Objectives

- Identify the characteristics of the natural exponential function $f(x) = e^x$, including the domain, range, intercept, asymptote, end behavior, and its graph.
- Sketch the graph of natural exponential functions using transformations.
- Solve natural exponential equations by relating the bases.
- Solve continuous compound interest application problems.
- Determine the present value of an investment using continuous compound interest.
- Solve population growth application problems.

Preliminaries

Consider the natural exponential function $f(x) = e^x$. List the following properties.

Domain: $(-\infty, \infty)$

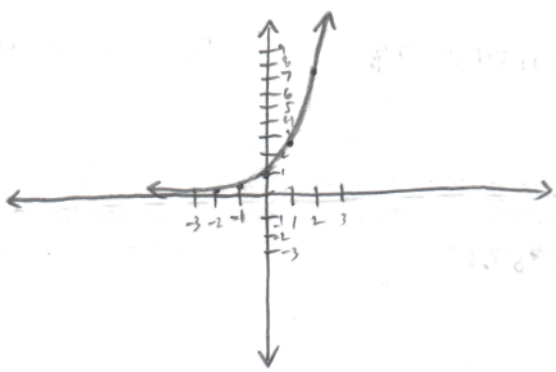
Range: $(0, \infty)$

y-intercept: $(0, 1)$ If $x=0$, $f(0) = e^0 = 1$

Asymptote: $y=0$ It is a horizontal asymptote

End behavior