \nu_\rho(\eta_0(x) = k)$$

and let the initial probability measure, μ^N, on configurations be the product measure satisfying

$$\mu^N(\eta_0(x) = k) = \Theta_{\rho_0(\frac{x}{N})}(k)$$

In other words, $\{\eta_0(x)\}_{x\in\mathbb{Z}}$ will be independent random variables with distributions $\Theta_{\rho_0(\frac{x}{N})}$.

4 Interacting Particle Systems

An interacting particle system is a Markov process in which particles move on a lattice. There may be a finite number of particles on the lattice, a finite but unspecified number of particles on the lattice, or an infinite number of particles on the lattice. When there are an infinite number of particles, extra care needs to be taken to make sure the process can be constructed.

The evolution of an interacting particle system with a finite number of particles is easily described by defining the generator L in terms of the rates $\lambda_{\eta,\zeta}$, where η and ζ are configurations.

$$Lf(\eta) = \sum_{\zeta\in E} \lambda_{\eta,\zeta}(f(\zeta) - f(\eta))$$

This formulation can be simplified—and sometimes given more of a physical interpretation—depending on the specific dynamics. See [9].

We will only consider systems where there is mass conservation (i.e. the total number of particles remains fixed) and only one particle can move at a time. If the current configuration of the system is η and a particle at location x moves to a new location y, then the new configuration of the system $\eta^{x,y}$ is defines as follows.

$$\eta^{x,y}(z) = \begin{cases} \eta(x) - 1 & \text{if } z = x \\ \eta(y) + 1 & \text{if } z = y \\ \eta(z) & \text{if } z \neq x, y \end{cases}$$

That is, there is one less particle at the site x, one more particle at the site y, and at every other site the number of particles is unchanged.

If only one particle is moving at a time, it follows that $\lambda_{\eta,\zeta} = 0$ whenever $\zeta \neq \eta^{x,y}$ for some x and y. Therefore,

$$Lf(\eta) = \sum_{\eta^{x,y}\in E} \lambda_{\eta,\eta^{x,y}}(f(\eta^{x,y}) - f(\eta)) = \sum_{x,y} \lambda_{\eta,\eta^{x,y}}(f(\eta^{x,y}) - f(\eta))$$

where the last equality holds if we define $\eta^{x,y} = \eta$ when $x = y$.

If our system consisted of only one particle, the dynamics of the system could be constructed by having that one particle hold an exponential alarm clock with a rate λ_x that may depend on the location x of the particle. When the alarm clocks goes off, the particle moves to a site y with some probability $p(x \to y)$ which may depend on x and y. We could then write $\lambda_{\eta,\eta^{x,y}} = \lambda_x p(x \to y)$. In light of this fact the rate $\lambda_{\eta,\eta^{x,y}}$ is often written as a product of two factors: One to describe a "rate" to be denoted $\lambda_{\eta,x,y}$ and one to describe the probability of making certain jumps to be denoted $p(x,y)$.

$$\lambda_{\eta,\eta^{x,y}} = \lambda_{\eta,x,y} p(x,y)$$

Nonetheless, when more than one particle is present the "rate" $\lambda_{\eta,x,y}$ and the probability $p(x,y)$ do not always have simple physical interpretations.

5 Classifying Interacting Particle Systems

Interacting particle systems are partially categorized according to additional properties of the function $p(x,y)$. We will restrict our attention to hydrodynamics for p which satisfies the following two conditions:

1. p is translation invariant: $p(x,y) = p(0, y-x)$. That is, the probability of "jumping" to a new location only depends to the relative position of the final and initial locations. Therefore, we define a function $p(d) = p(0,d)$. $p(d)$ is the probability of a particle "displacing by d" when it jumps. ($\sum_d p(d) = 1$.)
2. For the particle to actually move, we also require that $p(x,y) = 0$ if $x = y$. In terms of $p(d)$, we require $p(0) = 0$.

In terms of $p(d)$, the generator L is defined by

$$Lf(\eta) = \sum_{x,y} p(y-x)\lambda_{\eta,x,y}(f(\eta^{x,y}) - f(\eta)) = \sum_{x,d} p(d)\lambda_{\eta,x,x+d}(f(\eta^{x,x+d}) - f(\eta))$$

The interacting particle system may be further characterized by the following properties of $p(d)$.

There are the following properties concerning the range of $p(d)$.

Definition:

1. Nearest neighbor: $p(1) + p(-1) = 1$. ($p(d) = 0$ if $d \neq \pm 1$.)
2. Finite range: For some R, $p(d) = 0$ whenever $|d| > R$.
3. Long range: Not finite range.

And there are the following properties concerning the symmetry of $p(d)$.

Definition:

1. Symmetric: $p(d) = p(-d)$
2. Non-symmetric: $p(d) \neq p(-d)$
3. Asymmetric: $E_p(d) = \sum_d dp(d) \neq 0$
4. Mean-zero asymmetric: $E_p(d) = \sum_d dp(d) = 0$, but $p(d) \neq p(-d)$.

6 The Simple Exclusion Process

The simple exclusion process is one in which only one particle may occupy each site ($\eta(x) = 0$ or 1 for all $x \in \mathbb{Z}$) and the rate $\lambda_{\eta,x,y}$ is 1 whenever the original site x is occupied and the destination site y is unoccupied. That is,

$$\lambda_{\eta,x,y} = \left\{ \begin{array}{ll} 1 & \text{if } \eta(x) = 1 \text{ and } \eta(y) = 0 \\ 0 & \text{otherwise} \end{array} \right\} = \eta(x)(1 - \eta(y))$$

So the generator L for this process is defined by

$$Lf(\eta) = \sum_{x,d} p(d)\eta(x)(1 - \eta(x+d))(f(\eta^{x,x+d}) - f(\eta))$$

For an infinite number of particles, L generates a Markov semigroup with a core consisting of local functions [6]. (f is said to be a local function if if there exists a finite $\Lambda \subseteq \mathbb{Z}$ such that $f(\eta) = f(\zeta)$ whenever $\eta(x) = \zeta(x)$ for all $x \in \Lambda$. That is, a local function depends on configurations only up to a finite number of lattice sites.)

The simple exclusion process can be interpreted in the following way. Each particle holds an exponential alarm clock with rate 1. If a particle's alarm clock goes off when it is at location x, then it selects a displacement d with probability $p(d)$. If location $x+d$ is occupied ($1-\eta(x+d) = 0$), then the particle doesn't move. Otherwise, it relocates to $x + d$.

For the simple exclusion process, a class of invariant measures ν_ρ that are translation invariant product measures are those for which

$$\nu_\rho(\eta_0(x) = 1) = \rho, \quad \nu_\rho(\eta_0(x) = 0) = 1 - \rho$$

See [6].

7 The Zero Range Process

For the zero range process, any number of particles may occupy each site, but the interaction among particles is entirely local. That is, the range of interaction is zero. Hence, the name zero range. Formally, $\lambda_{\eta,x,y}$ does not depend on the destination site y; it is just a function of $\eta(x)$.

$$\lambda_{\eta,x,y} = g(\eta(x))$$

for some non-negative function g with $g(0) > 0$, and the generator L for this process is defined by

$$Lf(\eta) = \sum_{x,d} p(d)g(\eta(x))(f(\eta^{x,x+d}) - f(\eta))$$

For an infinite number of particles, there are several constructions depending on the type of g. The most general construction assumes that g is Lipschitz [1].

The zero range process can be interpreted in the following way. Each site x holds an exponential alarm clock with rate $g(\eta(x))$. (This rate will vary with time.) If a site's alarm clock goes off, then a displacement d is picked with probability $p(d)$, and a single particle moves from x to $x+d$. Note: If there is no particle at a location x, it doesn't make sense to have a particle move from x. Hence, we require that $g(0) = 0$.

For the zero range process, a class of invariant measures ν_ρ that are translation invariant product measures are those for which

$$\nu_\rho(\eta_0(x) = k) = \left(\frac{1}{\sum_{k=0}^{\infty} \frac{\lambda^k}{\prod_{j=1}^{k} g(j)}} \right) \frac{\lambda^k}{\prod_{j=1}^{k} g(j)}$$

See [1]. Here λ is a function of ρ, and we require that $\lambda \geq 0$ be small enough so that the sum will converge.

8 The Misanthrope Process

The misanthrope process is a more general interacting particle system that encompasses both the simple exclusion and many zero range processes. For the misanthrope process, $\lambda_{\eta,x,y}$ depends on both $\eta(x)$ and $\eta(y)$. Specifically, $\lambda_{\eta,x,y}$ is larger when there are more particles at the initial site x and is smaller when there are more particles at the destination site y. That is, particles avoid crowded sites. Hence, the name misanthrope. Formally, we define a function $b : \mathbb{N} \times \mathbb{N} \to [0, \infty)$ with the following properties:

(i) $b(0, m) = 0$ for all m

(ii) $b_m(n) : \mathbb{N} \to [0, \infty)$ defined by $b_m(n) = b(n, m)$ is a non-decreasing function for each m

(iii) $b^n(m) : \mathbb{N} \to [0, \infty)$ defined by $b^n(m) = b(n, m)$ is a non-increasing function for each n

And we let $\lambda_{\eta,x,y} = b(\eta(x), \eta(y))$. The generator L in this case is defined by

$$Lf(\eta) = \sum_{x,d} p(d)b(\eta(x), \eta(x+d))(f(\eta^{x,x+d}) - f(\eta))$$

See [2] for a construction of this process. Also see [8].

If we choose, the function b appropriately we can obtain the simple exclusion and zero range processes. To obtain the simple exclusion process, let $b(n, m) = 1(n = 1, m = 0)$. Then $b(\eta(x), \eta(y)) = \eta(x)(1 - \eta(y))$, and L will simplify to the generator for the simple exclusion process. To obtain the zero range process, let $b(n,m) = g(n)$. Then $b(\eta(x),\eta(y)) = g(\eta(x))$, and L will simplify to the generator for the zero range process. Note that this will only give us zero range processes for which g is non-decreasing.

9 Coupling Interacting Particle Systems

A coupled process is one in which multiple processes are defined on the same probability space. The processes may be independent but are usually made dependent to require certain properties of the coupled process. It also desired that after coupling each marginal process should be the same as the processes before coupling.

For interacting particle systems, the basic coupling is one in which each configuration develops on its own but whenever possible the same jump should be made for both processes. That is, at each jump time, the coupled process (η_t,ζ_t) can transition to one of three types of states: one in which a single particle moves from site x to y in the first process $(\eta^{x,y},\zeta)$; one in which a single particle moves from site x to y in the second process $(\eta,\zeta^{x,y})$; and one in which a particle moves from site x to y in both processes $(\eta^{x,y},\zeta^{x,y})$.

The formal generator $\tilde{L}$ for the basic coupled misanthrope process is defined below.

$$\begin{aligned}
\tilde{L}f(\eta,\zeta) \quad &= \sum_{x,y} p(y-x)\min\nolimits_{x,y}(f(\eta^{x,t},\zeta^{x,y}) - f(\eta,\zeta)) \\
&+ \sum_{x,y} p(y-x)(b(\eta(x),\eta(y)) - \min\nolimits_{x,y})(f(\eta^{x,y},\zeta) - f(\eta,\zeta)) \\
&+ \sum_{x,y} p(y-x)(b(\zeta(x),\zeta(y)) - \min\nolimits_{x,y})(f(\eta,\zeta^{x,y}) - f(\eta,\zeta))
\end{aligned}$$

where $\min_{x,y} = \min\{b(\eta(x),\eta(y)), b(\zeta(x),\zeta(y))\}$.

Since $b(n,m)$ is increasing in the first coordinate and decreasing in the second coordinate, we have the following two properties of the coupled process.

1. If $\eta(x) \leq \zeta(x)$ and $\eta(y) = \zeta(y)$, then $(b(\eta(x),\eta(y)) - \min_{x,y}) = 0$, and the transition $(\eta,\zeta) \to (\eta^{x,y},\zeta)$ is not possible.
2. If $\eta(x) = \zeta(x)$ and $\eta(y) \leq \zeta(y)$, then $(b(\zeta(x),\zeta(y)) - \min_{x,y}) = 0$, and the transition $(\eta,\zeta) \to (\eta,\zeta^{x,y})$ is not possible.

It follows that if $\eta(x) \leq \zeta(x)$ and $\eta(y) \leq \zeta(y)$, then we still have $\eta(x) \leq \zeta(x)$ and $\eta(y) \leq \zeta(y)$ after any allowable transition of particles from x to y. More generally, if $\eta_0(x) \leq \zeta_0(x)$ for all sites x, then $\eta_t(x) \leq \zeta_t(x)$ for all sites x at any time t. Any interacting particle system that satisfies this monotonicity property when coupled is called an attractive process.

The misanthrope process is an attractive process. As such, both the simple exclusion process and the zero range process with a non-decreasing g are attractive processes. Attractiveness is a useful property that can be used to prove several results about interacting particle systems. We will see later how it can be used.

10 Time Scaling

Now that we have defined several interacting particle systems on lattices that are embedded into continua of positions we shall want to take the limit of the dynamics as N approaches infinity. To compensate for the scaling of position, we will need to scale time as well to observe the same general dynamics. The time scaling is accounted for by multiply the generator by the scaling factor γ_N. Thus $\gamma_N L$ is the generator for the time scaled process.

To see why it is necessary to scale time, consider a system with just one particle on the lattice $\mathbb{Z}$ starting at 0. Suppose that on average the particle jumps once every unit interval of time, and it always jumps one lattice point to the right. Then at time t we would expect the particle to be at lattice point $x = \lfloor t \rfloor$, which corresponds to the macroscopic point $u = \frac{\lfloor t \rfloor}{N}$. If we let N go to infinity without scaling time, then at time t we would expect the particle to be at the macroscopic point $u = \lim_{N\to\infty} \frac{\lfloor t \rfloor}{N} = 0$. That is, we would see that the particle hasn't moved contrary to the natural dynamics in which the particle continually moves right.

We would expect that in the continuum limit the particle should slowly be moving to the right at a constant speed. This is indeed the case if we scale time appropriately. It often works to scale time by a factor of $\gamma_N \propto E[t_N]$ where $E[t_N]$ is the average time it takes the particle to move N lattice point away from its original location. In general, the appropriate scale factor can be determined when starting to preform calculations with the generator of the process. In any case, after scaling time by a factor of γ_N, t is called the macroscopic time, and $\gamma_N t$ is called the microscopic time.

In the above example with one particle moving to the right, $\gamma_N = N$. Then at microscopic time $\gamma_N t = Nt$ we would expect the particle to be at the microscopic point $x = \lfloor Nt \rfloor$, which corresponds to the macroscopic point $u = \frac{\lfloor Nt \rfloor}{N} \approx t$. If we then let N go to infinity, at the macroscopic time t we would expect the particle to be at the macroscopic point $u = \lim_{N\to\infty} \frac{\lfloor Nt \rfloor}{N} = t$. That is, we would see the particle moving to the right at a constant speed of 1.

In general, suppose that at time t we expect the particle to displace $d \in \mathbb{Z}$ lattice points from its original position corresponding to the macroscopic displacement $\Delta \in \mathbb{R}$ before scaling. Then at the microscopic time $\gamma_N t$ we expect the particle to be displace Nd lattice points, which still corresponds to the macroscopic displacement Δ after scaling.

Lastly, we discuss three common scaling factors. We have already seen the scaling factor $\gamma_N = N$. Scaling in this way is called Euler scaling. It is common when the particles have a natural drift in their motion. Next, we have the scaling factor associated with a simple symmetric random walk that models Brownian motion. There is no drift is in this case, and the average time it takes to travel N lattice points is N^2. Therefore, $\gamma_N = N^2$. This is called diffusive scaling, because it often models the way things diffuse (e.g. heat transfer). Lastly, if the process is long range (e.g. $p(d) = 1(d > 0)\frac{1}{d^{1+\alpha}}$ with $\alpha < 1$) we will see a scaling factor of the form $\gamma_N = N^\alpha$.

11 Empirical Density and Process Measures

Now that we have introduced the processes and the relevant scaling, we will want deduce the hydrodynamic equation that process satisfies in the limit as N goes to infinity. Specifically, we will want to show that in some sense the process η_t converges to a density $\rho(t, u)$, which is the deterministic solution of a hydrodynamic equation.

The density $\rho(t, u)$ corresponds to the process η_t. More specifically, integrating a test function $g(t, u)$ against $\rho(t, u)du$ (i.e. $\int g(t, u)\rho(t, u)du$) corresponds to the sum $\frac{1}{N}\sum_x \eta_t(x)g(t, \frac{x}{N})$. It is easier to describe this sum with the empirical measure

$$\pi_t^N = \frac{1}{N}\sum_{x\in\mathbb{Z}} \eta_t(x)\delta_{\frac{x}{N}}$$

The following notation then is used for integration

$$\langle \pi_t^N, g(t, u)\rangle = \frac{1}{N}\sum_{x\in\mathbb{Z}} \eta_t(x)g\left(t, \frac{x}{N}\right)$$

There is a one-to-one correspondence between the process η_t and the process π_t^N, but it will be easier to work with the empirical measure.

The empirical measure π_t^N will also be a Markov process with an initial distribution that corresponds to the initial distribution μ^N of the process η_t. π_t^N is a càdlàg function of t (i.e. right continuous with left limits) taking values in M_+, the set of positive Radon measures on $\mathbb{R}$. The Skorokhod space $D([0, T], M_+)$ is the set of all such functions, but for bounded time T. Let $\{P^N\}_{N=1}^\infty$ be probability measures on $D([0, T], M_+)$ that correspond to the processes $\{\pi_t^N\}_{N=1}^\infty$ with the appropriate time scaling and initial distributions.

We could try to show that $\langle \pi_t^N, g(t, u)\rangle$ converges to $\int g(t, u)\rho(t, u)du$ in something like convergence in probability. For example, we could try to show that

$$\lim_{N\to\infty} P^N\left(\left|\langle \pi_t^N, g(t, u)\rangle - \int g(t, u)\rho(t, u)du\right| > \epsilon_0\right) = 0$$

for all $\epsilon_0 > 0$ and test functions g where $\rho(t, u)$ is the unique solution of some hydrodynamic equation satisfying the initial conditions. But it is common to prove a different type of convergence: weak convergence. That for any T, P^N converges weakly to a measure P supported on the absolutely continuous $\pi_t = \rho(t, u)du$ measure valued trajectory where the density $\rho(t, u)$ is the unique solution of some hydrodynamic equation satisfying the initial conditions. It then follows that the measures P^N defined on $D([0,\infty), M_+)$ also converges to that P.

The result is difficult to prove directly, because it is difficult to show that P^N converges at all. Instead a common trick is to first show that $\{P^N\}_{N=1}^{\infty}$ is weakly relatively compact within the space $\mathcal{P}(D([0,T], M_+))$ of probability measures on $D([0,T], M_+)$. That is, its closure is compact with respect to the weak topology. If the weak topology on $\mathcal{P}(D([0,T], M_+))$ is metrizable, then compactness is equivalent to sequential compactness. And one could prove that P^N converges weakly by proving that every convergent subsequence of P^N converges to the same limit. Essentially, we can deal only with convergent subsequences if we can show that $\{P^N\}_{N=1}^{\infty}$ is weakly relatively compact.

12 Relevant Metrics

To define weak convergence on the P^N, we must have a metric on the space $D([0,T], M_+)$. Furthermore, to describe what it means for $\{P^N\}_{N=1}^{\infty}$ to be weakly relatively compact we must introduce a topology on $\mathcal{P}(D([0,T], M_+))$, which we will want to be metrizable.

First, we shall introduce a metric on $D([0,T], M_+)$. Recall that $D([0,T], M_+)$ is not a space of continuous functions but of càdlàg functions. Specifically, two realizations of the empirical measure $\{\pi_t^N : t \in [0,T]\} \in D([0,T], M_+)$ are similar if the same jumps occur at roughly the same time. Therefore, we shall not equip $D([0,T], M_+)$ with a uniform metric, rather we will use the Skorokhod metric defined below.

First note that the weak topology on M_+ is metrizable. Let δ be a metric on M_+ that corresponds to the weak topology. Next, let Λ be the set of strictly increasing functions on $[0,T]$ into itself. For every $\lambda \in \Lambda$, we define

$$\|\lambda\| = \sup_{s \neq t} \left| \ln \left(\frac{\lambda(t) - \lambda(s)}{t - s} \right) \right|$$

Basically, the function $\lambda : [0,T] \to [0,T]$ will adjust time, and $\|\lambda\|$ quantifies the extent to which λ adjusts time. For example, if $\lambda(t) = t$ time is unchanged and $\|\lambda\| = 0$. We then define the Skorokhod metric by comparing $\mu : [0,T] \to M_+$ at time t with $\nu : [0,T] \to M_+$ at the adjusted time $\lambda(t)$. Formally, the Skorokhod metric is defined as follows.

$$d(\mu, \nu) = \inf_{\lambda \in \Lambda} \max\{\|\lambda\|, \sup_{0 \leq t \leq T} \delta(\mu_t, \nu_{\lambda(t)})\}$$

By introducing λ, we have allowed for $\delta(\mu_t, \nu_{\lambda(t)})$ to be small when jumps occur at nearby times. The term $\|\lambda\|$ is also needed so that $d(\mu, \nu)$ will not equal zero when $\mu \neq \nu$ but $\mu_t = \nu_{\lambda(t)}$ for some λ. With the respect to the Skorokhod metric, d, the Skorokhod space, $D([0,T], M_+)$, is a complete separable metric space.

It follows by Prokhorov's work that $\mathcal{P}(D([0,T],M_+))$ with the weak topology is metrizable. Moreover, we may characterize its compact subsets in terms of tightness. That is, $\{P^N\}_{N=1}^\infty$ is weakly relatively compact iff it is tight.

Other metrics could potentially have been used instead. The key choice of metric depends on a later step in the calculation. We will eventually show that for every weakly convergent subsequence P^{N_k},

$$\lim_{k\to\infty} P^{N_k}(f(\{\pi_t^{N_k} : 0 \le t \le T\}) > \epsilon_0) = 0$$

where f is a function on $D([0,T],M_+)$. We will want to conclude that for the limit point P^* of P^{N_k},

$$P^*(f(\{\pi_t : 0 \le t \le T\}) > \epsilon_0) = 0$$

This follows if the event $\{f(\{\pi_t : 0 \le t \le T\} > \epsilon_0\}$ is an open set in $D([0,T],M_+)$. Therefore, we will want to choose a metric on $D([0,T],M_+)$ so that the resulting function f will be continuous. The Skorokhod metric is one that commonly works.

Note that the uniform metric is stronger than the Skorokhod metric. In particular, if one could prove that $\{P^N\}_{N=1}^\infty$ converges weakly when the uniform metric was used, it follows that $\{P^N\}_{N=1}^\infty$ would converge weakly when the Skorokhod metric is used.

[Lastly, we could have written π_t instead of π_t^N (or $\pi_t^{N_k}$) in the early probabilities because the N (or N_k) is already accounted for in the probability measure P^N (or P^{N_k}). Nonetheless, it is common to write the superfluous N to aid in later calculations. We can, however, no longer continue to write the superscript once the limit point P^* is used.]

13 Martingales

Two types of martingales are repeatedly used to prove that the P^N converge to the desired limit. They are used both to prove tightness and to prove that convergent subsequences have the appropriate limit.

Let X_t be a continuous time Markov process taking values in $\mathcal{X}$, and let $F : [0,\infty) \times \mathcal{X} \to \mathbb{R}$ be a smooth function in time that is in the domain of L for each t. Then the following are mean zero martingales.

1. $M^F(t) = F(t,X_t) - F(0,X_0) - \int_0^t \frac{\partial F}{\partial s}(s,X_s) + LF(s,X_s)ds$
2. $(M^F(t))^2 - \langle M^F\rangle_t$ where $\langle M^F\rangle_t = \int_0^t LF^2(s,X_s) - 2F(s,X_s)LF(s,X_s)ds$

The quantity $\langle M^F \rangle_t$ is called the quadratic variation of the martingale $M^F(t)$. The function F needs to be chosen with some creativity depending on the result one wants to prove. In proving tightness and one of the main results, the function $F(t, \pi_t^N) = \langle \pi_t^N, g_t \rangle$ is used where $\langle \ , \ \rangle$ indicates spatial integration and $g_t(u) = g(t, u)$ is a test function. For this choice of F, we will denote $M^F(t)$ and $\langle M^F \rangle_t$ by $M_t^{N,g}$ and $\langle M^{N,g} \rangle_t$, respectively. We will also denote $\frac{\partial F}{\partial s}(s, \pi_s^N)$ by $\partial_s \langle \pi_s^N, g_s \rangle$, which will equal zero if g doesn't depend on time.

The martingale $M_t^{N,g}$ is primarily used because it resembles the weak formulation of a hydrodynamic equation. The martingale $(M^{N,g}(t))^2 - \langle M^{N,g} \rangle_t$ is primarily used to prove results about convergence by providing estimates on the variance of $M^{N,g}(t)$. Specifically, we will want to show that

$$\limsup_{N \to \infty} P^N \left(\left| M_t^{N,g} \right| > \epsilon_0 \right) = 0$$

If $\langle M^{N,g} \rangle_t$ can be bounded appropriately, that can be shown by relying on the following string of inequalities.

$$P^N \left(\left| M_t^{N,g} \right| > \epsilon_0 \right) \leq \frac{1}{\epsilon_0^2} E^N \left(\left(M_t^{N,g} \right)^2 \right) = \frac{1}{\epsilon_0^2} E^N \left(\langle M^{N,g} \rangle_t \right)$$

14 Tightness

We now return to showing that the sequence $\{P^N\}_{N=1}^{\infty}$ is tight or weakly relatively compact.

Proposition 1: *$\{P^N\}_{N=1}^{\infty}$ is tight if for all test functions g the following hold:*

1. *For all $t \in [0, T]$ and every $\epsilon > 0$, there is a compact $K \subseteq \mathbb{R}$ such that*

$$\sup_N P^N [\langle \pi_t^N, g \rangle \notin K] \leq \epsilon$$

2. *Aldous's criterion is satisfied:*

$$\limsup_{\gamma \to 0^+} \limsup_{N \to \infty} \sup_{\theta \leq \gamma} \sup_{\tau \leq T} P^N (|\langle \pi_{\tau+\theta}^N, g \rangle - \langle \pi_\tau^N, g \rangle| > \epsilon) = 0$$

where τ is a stopping time.

The above proposition can be used to prove tightness when the Skorokhod metric is used. Alternatively, one can sometimes show tightness in the uniform metric, which would imply tightness in the Skorokhod metric. The advantage of the latter approach is that it also implies that the limit points P^* are supported on continuous paths. We now proceed to show below how the two conditions above can be verified.

Condition 1 can be proven directly by bounding $\langle \pi_t^N, g\rangle$. If the number of particles per site is bounded, the result follows immediately. If not, the entropy inequality and the following bound help.

$$|\langle \pi_t^N, g\rangle| = \left| \frac{1}{N} \sum_{x \in \mathbb{Z}} g\left(\frac{x}{N}\right) \eta_t(x) \right| \leq \|g\| \frac{1}{N} \sum_{|x| \leq RN} \eta_t(x)$$

Here we used the fact that a continuous function g with compact support (supp $(g) \subseteq [-R, R]$) is bounded by $\|g\|$.

In proving condition 2, we will need to bound, in some sense, the term $\langle \pi_{\tau+\theta}^N, g\rangle - \langle \pi_\tau^N, g\rangle$. This is more easily done by rewriting $\langle \pi_t^N, g\rangle$ as

$$\langle \pi_t^N, g\rangle = \langle \pi_0^N, g\rangle + \int_0^t \gamma_N L\langle \pi_s^N, g\rangle ds + M_t^{N,g}$$

Note that the "missing" term $\partial_s\langle \pi_s^N, g\rangle$ equals zero because g doesn't depend on time. We can then "bound" $\langle \pi_{\tau+\theta}^N, g\rangle - \langle \pi_\tau^N, g\rangle$ by "bounding" both the difference between the integral term and the difference between the martingales. This is stated more precisely below.

$$\limsup_{\gamma \to 0^+} \limsup_{N \to \infty} \sup_{\theta \leq \gamma} \sup_{\tau \leq T} P^N(|\langle \pi_{\tau+\theta}^N, g\rangle - \langle \pi_\tau^N, g\rangle| > \epsilon)$$

$$\leq \limsup_{\gamma \to 0^+} \limsup_{N \to \infty} \sup_{\theta \leq \gamma} \sup_{\tau \leq T} P^N \left(\left| \int_\tau^{\tau+\theta} \gamma_N L\langle \pi_s^N, g\rangle ds \right| > \epsilon/2 \right)$$

$$+ \limsup_{\gamma \to 0^+} \limsup_{N \to \infty} \sup_{\theta \leq \gamma} \sup_{\tau \leq T} P^N \left(\left| M_{\tau+\theta}^{N,g} - M_\tau^{N,g} \right| > \epsilon/2 \right)$$

We can therefore prove the Aldous's criterion by showing that the last two terms equal zero. The first can often be shown to go to zero by directly calculating and then bounding the itegrand. The second cannot often be bounded directly because we don't know enough about $M_t^{N,g}$ without reverting back to terms of the form $\langle \pi_t^N, g\rangle$. Instead we relate $M_t^{N,g}$ to $\langle M_{\tau+\theta}^{N,g}\rangle$ using the Burkholder-Davis-Gundy inequality. Below are calculations similar to the Burkholder-Davis-Gundy inequality that show how $\langle M_{\tau+\theta}^{N,g}\rangle$ can be used to bound the second term.

$$P^N(|M_{\tau+\theta}^{N,g} - M_\tau^{N,g}| > \epsilon/2) \leq \frac{4}{\epsilon^2} E^N[(M_{\tau+\theta}^{N,g} - M_\tau^{N,g})^2]$$

by Chebyshev's inequality. The right side of the inequality can be multiplied and simplified by relying on the equality $E[M_{\tau+\theta}^{N,g} M_\tau^{N,g}] = E[(M_\tau^{N,g})^2]$, which holds because $M_t^{N,g}$ is a martingale. After simplification, the right side of the above inequality equals

$$\frac{4}{\epsilon^2} E^N[(M_{\tau+\theta}^{N,g})^2 - (M_\tau^{N,g})^2] = \frac{4}{\epsilon^2} E^N[\langle M^{N,g}\rangle_{\tau+\theta} - \langle M^{N,g}\rangle_\tau]$$

Therefore, it sufficient to prove that expected value of the difference in the quadratic variation goes to zero after taking the appropriate limits.

15 Substitution Lemma

We are almost ready to discuss the limits of weakly convergent subsequences of P^N. First we mention the following lemma that is used repeatedly throughout later proofs. The lemma is used to replace an old expression O_k by a new expression N_k in certain types of probability statements. I have dubbed this lemma the substitution lemma, but it does not have a formal name.

Substitution Lemma: Suppose $\lim_{k\to\infty} E[|D_k|] = 0$ where $D_k = O_k - N_k$.

For all $\epsilon_0 > 0$, if $\lim\limits_{k\to\infty} P(|O_k| > \epsilon_0) = 0$, then $\lim\limits_{k\to\infty} P(|N_k| > \epsilon_0) = 0$

Proof: $P(|N_k| > \epsilon_0) \leq P(|D_k| + |O_K| > \epsilon_0) \leq P(|D_k| > \epsilon_0/2) + P(|O_K| > \epsilon_0/2) \leq \frac{2}{\epsilon_0} E[|D_k|] + P(|O_K| > \epsilon_0/2)$ which goes to zero as k goes to infinity. QED

The substitution lemma has many variations that are used in practice but all are similarly easy to prove.

16 Previous Results and Current Work

Results have already been proven for nearest neighbor symmetric processes. In the case of simple exclusion, the resulting hydrodynamic equation is just the heat equation [5]. Rezakhanlou proved results for finite range asymmetric simple exclusion and zero range processes [7]. And Jara prove results for long range symmetric simple exclusion and zero range processes [4].

I have been working on the long range asymmetric simple exclusion model. Specifically, we are working with $p(d) = 1(d > 0)\frac{1}{d^{1+\alpha}}$ where $\alpha > 0$. This splits in to three cases: $\alpha < 1, \alpha = 1$, and $\alpha > 1$. The $\alpha = 1$ case is similar to the $\alpha > 1$ case, so I will use the $\alpha < 1$ case and $\alpha > 1$ case to illustrate the next steps in the proof of hydrodynamic limits. Note that when $\alpha < 1$ the appropriate time scaling is $\gamma_N = N^\alpha$, and when $\alpha > 1$ the appropriate time scale is just $\gamma_N = N$. When $\alpha = 1$, an extra $(\ln N)^{-1}$ factor is present.

17 Example Calculations

The $\alpha > 1$ case has added complication in the proof so we will illustrate the next steps with the $\alpha < 1$ case. Our goal will be to prove results similar to the following.

Theorem 2: *When $\alpha < 1$, every limit point P^* of $\{P^N\}_{N=1}^\infty$ is supported on absolutely continuous measures $\pi_t = \rho(t, u)du$ whose densities are weak solutions of the hydrodynamic equation*

$\partial_t \rho = \mathcal{L}(\rho)$ *where*

$$\mathcal{L}(\rho(u)) = \int_0^\infty \frac{\rho(u-v)(1-\rho(u)) - \rho(u)(1-\rho(u+v))}{v^{1+\alpha}} dv$$

The resulting hydrodynamic equation needs to be studied in detail to determine what additional conditions, if any, are necessary to show that the weak solution(s) are all in fact the unique strong solution of the hydrodynamic equation. If the equation does not have a unique strong solution, we may be able to show that the weak solution(s) are all, say, the unique entropy solution of the equation.

In any case, it is necessary to prove that every limit point P^* is also supported on absolutely continuous measures whose densities satisfy these other conditions. It will then follow that every limit point P^* is concentrated the absolutely continuous measures whose density is the unique strong solution of the hydrodynamic equation. As there is only one such probability measure, all the limits points P^* must in fact be equal. We will have thus proven that P^N converges weakly to the absolutely continuous $\pi_t = \rho(t,u)du$ measure whose densities $\rho(t,u)$ is the unique solution of some hydrodynamic equation satisfying the initial conditions.

We have not yet worked out what additional conditions besides the above theorem might need to be verified. For the time being, I will simply outline the proof of the above theorem. To avoid cumbersome notation I will abuse notation and write P^N instead of P^{N_k} to denote an arbitrary weakly convergent subsequence in the following sections.

Proof: First, we verify that

$$\limsup_{N\to\infty} P^N \left(\left| M_t^{N,g} \right| > \epsilon_0 \right) = 0$$

for all $\epsilon_0 > 0$. This is proven by relating $M_t^{N,g}$ to its quadratic variation $\langle M^{n,g} \rangle_t$ as shown earlier, and then bounding $\langle M^{n,g} \rangle_t$, the details of which are omitted. Therefore,

$$P^N \left(\left| \langle \pi_t^N, g_t \rangle - \langle \pi_0^N, g_0 \rangle - \int_0^t \partial_s \langle \pi_s^N, g_s \rangle ds - \int_0^t N^\alpha L \langle \pi_s^N, g_s \rangle ds \right| > \epsilon_0 \right)$$

goes to zero as N goes to infinity. Since g has compact support in time, we may choose t large enough so that g_t, and hence $\langle \pi_t^N, g_t \rangle$, is zero. For such t,

$$\limsup_{N\to\infty} P^N \left(\left| \langle \pi_0^N, g_0 \rangle + \int_0^t \partial_s \langle \pi_s^N, g_s \rangle ds + \int_0^t N^\alpha L \langle \pi_s^N, g_s \rangle ds \right| > \epsilon_0 \right) = 0$$

That step could be done later, but it saves space to do it already. Next, we repeatedly use the substitution lemma to replace $\int_0^t N^\alpha L \langle \pi_s^N, g_s \rangle ds$ by similar terms. Our goal will be to have

$$\langle \pi_0^N, g_0 \rangle + \int_0^t \partial_s \langle \pi_s^N, g_s \rangle ds + \int_0^t N^\alpha L \langle \pi_s^N, g_s \rangle ds$$

look like the weak formulation of a hydrodynamic equation. By direct calculation we find that

$$\int_0^t N^\alpha L\langle \pi_s^N, g_s\rangle ds = \int_0^t \frac{N^\alpha}{N} \sum_{x\in\mathbb{Z}} \sum_{d=1}^{\infty} \frac{1}{d^{1+\alpha}} \eta_s^d(x) \left[g_s\left(\frac{x+d}{N}\right) - g_s\left(\frac{x}{N}\right)\right] ds$$

where $\eta_s^d(x) = \eta_s(x)(1-\eta_s(x+d))$. We begin by using the substitution lemma to preform several technical replacements that prevent terms from blowing up later on. First we will limit the sum over d to the sum in which d is at least $\epsilon N + 1$ and at most DN where first N goes to infinity, then ϵ will go to zero, and finally D will go to infinity. To use the substitution lemma, we find a simple bound that relies on g being a smooth function with compact support.

At this step it is also common to try and replace the part with g by an operator acting on g. For example, in the $\alpha > 1$ case one can replace $\left(g_s\left(\frac{x+d}{N}\right) - g_s\left(\frac{x}{N}\right)\right) / \left(\frac{d}{N}\right)$ by $g'(x/N)$ at this step. Such a replacement is not possible here because the particle interactions are long range and it is not possible to bound d to be at most ϵN.

Instead we will continue to the three major replacements. Collectively these three replacements are often called the replacement lemma. The purpose of these replacements is to **close the equation**. That is, we want every term in the martingale $M_t^{N,g}$ to be written in terms of π^N. But after evaluating $N^\alpha L\langle \pi_s^N, g_s\rangle$ we obtained an $\eta_s^d(x)$ instead. Specifically, we want to replace $\eta_s^d(x)$ by a function of π_s^N. Only rarely will the equations be closed already.

The first of the three major replacement involves substituting the part with η, in this case $\eta_s^d(x)$, with it average over l-blocks: $\frac{1}{2l+1}\sum_{|y|\le l} \eta_s^d(x+y)$ where l will go to infinity after N goes to infinity and before ϵ and D go to there respective limits. This substitution relies on a discrete integration by parts as well as the smoothness and compact support of g.

An l-block consists of the microscopic locations $[x-l, x+l]$, all of which will correspond to a single macroscopic point as N goes to infinity. For that reason, an l-block is considered microscopically large. The average over l-blocks serves to average out the variation in the η's. Specifically, at microscopic points corresponding to similar macroscopic location, the $\eta_s(x)$ should sort of be i.i.d. random variables, and by taking the average over l-blocks we will eventually be able to replace η_s by the density ρ using something like a law of large numbers.

After all of the above substitutions we have $\limsup_{D\to\infty} \limsup_{\epsilon\to 0^+} \limsup_{l\to\infty} \limsup_{N\to\infty}$

$$P^N\Bigg(\Bigg|\langle \pi_0^N, g_0\rangle + \int_0^t \partial_s\langle \pi_s^N, g_s\rangle ds$$

$$+ \int_0^t \frac{N^\alpha}{N} \sum_{x\in\mathbb{Z}} \sum_{d=\epsilon N+1}^{DN} \frac{1}{d^{1+\alpha}} \frac{1}{2l+1} \sum_{|y|\le l} \eta_s^d(x+y) \left[g_s\left(\tfrac{x+d}{N}\right) - g_s\left(\tfrac{x}{N}\right)\right] ds\Bigg| > \epsilon_0\Bigg) = 0$$

Next, we preform the second major replacement. If we view the superscript d as a function, then $\frac{1}{2l+1}\sum_{|y|\le l}\eta_s^d(x+y)$ is an average of a function of η. Our goal is to replace that average of a function of η with the function of the average. That is, we want to replace

$$\frac{1}{2l+1}\sum_{|y|\le l}\eta_s^d(x+y) \quad \text{by} \quad (\eta_s^l(x))^d = \eta_s^l(x)(1-\eta_s^l(x+d))$$

where $\eta_s^l(x) = \frac{1}{2l+1}\sum_{|y|\le l}\eta_s(x+y)$. The 1-block estimate in conjunction with the substitution lemma allow for such a replacement. The 1-block estimate is both important and rather technical so it will discuss in its own section.

After the 1-block replacement, we have $\limsup\limits_{D\to\infty}\limsup\limits_{\epsilon\to 0^+}\limsup\limits_{l\to\infty}\limsup\limits_{N\to\infty}$

$$P^N\Big(\Big|\langle\pi_0^N,g_0\rangle+\int_0^t\partial_s\langle\pi_s^N,g_s\rangle ds$$

$$+\int_0^t\frac{N^\alpha}{N}\sum_{x\in\mathbb{Z}}\sum_{d=\epsilon N+1}^{DN}\frac{1}{d^{1+\alpha}}(\eta_s^l(x))^d\left[g_s\left(\tfrac{x+d}{N}\right)-g_s\left(\tfrac{x}{N}\right)\right]ds\Big|>\epsilon_0\Big)=0$$

The third and last major replacement is the 2-blocks replacement. We will work to replace $\eta_s^l(x)$ by $\eta_s^{\epsilon' N}(x)$ where ϵ' will go to zero after N goes to infinity but before l does. By the substitution lemma, we just need to show that $\limsup\limits_{D\to\infty}\limsup\limits_{\epsilon\to 0^+}\limsup\limits_{l\to\infty}\limsup\limits_{\epsilon'\to 0^+}\limsup\limits_{N\to\infty}$

$$E^N\left[\int_0^t\frac{1}{N^2}\sum_{x\in\mathbb{Z}}\sum_{d=\epsilon N+1}^{DN}\left|(\eta_s^l(x))^d-(\eta_s^{\epsilon' N}(x))^d\right|\left|\frac{g_s\left(\frac{x}{N}+\frac{d}{N}\right)-g_s\left(\frac{x}{N}\right)}{(\frac{d}{N})^{1+\alpha}}\right|ds\right]=0$$

By using that g is smooth and has compact support and expanding the expression $(\eta_s^l(x))^d-(\eta_s^{\epsilon' N}(x))^d$, it can be shown that it is sufficient to prove that $\limsup\limits_{D\to\infty}\limsup\limits_{l\to\infty}\limsup\limits_{\epsilon'\to 0^+}\limsup\limits_{N\to\infty}$

$$E^N\left[\int_0^t\frac{1}{N}\sum_{|x|\le(R+D)N}\left|\eta_s^l(x)-\eta_s^{\epsilon' N}(x)\right|ds\right]=0$$

where R is a constant chosen so that supp $g\subseteq[-R,R]$. The above estimate is called the 2-blocks estimate and will also be discussed in its own section.

An $\epsilon' N$-block consists of the microscopic locations $[x-\epsilon' N, x+\epsilon' N]$ which corresponds to a macroscopic interval $[u-\epsilon', u+\epsilon']$ as N goes to infinity. For that reason, an $\epsilon' N$-block is considered macroscopically small. It is that macroscopic feature that allows the equation to be closed.

Specifically, we can write $\eta_s^{\epsilon' N}(x)$ in terms of π_s^N as follows:

$$\eta_s^{\epsilon' N}(x)=\frac{2\epsilon' N}{2\epsilon' N+1}\left(\pi_s^N*\iota_{\epsilon'}\right)\left(\frac{x}{N}\right)$$

where $\iota_{\epsilon'} = \frac{1}{2\epsilon'}1_{[-\epsilon',\epsilon']}$. We will therefore be able to replace $\eta_s^{\epsilon' N}(x)$ by $(\pi_s^N * \iota_{\epsilon'})(\frac{x}{N})$ giving that $\limsup_{D\to\infty} \limsup_{\epsilon\to 0^+} \limsup_{\epsilon'\to 0^+} \limsup_{N\to\infty}$

$$P^N\Big(\Big|\langle \pi_0^N, g_0\rangle + \int_0^t \partial_s \langle \pi_s^N, g_s\rangle ds$$

$$+ \int_0^t \frac{1}{N}\sum_{x\in\mathbb{Z}} \frac{1}{N}\sum_{d=\epsilon N+1}^{DN} ((\pi_s^N * \iota_{\epsilon'})(\tfrac{x}{N}))^d \left[\frac{g_s(\frac{x}{N}+\frac{d}{N}) - g_s(\frac{x}{N})}{(\frac{d}{N})^{1+\alpha}}\right] ds\Big| > \epsilon_0\Big) = 0$$

where $((\pi_s^N * \iota_{\epsilon'})(\frac{x}{N}))^d = (\pi_s^N * \iota_{\epsilon'})(\frac{x}{N})(1 - (\pi_s^N * \iota_{\epsilon'})(\frac{x+d}{N}))$.

Now that the equation is closed we can use the substitution lemma to preform the following standard replacements to obtain the weak formulation of the hydrodynamic equation. First, we replace the two Riemann sums with integrals. The limits of one of these integrals will be ϵ and D. We will be able to replace those limits by 0 and ∞, respectively. After these replacements, we have $\limsup_{\epsilon'\to 0^+} \limsup_{N\to\infty}$

$$P^N\Big(\Big|\langle \pi_0^N, g_0\rangle + \int_0^t \partial_s \langle \pi_s^N, g_s\rangle ds + \int_0^t \int_{\mathbb{R}} \int_0^\infty (\pi_s^N * \iota_{\epsilon'}(u))(1 - (\pi_s^N * \iota_{\epsilon'}(u+v))$$
$$\left(\frac{g_s(u+v) - g_s(u)}{v^{1+\alpha}}\right) dv du ds\Big| > \epsilon_0\Big) = 0$$

Lastly, we need to remove the dependence on ϵ'. To do so, we first need take the limit as N goes to infinity. Recall that we used N instead of N_k to denote a subsequence for which P^N converge. Therefore, $\limsup_{\epsilon'\to 0^+}$

$$P^*\Big(\Big|\langle \pi_0, g_0\rangle + \int_0^t \partial_s \langle \pi_s, g_s\rangle ds + \int_0^t \int_{\mathbb{R}} \int_0^\infty (\pi_s * \iota_{\epsilon'}(u))(1 - (\pi_s * \iota_{\epsilon'}(u+v))$$
$$\left(\frac{g_s(u+v) - g_s(u)}{v^{1+\alpha}}\right) dv du ds\Big| > \epsilon_0\Big) = 0$$

for every limit point P^*. We have skipped proving a separate lemma that P^* is supported on measures π_s that are absolutely continuous with respect to Lebesgue measure. Using that fact, we may write $\pi_s = \rho_s(u)du$ for some function $\rho_s(u) = \rho(s,u)$. We now use the substitution lemma one last time to replace $(\pi_s * \iota_{\epsilon'})(u) = \frac{1}{2\epsilon'}\int_{-\epsilon'}^{\epsilon'} \rho_s(u+v)dv$ by $\rho_s(u)$. The criterion for the substitution lemma follows almost immediately from the Fatou-Lebesgue convergence theorem and the following fact: P^*-a.s.

$$\limsup_{\epsilon'\to 0^+} \left|\frac{1}{2\epsilon'}\int_{-\epsilon'}^{\epsilon'} \rho_s(u+v)dv - \rho_s(u)\right| = 0 \quad u\text{-a.s.}$$

Therefore,

$$P^*\Big(\Big|\int_{\mathbb{R}}\rho_0(u)g_0(u)du+\int_0^\infty\int_{\mathbb{R}}\rho_s(u)\partial_s g_s(u)duds$$

$$+\int_0^\infty\int_{\mathbb{R}}\int_0^\infty\rho_s(u)(1-\rho_s(u+v))\left(\tfrac{g_s(u+v)-g_s(u)}{v^{1+\alpha}}\right)dvduds\Big|>\epsilon_0\Big)=0$$

where we used that g has compact support in time to replace t by ∞.

It then follows that for every limit point P^*

$$\int_{\mathbb{R}}\rho_0(u)g_0(u)du+\int_0^\infty\int_{\mathbb{R}}\rho_s(u)\partial_s g_s(u)duds$$

$$+\int_0^\infty\int_{\mathbb{R}}\int_0^\infty\rho_s(u)(1-\rho_s(u+v))\left(\tfrac{g_s(u+v)-g_s(u)}{v^{1+\alpha}}\right)dvduds=0$$

with probability 1. The above is the weak formulation of the hydrodynamic equation

$$\partial_t\rho=\mathcal{L}(\rho)$$

That is, every limit point P^* is supported on absolutely continuous measures, $\pi_s=\rho_s(u)du$, whose densities $\rho_s(u)$ are weak solutions of the above equation. QED

[Note that we haven't actually proved that the densities $\rho(t,u)$ are weak solutions of the above hydrodynamic equation, because we only integrated with respected to one test function g. The result will follow, however, by introducing a supremum over all test functions g satisfying common bounds, and then taking a union over a countable number of such bounds. For example, we might introduce a supremum over all g with $|g|<m$ throughout the proof, and at the end take a union over all $m\in\mathbb{N}$.

Alternatively, we can take a union over all g in a countable dense subset of test functions. It then follows that the weak formulation holds for all g with probability 1.]

18 Dirichlet Form

Before proving 1-block and 2-block estimates, we discuss the Dirichlet form and related results, which are essential for the standard proofs of the 1-block and 2-blocks estimates.

Recall that μ^N is the initial probability measure on configurations when space is scaled by a factor of $1/N$. Let P_t be the semigroup associated to the process with the appropriate time scaling. Then $\mu_t^N=\mu^N P_t$ is the probability measure on configurations for the process at time t. Let ν_λ be the translation invariant equilibrium product measure with density λ.

We define the entropy of μ_t^N with respect to ν_λ as follows:

$$H(\mu_t^N \mid \nu_\lambda) = \int f_t^N \ln(f_t^N) d\nu_\lambda$$

where $f_t^N = d\mu_t^N / d\nu_\lambda$. If μ_t^N is not absolutely continuous with respect to ν_λ and f_t^N cannot be defined, then the entropy is taken to be infinite.

The proofs of the 1-block estimate and 2-block estimates rely on assumptions about the entropy. It is common to require that $H(\mu_0^N \mid \nu_\lambda) \leq CN^n$ where C is a constant and n is the number of spacial dimensions. This bound holds if the process is defined on a torus, and it holds on $\mathbb{R}$ if ρ_0 equals a positive constant outside a compact set. The entropy can be shown to decrease in time. Therefore, $H(\mu_t^N \mid \nu_\lambda)$ will be finite for all t, and the above formula for entropy may be used.

The density f_t^N satisfies the forward Kolmogorov equation

$$\frac{\partial f_t^N}{\partial t} = \gamma_N L^* f_t^N$$

where L^* is the adjoint of L. This equation along with other manipulations allows us to bound the derivative of$H(\mu_t^N \mid \nu_\lambda)$ with respect to time as follows:

$$\frac{\partial H(\mu_t^N \mid \nu_\lambda)}{\partial t} \leq -2\gamma_N D(f_t^N)$$

where $D(f) = -\int \sqrt{f} L^{sym} \sqrt{f} d\nu_\lambda$ is the Dirichlet form, and $L^{sym} = (L + L^*)/2$ is the symmetric part of L. By integrating both sides over time, we obtain the following bound

$$H(\mu_t^N \mid \nu_\lambda) + 2\gamma_N \int_0^t D(f_s^N) ds \leq H(\mu_0^N \mid \nu_\lambda) \leq CN^n$$

Since the entropy is always positive, we can remove the entropy term on the left side and then divide by $2\gamma_N t$ giving us

$$\frac{1}{t} \int_0^t D(f_s^N) ds \leq \frac{CN^n}{2\gamma_N t}$$

Finally, the Dirichlet form is a convex function, so

$$D\left(\frac{1}{t} \int_0^t f_s^N ds\right) \leq \frac{CN^n}{2\gamma_N t}$$

It is useful to have a bound on the Dirichlet form of $\bar{f}_t^N = \frac{1}{t} \int_0^t f_s^N ds$ because expectations can be written in terms of it. Specifically,

$$E^N\left[\int_0^t h(\eta_s)ds\right] = \int\int_0^t h(\eta_s)dsdP^N = \int_0^t\int h(\eta)\mu_s^N(d\eta)ds$$
$$= \int_0^t\int h(\eta)f_s^N(d\eta)d\nu_\lambda ds = \int h(\eta)\int_0^t f_s^N(d\eta)dsd\nu_\lambda = t\int h(\eta)\bar{f}_t^N(\eta)\nu_\lambda(d\eta)$$

The method of proof for the 1-block and 2-blocks estimates involves rewriting the relevant expected values as integrals involving $\bar{f}_t^N$. Since we don't know much about $\bar{f}_t^N$, those integrals are then bounded by replacing $\bar{f}_t^N$ with an arbitrary density f and taking a supremum over all f that have the same bound on their Dirichlet form. For reasons that will be clear later, we will then manipulate f and introduce a new Dirichlet form that will go to zero. Below we show how to construct such a Dirichlet form.

First we define the pieces of the generator and Dirichlet form corresponding to jumps from x to y. They will be denoted $L_{x,y}$ and $D^{x,y}(f)$, respectively. Using the notation from section 4,

$$L_{x,y}f(\eta) = p(y-x)\lambda_{\eta,x,y}(f(\eta^{x,y}) - f(\eta)), \quad D^{x,y}(f) = -\int \sqrt{f}L_{x,y}^{sym}\sqrt{f}d\nu_\lambda$$

$D^{x,y}(f)$ can also be written as

$$D^{x,y}(f) = \frac{1}{2}\int p(y-x)\lambda_{\eta,x,y}(\sqrt{f(\eta^{x,y})} - \sqrt{f(\eta)})^2 d\nu_\lambda$$

Roughly speaking, $D^{x,y}(f)$ measures how much $f(\eta)$ can vary as one particle is moved from x to y or vice versa. In particular, if $D^{x,y}(f) = 0$ then $f(\eta) = f(\eta^{x,y})$ assuming $p(y-x)\lambda_{\eta,x,y} \neq 0$. The full Dirichlet form $D(f) = \sum_{x,y} D^{x,y}(f)$. Our new Dirichlet form will be a sum of just some of the $D^{x,y}(f)$. If our new Dirichlet form is zero, then f will be constant on certain regions of its domain.

The constructions and bounds for the new Dirichlet forms in the $\alpha < 1$ case have extra technicalities. Therefore, I will present the construction and bound of the Dirichlet form for the 1-block estimate when $\alpha > 1$. In the next section, I will return to the 1-block estimate for the $\alpha < 1$ case.

First, let
$$\bar{f}_N(\eta) = \frac{1}{2RN+1}\sum_{|x|\le RN}\tau_x f(\eta)$$
where τ_x is a shift operator. [$\tau_x f(\eta) = f(\tau_x\eta)$ and $\tau_x\eta(y) = \eta(x+y)$.] Let ξ be a configuration supported of $[-l, l+d]$; let $\nu_\lambda^{l,d}(d\xi)$ be the measure ν_λ restricted to such configurations; and let $\bar{f}_{l,d}(\xi)$ be the conditional expectation of $\bar{f}_N(\eta)$ given such a configuration. Define

$$D_{l,d}(f) = \sum_{y\in[-l,l+d]}\bar{D}^{-l,y}(f)$$

where $\bar{D}^{x,y}(f)$ is like $D^{x,y}(f)$ except with integration with respect to $\nu_\lambda^{l,d}$ instead of ν_λ. By convexity, we have the relation $\bar{D}^{x,y}(\bar{f}_{l,d}) \leq D^{x,y}(\bar{f}_N)$.

We now prove the following bound on $D_{l,d}(\bar{f}_{l,d})$ assuming that $D(f) \leq \frac{C_0 N}{N}$. Here $C_0 = C/(2t)$; $N = N^1$ is on the numerator because we are in one dimension; and $\gamma_N = N$ when $\alpha > 1$ accounts for the N on the denominator.

Lemma 3: *If $D(f) \leq C_0$, then $D_{l,d}(\bar{f}_{l,d}) \leq \frac{C_1}{N}$ where $C_1 = \frac{C_0}{2R}$*

Proof:

$$D_{l,d}(\bar{f}_{l,d}) = \sum_{y \in [-l,l+d]} \bar{D}^{-l,y}(\bar{f}_{l,d}) \leq \sum_{y \in [-l,l+d]} D^{-l,y}(\bar{f}_N)$$

Since $\bar{f}_N$ is an average of the $\tau_x f$, the above is bounded by

$$\sum_{y \in [-l,l+d]} \frac{1}{2RN+1} \sum_{|x| \leq RN} D^{-l,y}(\tau_x f)$$

by the convexity of the Dirichlet form. $D^{-l,y}(\tau_x f) = D^{-l+x,y+x}(f)$. So we also have the bound

$$\frac{1}{2RN+1} \sum_{\substack{y \in [-l,l+d] \\ |x| \leq RN}} D^{-l+x,y+x}(f)$$

Note that we are not summing over all pieces of the Dirichlet form and no piece is summed over twice. Therefore, the sum is bounded above by the full Dirichlet form, and we have the bound

$$\frac{1}{2RN+1} D(f) \leq \frac{1}{2RN+1} \frac{C_0 N}{N} \leq \frac{C_1}{N}$$

QED

The same proof will also work in higher dimensions ($n > 1$), since the factor of N^n in the numerator will be offset by $(2RN+1)^n$ in the denominator.

Using the above bound, we will be able to have $D_{l,d}(f) = 0$ after N goes to infinity. Specifically, suppose that $D_{l,d}(f_N) \leq C_1/N$. By the lower semi-continuity of the Dirichlet form, $D_{l,d}(f) = 0$ where $f = \limsup_{N \to \infty} f_N$.

If $D_{l,d}(f) = 0$, then in particular, $\bar{D}^{-l,y}(f)$ and $\bar{D}^{-l,z}(f)$ must both be zero for any $y, z \in [-l, l+d]$. Therefore, a particle can be moved from y to $-l$ and then $-l$ to z without changing the value of f. This moving of one particle from y to z can be repeated to show that $f = f(\xi)$ is constant along the hyperplanes $H_j = \{\xi : \sum_{-l \leq y \leq l+d} \xi(y) = j\}$, where $j = 0, 1, ..., 2l + d + 1$. It is this constancy along hyperplanes that is used in the standard proofs of the 1-block and 2-block estimates.

19 1-block estimate ($\alpha < 1$)

The following estimate in conjunction with the substitution lemma allows us to preform to 1-block replacement.

1-block estimate:

$$\limsup_{\epsilon\to 0^+}\limsup_{l\to\infty}\limsup_{N\to\infty}$$

$$E^N\left[\int_0^t \frac{N^\alpha}{N}\sum_{d=\epsilon N+1}^{\infty}\frac{1}{d^{1+\alpha}}\left|\sum_{x\in\mathbb{Z}} H_{x,d,l}(\eta_s)\left[g_s\left(\frac{x+d}{N}\right)-g_s\left(\frac{x}{N}\right)\right]\right| ds\right]=0$$

where $H_{x,d,l}(\eta)=\frac{1}{2l+1}\sum_{|y|\le l}\eta^d(x+y)-\left(\eta^l(x)\right)^d$

Proof:

Using discrete integration by parts and some other manipulations it will be enough to show that

$$E^N\left[\int_0^t \sum_{d=\epsilon N+1}^{\infty}\frac{N^\alpha}{d^{1+\alpha}}\frac{1}{N}\sum_{|x|\le RN}|H_{x,d,l}(\eta_s)|\,ds\right]$$

goes to zero. The expected value can be written with the density $\bar{f}_t^N$ as follows

$$t\int\sum_{d=\epsilon N+1}^{\infty}\frac{N^\alpha}{d^{1+\alpha}}\frac{1}{N}\sum_{|x|\le RN}|H_{x,d,l}(\eta)|\,\bar{f}_t^N(\eta)\nu_\lambda(d\eta)$$

It will be enough to show that

$$\sup_{D(f)\le\frac{C_0 N}{N^\alpha}}\int\sum_{d=\epsilon N+1}^{\infty}\frac{N^\alpha}{d^{1+\alpha}}\frac{1}{N}\sum_{|x|\le RN}|H_{x,d,l}(\eta)|\,f(\eta)\nu_\lambda(d\eta)$$

goes to zero. Since ν_λ is a translation invariant measure, we may replace $H_{x,d,l}(\eta)\cdot f(\eta)=\tau_x H_{0,d,l}(\eta)\cdot f(\eta)=\tau_x(H_{0,d,l}(\eta)\cdot\tau_{-x}f(\eta))$ by $H_{0,d,l}(\eta)\cdot\tau_{-x}f(\eta)$. $H_{0,d,l}(\eta)$ can then be factored out of the sum over x to give

$$\sup_{D(f)\le\frac{C_0 N}{N^\alpha}}\int\sum_{d=\epsilon N+1}^{\infty}\frac{N^\alpha}{d^{1+\alpha}}\frac{2RN+1}{N}|H_{0,d,l}(\eta)|\,\bar{f}_N(\eta)\nu_\lambda(d\eta)$$

where $\bar{f}_N(\eta)$ is defined as in the previous section. The $(2RN+1)/N$ can be bounded above by $2R+1$, and hence, ignored.

At this stage, there is a trick that is not part of the standard 1-block proof. The trick involves rewriting $H_{0,d,l}(\eta)$ as follows:

$$
\begin{aligned}
H_{0,d,l}(\eta) &= \tfrac{1}{2l+1} \sum_{|y|\le l} \left(\eta^d(y) - \left(\eta^l(0)\right)^d\right) \\
&= \tfrac{1}{2l+1} \sum_{|y|\le l} \left(\eta(y)(1-\eta(y+d)) - \eta^l(0)(1-\eta^l(d))\right) \\
&= -\tfrac{1}{2l+1} \sum_{|y|\le l} \left(\eta(y)\eta(y+d) - \eta^l(0)\eta^l(d)\right) \\
&= -\tfrac{1}{2l+1} \sum_{|y|\le l} \left(\eta(y) - \eta^l(0)\right)\left(\eta(y+d) - \eta^l(d)\right) \\
&= -\tfrac{1}{2l+1} \sum_{|y|\le l} \left(\xi(y) - \xi^l(0)\right)\left(\zeta(y) - \zeta^l(0)\right)
\end{aligned}
$$

where ξ and ζ are configurations on $[-l,l]$ that equal η and $\tau_d\eta$, respectively, on $[-l,l]$. We will denote the last line by $\mathcal{H}_l(\xi,\zeta)$. Let $\nu_\lambda^2(d\xi,d\zeta)$ be the measure on pairs of configurations (ξ,ζ) related to the measure ν_λ, and let $\bar{f}_{l,d}(\xi,\zeta)$ be the conditional expectation of $\bar{f}_N(\eta)$ given configurations η that equal ξ on $[-l,l]$ and ζ on $[-l+d,l+d]$. We then have

$$
\sup_{D(f)\le \frac{C_0 N}{N^\alpha}} \int \sum_{d=\epsilon N+1}^{\infty} \frac{N^\alpha}{d^{1+\alpha}} \left|\mathcal{H}_l(\xi,\zeta)\right| \bar{f}_{l,d}(\xi,\zeta)\nu_\lambda^2(d\xi,d\zeta)
$$

Since $\sum_{d=\epsilon N+1}^{\infty} \frac{N^\alpha}{d^{1+\alpha}} \le \frac{1}{\alpha\epsilon^\alpha}$, the previous line is bounded above by

$$
\frac{1}{\alpha\epsilon^\alpha} \sup_{D(f)\le \frac{C_0 N}{N^\alpha}} \int \left|\mathcal{H}_l(\xi,\zeta)\right| \bar{\bar{f}}_{N,l,\epsilon}(\xi,\zeta)\nu_\lambda^2(d\xi,d\zeta)
$$

where we have define

$$
\bar{\bar{f}}_{N,l,\epsilon} = \left(\sum_{d=\epsilon N+1}^{\infty} \frac{N^\alpha}{d^{1+\alpha}} \bar{f}_{l,d}\right) \Big/ \sum_{d=\epsilon N+1}^{\infty} \frac{N^\alpha}{d^{1+\alpha}}
$$

After these tricks, the standard proof of the 1-block involves introducing a Dirichlet form D_l^2 such that $D_l^2(\bar{\bar{f}}_{N,l,\epsilon})$ is bounded by a quantity that goes to zero as N goes to infinity. We will skip the details of this Dirichlet form and its bound. But it will be enough to show that the following goes to zero as N and then l go to infinity.

$$
\sup_{D_l^2(f)\le \frac{C_2}{N^\alpha}} \int \left|\mathcal{H}_l(\xi,\zeta)\right| f(\xi,\zeta)\nu_\lambda^2(d\xi,d\zeta)
$$

f is a function of configurations (ξ,ζ) each of which is defined on a finite number of sites and has a bounded number of particles per site. (For simple exclusion, each site has at most one particle.) Therefore, the domain of f is finite. It follows that the set of all f satisfying $D_l^2(f) \le C_2/N^\alpha$ is a compact set, and that the integral integral above, which is in fact a sum, is a continuous function of f. Therefore, the supremum is attained by some function $f_{N,l}$. If we then let N go to infinity, $f_{N,l}$ will approach a function whose Dirichlet form is zero. Therefore, as N goes to infinity the above expression is bounded by the supremum over f with $D_l^2(f) = 0$:

$$\sup_{D_l^2(f)=0} \int |\mathcal{H}_l(\xi,\zeta)|\, f(\xi,\zeta)\nu_\lambda^2(d\xi, d\zeta)$$

The Dirichlet form D_l^2 was chosen so that $D_l^2(f) = 0$ implies that f is constant ($f = C_{j_1,j_2}$) along each of the hyperplanes

$$H^2_{j_1,j_2} = \left\{ (\xi,\zeta) : \sum_{|y|\le l} \xi(y) = j_1, \sum_{|y|\le l} \zeta(y) = j_2 \right\}$$

where $j_1, j_2 = 0, 1, ..., 2l + 1$. Because f is a probability density, we know that the constants C_{j_1,j_2} are non-negative and sum to 1. Therefore, the last expression can be written as a supremum over all possible weighted averages of the integral over hyperplanes:

$$\sup_{\sum C_{j_1,j_2}=1} \sum_{j_1,j_2=0}^{2l+1} C_{j_1,j_2} \int_{H^2_{j_1,j_2}} |\mathcal{H}_l(\xi,\zeta)|\, \nu_\lambda^2(d\xi, d\zeta)$$

This can be bounded by the supremum of the integral over possible hyperplanes. We will write those integrals in terms of the measure ν^{l,j_1,j_2}, which is the measure ν_λ^2 restricted to the hyperplane $H^2_{j_1,j_2}$. Because of the restriction, ν^{l,j_1,j_2} does not depend on λ. We then need to show that

$$\sup_{0\le j_1.j_2\le 2l+1} \int |\mathcal{H}_l(\xi,\zeta)|\, \nu^{l,j_1,j_2}(d\xi, d\zeta)$$

goes to zero as l goes to infinity.

If the integrand did not depend on the value of ξ and ζ on l-blocks, we could let l go to infinity while $j_i/(2l + 1)$ goes to ρ_i. The measure $\nu^{l,j_1,j_2}(d\xi, d\zeta)$ would then be equivalent to the measure $\nu_{\rho_1}(d\xi) \times \nu_{\rho_2}(d\zeta)$ by what's called the equivalence on ensembles.

We can have the ξ and ζ in the integrand depend only on a k-block as follows. We could have chosen l such that an l-block could be partitioned into q disjoint k-blocks. That is, $2l + 1 = q(2k + 1)$, and the limit as l goes to infinity is the same as the limit as q and then k go to infinity. If we let $B_1, .., B_q$ be the q k-blocks, then

$$\mathcal{H}_l(\xi,\zeta) = \frac{1}{q} \sum_{i=1}^{q} \frac{1}{2k+1} \sum_{y\in B_i} \left(\xi(y) - \xi^l(0)\right) \left(\zeta(y) - \zeta^l(0)\right)$$

Under the measure ν^{l,j_1,j_2}, the distribution of $\frac{1}{2k+1}\sum_{y\in B_i}(\xi(y)-\xi^l(0))(\zeta(y)-\zeta^l(0))$ doesn't depend on i. Therefore, we can bound the previous supremum by

$$\sup_{0\le j_1, j_2\le 2l+1}\int\left|\frac{1}{2k+1}\sum_{|y|\le k}\left(\xi(y)-\frac{j_1}{2l+1}\right)\left(\zeta(y)-\frac{j_1}{2l+1}\right)\right|\nu^{l,j_1,j_2}(d\xi,d\zeta)$$

where we used the fact that $\xi^l(0)$ and $\zeta^l(0)$ equal $\frac{j_1}{2l+1}$ and $\frac{j_1}{2l+1}$, respectively. We can then take the limit as l goes to infinity to get

$$\sup_{0\le\rho_1,\rho_2\le 1}\int\left|\frac{1}{2k+1}\sum_{|y|\in k}(\xi(y)-\rho_1)(\zeta(y)-\rho_2)\right|(\nu_{\rho_1}\times\nu_{\rho_2})(d\xi,d\zeta)$$

And this goes to zero uniformly in ρ as k goes to infinity by the law of large numbers, since $E_{\nu_{\rho_1}\times\nu_{\rho_2}}[(\xi(y)-\rho_1)(\zeta(y)-\rho_2)]=\mathrm{Cov}_{\nu_{\rho_1}\times\nu_{\rho_2}}[\xi(y)\zeta(y)]=0$.

The last step is rather interesting. Normally, the 1-block estimate ends by showing that an average of a function of the $\xi(y)$ converges to its expected value. It's rather curious that for the $\alpha<1$ case we end up with term that looks like a covariance. QED

Lastly, we note one additional technicality that can arise when proving the 1-block estimate. It was necessary in the proof that the set of densities f eventually had a finite domain. For this, it's necessary that the number of particles per site be bounded. While this holds trivially for the simple exclusion process, an extra step is needed for the zero range process to introduce an indicator function that limits the number of particles per site.

20 2-blocks estimate ($\alpha<1$)

Next we proceed to explain the 2-block estimate. The proof is similar to the 1-block estimate, so we will give only a brief overview of the key differences.

2-blocks estimate:

$\limsup_{l\to\infty}\limsup_{\epsilon'\to 0^+}\limsup_{N\to\infty}$

$$E^N\left[\int_0^t\frac{1}{N}\sum_{|x|\le(R+D)N}\left|\eta_s^l(x)-\eta_s^{\epsilon'N}(x)\right|ds\right]=0$$

Proof: The key step in the 2-blocks estimate is to replace $\eta_s^{\epsilon'N}(x)$ by an average, more or less, of the η_s^l over an $\epsilon'N$-block. Specifically, we will use a version of the substitution lemma to replace $\eta_s^{\epsilon'N}(x)$ by

$$\frac{1}{2\epsilon'N+1}\sum_{2l<|y|\le\epsilon'N}\eta_s^l(x+y)$$

After some manipulations, it will be enough to show that

$$\limsup_{l\to\infty}\limsup_{\epsilon'\to 0^+}\limsup_{N\to\infty}$$

$$\sup_{2l<|y|\le\epsilon' N} E^N\left[\int_0^t \frac{1}{N}\sum_{|x|\le RN}\left|\eta_s^l(x)-\eta_s^l(x+y)\right| ds\right]=0$$

Written this way, it is clear why we call the result the 2-blocks estimate; we are essentially showing that η_s averaged over two different l-blocks must be similar. It is important that $|y|>2l$ so that the two l-blocks will be disjoint.

From here, the proof of the 2-blocks estimates proceeds in the same way as the 1-block estimate. There are only two main differences. First, a different Dirichlet form, D_l^*, is needed. In particle, one of the terms in the Dirichlet form must correspond to a bond between the two l-blocks. Such a bond may not naturally exist for example if $p(d)$ has finite range. Instead, it will need to be introduced with some trickery. Finding an appropriate term for this bond an bounding it appropriately is the most difficult part of the 2-blocks estimate. Second, the supremum over y will go away once the new Dirichlet form is introduced.

We will eventually end with needing to show the following:

$$\limsup_{l\to\infty}\sup_{D_l^*(f)=0}\int\left|\xi_1^l(0)-\xi_2^l(0)\right| f(\xi_1,\xi_2)\nu_\lambda^{2,l}(d\xi_1,d\xi_2)=0$$

where thanks to the added term in the Dirichlet form, $D_l^*(f)=0$ implies that f is constant ($f=C_j$) along each of the hyperplanes $H_j^2=\{(\xi_1,\xi_2):\sum_{|y|\le l}(\xi_1(y)+\xi_2(y))=j\}$, where $j=0,1,...,4l+2$.

The constancy on hyperplanes will lead us to prove that

$$\limsup_{l\to\infty}\sup_{0\le j\le 4l+2}E_{\nu^{2,l,j}}\left|\xi_1^l(0)-\xi_2^l(0)\right|=0$$

where $\nu^{2,l,j}$ is the measure on configurations (ξ_1,ξ_2) that randomly distribute j particles between the two l-blocks. We do not need to worry about the equivalence of ensembles here and can simply replace both $\xi_1^l(0)$ and $\xi_2^l(0)$ by $\frac{j}{4l+2}$ as l goes to infinity. This follows because $E_{\nu^{2,l,j}}(\xi_i^l(0))=\frac{j}{4l+2}$ and $\mathrm{Var}_{\nu^{2,l,j}}(\xi^l(0))$ goes to zero uniformly in j as l goes to infinity. The 2-block estimate is then proved. QED

21 Young Measures

Until now, we have discussed the details of the simple exclusion process with $p(d)=1(d>0)\frac{1}{d^{1+\alpha}}$ and $\alpha<1$. We now discuss some of the complications that arise in the $\alpha>1$ case. When $\alpha>1$, the expected jumps size $\sum_d dp(d)$ is finite. The proof, as a result, is similar to Rezakhanlou's proof when p is asymmetric and has finite range.

When p is asymmetric and the expected jump size is finite, it has in general not been possible to prove the 2-blocks estimate. (Rezakhanlou was able to prove a 2-blocks estimate using coupling and a nonstandard proof when the process has nearest neighbor jumps.) Without a 2-blocks estimate, Young measures can be used to circumnavigate the issue.

We define the Young measures as

$$\pi_t^{N,l}(du, d\lambda) = \frac{1}{N} \sum_{x \in \mathbb{Z}} \delta_{\frac{x}{N}}(du) \delta_{\eta_t^l(x)}(d\lambda)$$

and use the following notation for integration

$$\langle \pi_t^{N,l}, h(t, u, \lambda) \rangle = \frac{1}{N} \sum_{x \in \mathbb{Z}} h\left(t, \frac{x}{N}, \eta_t^l(x)\right)$$

Like the empirical measures π_t^N, the Young measures $\pi_t^{N,l}$ will also be a Markov process. We can construct probability measures $\{Q^{N,l}\}$ that correspond to the processes $\{\pi_t^{N,l}\}$ with the appropriate time scaling and initial distributions. The details of the space on which the $Q^{N,l}$ is defined and the underlying topology will be skipped. Suffice to say, it is simpler to prove that the $\{Q^{N,l}\}$ is tight. Letting N go to infinity we obtain limit points $Q^{*,l}$ for each l. If we then let l go to infinity, we will obtain limit points Q^*. Our first goal will be to prove the following theorem about the limit points Q^*.

Theorem 4: *Every limit point Q^* of $\{Q^{N,l}\}$ is supported on measures $\pi_s(du, d\lambda)$ absolutely continuous in the first coordinate whose densities $\rho(t, u, d\lambda)$ are measured valued solutions of the hydrodynamic equation*

$$\partial_t \rho + \gamma_\alpha \partial_u \Phi(\rho) = 0$$

where $\gamma_\alpha = \sum_{d=1}^{\infty} \frac{1}{d^\alpha}$ and $\Phi(\rho) = \rho(1 - \rho)$.

I prefer the terminology "measure weak solution" to "measure valued solution" because of the close connection between measure valued solutions and weak solutions. Specifically, suppose $\rho(s, u)$ is a weak solution of a differential equation, and consider the weak formulation of the differential equation. The measure weak formulation is obtained by replacing $\rho(s, u)$ everywhere it appears in the weak formulation by λ and then integrating against $\rho(s, u, d\lambda)$ from $\lambda = 0$ to infinity. If $\rho(s, u, d\lambda)$ is a solution of the resulting equation, it is called a measure valued solution.

Proof: To show that every limit point Q^* of $\{Q^N\}_{N=1}^{\infty}$ is supported on the measures stated above we begin as in the $\alpha < 1$ case with the measures P^N. After the 1-block replacement, we will have shown that

$$P^N\left(\left|\langle\pi_0^N,g_0\rangle+\int_0^t\partial_s\langle\pi_s^N,g_s\rangle ds+\gamma_\alpha\int_0^t\frac{1}{N}\sum_{x\in\mathbb{Z}}\Phi(\eta_s^l(x))g_s'\left(\frac{x}{N}\right)ds\right|>\epsilon_0\right)$$

goes to zero as N and then l go to infinity. In the above expression, $g_s'(u)$ denotes $\partial_u g(s,u)$.

The integrand of the second integral can be written in terms of Young measures

$$\frac{1}{N}\sum_{x\in\mathbb{Z}}\Phi(\eta_s^l(x))g_s'\left(\frac{x}{N}\right)=\langle\pi_s^{N,l},g_s(u)\Phi(\lambda)\rangle$$

Using the substitution lemma, we can further replace $\langle\pi_0^N,g_0\rangle$ by $\langle\pi_0^{N,l},g_0(u)\lambda\rangle$ and $\int_0^t\partial_s\langle\pi_s^N,g_s\rangle ds$ by $\int_0^t\partial_s\langle\pi_s^{N,l},g_s(u)\lambda\rangle ds$. Because of the close connection between P^N and $Q^{N,l}$, it follows that

$$Q^{N,l}\left(\left|\langle\pi_0^{N,l},g_0(u)\lambda\rangle+\int_0^t\partial_s\langle\pi_s^{N,l},g_s(u)\lambda\rangle ds+\gamma_\alpha\int_0^t\langle\pi_s^{N,l},g_s'(u)\Phi(\lambda)\rangle ds\right|>\epsilon_0\right)$$

goes to zero as N and then l go to infinity. As with P^N, we will abuse notation and let N and l denote subsequences along which $Q^{N,l}$ converges. Therefore,

$$Q^*\left(\left|\langle\pi_0,g_0(u)\lambda\rangle+\int_0^t\partial_s\langle\pi_s,g_s(u)\lambda\rangle ds+\gamma_\alpha\int_0^t\langle\pi_s,g_s'(u)\Phi(\lambda)\rangle ds\right|>\epsilon_0\right)=0$$

for every limit point Q^*. We have again skipped the separate lemma that Q^* is supported on measures $\pi_s=\pi_s(du,d\lambda)$ that are absolutely continuous in the first coordinate. Using this fact, we may write $\pi_s=\rho_s(u,d\lambda)du$. We can also use the fact that g has compact support to replace t by ∞. It then follows that for every limit point Q^*

$$\int_{\mathbb{R}}g_0(u)\int_0^\infty\lambda\rho_0(u,d\lambda)du+\int_0^\infty\int_{\mathbb{R}}\partial_s g_s(u)\int_0^\infty\lambda\rho_s(u,d\lambda)duds$$
$$+\gamma_\alpha\int_0^\infty\int_R\int_0^\infty g_s'(u)\Phi(\lambda)\rho_s(u,d\lambda)duds=0$$

with probability 1. The above is the "measure-weak" formulation of the hydrodynamic equation

$$\partial_t\rho+\gamma_\alpha\partial_u\Phi(\rho)=0$$

meaning that $\rho(t,u,d\lambda)$ is a measure valued solution of the above partial differential equation.

That is, every limit point Q^* is supported on measures, $\pi_s(du,d\lambda)$, absolutely continuous in their first coordinate whose densities $\rho(s,u,d\lambda)$ are measured valued solutions of the above equation. QED

22 Entropy Condition

Unfortunately, we cannot guarantee that there is a unique weak measured valued solution. In fact, $\partial_t \rho + \gamma_\alpha \partial_u \Phi(\rho) = 0$ does not even have a strong solution. It does, however, have a unique entropy solution if ρ_0 is bounded.

A weak solution $\rho(t, u)$ is an entropy solution if

1. $\partial_t |\rho - c| + \gamma_\alpha \partial_u[\text{sgn}(\rho - c)(\Phi(\rho) - \Phi(c))] \leq 0$ where c is any real number.
2. $\lim_{t \to 0^+} \int_{-R}^{R} |\rho_t(u) - \rho_0(u)| du = 0$ for all $R > 0$

Condition 1 above is meant in the weak sense. Kružkov proved that there is a unique entropy solution if ρ_0 is bounded.

DiPerna proved a similar result for measure valued solutions. We say that $\rho(t, u, d\lambda)$ is a measure valued entropy solution if it measure weakly satisfies condition 1 and 2. Then provided one technical condition is satisfied $\rho(t, u, d\lambda)$ is the Dirac measure $\delta_{\rho(t,u)}(d\lambda)$ where $\rho(t, u)$ is the unique entropy solution.

Our final goal then is to prove that every limit point Q^* of $\{Q^{N,l}\}$ is supported on the measure $\pi_s(du, d\lambda)$ absolutely continuous in the first coordinate whose densities $\rho(t, u, d\lambda)$ is the Dirac measure $\delta_{\rho(t,u)}(d\lambda)$ where $\rho(t, u)$ is the unique entropy solution. To prove this we will have to show that every limit point Q^* is supported on measures $\pi_s(du, d\lambda) = \rho(t, u, d\lambda)du$ for which $\rho(t, u, d\lambda)$ satisfies the conditions of DiPerna's uniqueness theorem.

Showing that $\rho(t, u, d\lambda)$ satisfies the technical condition is trivial for the simple exclusion process. A version of the second condition can be proven with multiple coupling arguments. I will only present an outline of the proof involving the first condition in detail.

Theorem 5: *Every limit point Q^* is supported on measures, $\pi_s(du, d\lambda)$, absolutely continuous in their first coordinate whose densities $\rho(s, u, d\lambda)$ are satisfy the measure weak formulation of the entropy condition*

$$\partial_t |\rho - c| + \gamma_\alpha \partial_u[sgn(\rho - c)(\Phi(\rho) - \Phi(c))] \leq 0$$

where c is any real number. ($c \in [0, 1]$ is sufficient in our case.)

Proof: Before being able to use Young measures we will need to prove the following microscopic version of the entropy inequality.

Microscopic Entropy Inequality: For all $c \in [0, 1], \epsilon_0 > 0, t \geq T$

$$\liminf_{l\to\infty}\liminf_{N\to\infty} P^N\Big(\int_0^t \tfrac{1}{N}\sum_{x\in\mathbb{Z}} \left|\eta_s^l(x)-c\right| \partial_s g_s\left(\tfrac{x}{N}\right) ds + \gamma_\alpha \int_0^t \tfrac{1}{N}\sum_{x\in\mathbb{Z}} \operatorname{sgn}(\eta_s^l(x)-c)\left(\Phi(\eta_s^l(x))-\Phi(c)\right) g_s'\left(\tfrac{x}{N}\right) ds \geq -\epsilon_0\Big) = 1$$

where g is non-negative.

One can then write the previous equality in terms of Young measures and the $Q^{N,l}$ to deduce that $\liminf_{l\to\infty}\liminf_{N\to\infty}$

$$Q^{N,l}\Big(\int_0^t \langle \pi_s^{N,l}, \partial_s g_s\,(u)\,|\lambda-c|\rangle ds + \gamma_\alpha \int_0^t \langle \pi_s^{N,l}, g_s'\,(u)\,q(\lambda,c) ds \geq -\epsilon_0\Big) = 1$$

where $q(\lambda,c) = \operatorname{sgn}(\lambda-c)(\Phi(\lambda)-\Phi(c))$. The conclusion of the theorem then follows by the same method as before.

To prove the microscopic entropy inequality, one needs a martingale that looks like it. Our original martingale $M_t^{N,g}$ will not work. Instead we will need to look for a different martingale.

To begin, consider the coupled process (η_t,ξ_t). Define the coupled empirical measure by

$$\tilde{\pi}_t^N = \frac{1}{N}\sum_{x\in\mathbb{Z}} |\eta_t(x)-\xi_t(x))|\delta_{\frac{x}{N}}$$

We can construct probability measures $\tilde{P}^N$ corresponding to the coupled process (η_t,ξ_t). Finally, we define the martingale $\tilde{M}_t^{N,g} = M^F(t)$ where $F(t,\tilde{\pi}_t^N) = \langle \tilde{\pi}_t^N, g_t\rangle$. Using that martingale, one can prove a coupled version of the microscopic entropy inequality.

Coupled Microscopic Entropy Inequality: For all $c\in[0,1], \epsilon>0, t\geq T$

$$\liminf_{l\to\infty}\liminf_{N\to\infty} \tilde{P}^N\Big(\int_0^t \tfrac{1}{N}\sum_{x\in\mathbb{Z}} \left|\eta_s^l(x)-\xi_s^l(x)\right| \partial_s g_s\left(\tfrac{x}{N}\right) ds + \gamma_\alpha \int_0^t \tfrac{1}{N}\sum_{x\in\mathbb{Z}} \operatorname{sgn}(\eta_s^l(x)-\xi_s^l(x))\left(\Phi(\eta_s^l(x))-\Phi(\xi_s^l(x))\right) g_s'\left(\tfrac{x}{N}\right) ds \geq -\epsilon_0\Big) = 1$$

where g is a non-negative test function.

If we choose ξ_0 to be distributed according to the translation invariant equilibrium measure ν_c, then the microscopic entropy inequality follows from the coupled version. Essentially, ξ_t^l serves as an approximation of c.

To prove the coupled microscopic entropy inequality, we begin by showing

$$\liminf_{N\to\infty} \tilde{P}^N\left(-\tilde{M}_t^{N,g} \geq -\epsilon_0\right) = 1$$

$-\tilde{M}_t^{N,g}$ can be explicitly calculated and then naturally bounded above by

$$\int_0^t \frac{1}{N} \sum_{x\in\mathbb{Z}} |\eta_s(x) - \xi_s(x)| \partial_s g_s(\tfrac{x}{N}) ds$$
$$+ \int_0^t \sum_{x\in\mathbb{Z}} \sum_{d=1}^{\infty} \frac{1}{d^{1+\alpha}} (\eta_s^d(x) - \xi_s^d(x)) F_{x,d}(\eta_s, \xi_s) \left[g_s\left(\tfrac{x+d}{N}\right) - g_s\left(\tfrac{x}{N}\right)\right] ds$$

which must also be greater than $-\epsilon_0$ with high probability. We can then continually use the substitution lemma until after the 1-block replacement, which will complete the proof. The 1-block replacement is quite interesting. in that it requires a replacement in three separate terms done in three separate ways. For one of the terms the replacement holds trivially. For another term, the standard 1-block estimate will work. But for the last term, coupling arguments are used to make the 1-block replacement. Specifically, one needs to prove what I have dubbed the ordering lemma:

$$\limsup_{N\to\infty} \tilde{E}^N \left[\int_0^t \frac{1}{N} \sum_{|x|\leq RN} G_{x,d}(\eta_s, \xi_s) ds \right] = 0$$

for all $d > 1$, where $G_{x,d}(\eta_s, \xi_s)$ is the indicator function that is one if η_s and ξ_s are not ordered on $\{x, x+d\}$. That is, $G_{x,d}(\eta_s, \xi_s)$ is one if $\eta_s(x) < \xi_s(x)$ and $\eta_s(x+d) > \xi_s(x+d)$ or vice versa.

If the initial distribution of the configurations was such that $\eta_0(x) \leq \xi_0(x)$ we would know that $\eta_t(x) \leq \xi_t(x)$ for all later times by attractiveness. The 1-block replacement would then be straightforward. The ordering lemma tells us that even though we have not started from such configurations in some sense η and ξ will be ordered as N goes to infinity. QED

Once we know that the $\{Q^{N,l}\}$ converges weakly to the measure Q supported on $\delta_{\rho(t,u)}(d\lambda)du$ where $\rho(t,u)$ is the unique entropy solution, we can prove that $\{P^N\}$ converges weakly to the measure P supported on $\rho(t,u)du$ by relating P^N to $Q^{N,l}$. Specifically, we know that

$$Q^{N,l}\left(\left| \int_0^t \langle \pi_s^{N,l}, g_s\lambda\rangle ds - \int_0^t \int \rho(s,u) g(s,u) du ds \right| > \epsilon_0 \right)$$

converges to zero, where $\rho(s,u)$ is the unique entropy solution. Using the substitution lemma, the above probability can be related to P^N to show that

$$P^N\left(\left| \int_0^t \langle \pi_s^N, g_s\rangle ds - \int_0^t \int \rho(s,u) g(s,u) du ds \right| > \epsilon_0 \right)$$

converges to zero as N goes to infinity. By choosing a subsequence of $\{P^N\}_{N=1}^{\infty}$ that converges weakly, we can conclude that for every limit point P^*

$$P^*\left(\int_0^t \langle \pi_t, g_t\rangle ds = \int_0^t \int \rho(t,u) g(t,u) du ds \right) = 1$$

Since g is an arbitrary test function and P^* is supported on absolutely continuous measures, it follows that P^* is supported on $\rho(t,u)du$ where $\rho(t,u)$ equals the unique entropy solution for almost all times t. One can show that $\rho(t,\cdot)$ is weakly continuous in t, from which it follows that P^* is supported on $\rho(t,u)du$ where $\rho(t,u)$ will equal the entropy solution for all time.

23 Future Work

Rezakhanlou's proof for the asymmetric finite range case relies heavily on attractiveness. The proof we have developed attempts to eliminate the need for attractiveness. Due to the different approach, there are a few more details that still need to be worked out in the $\alpha > 1$ case. In the $\alpha < 1$ case, we hope to determine additional conditions that would guarantee a unique strong solution. It will also be interesting to extend these results to other processes, namely, the zero range process.

In addition to this problem, there are other interesting problems within interacting particle systems of interest. A few problems I was thinking of working on next include interacting particle systems with a random environment, the slow bond problem, and fluctuations.

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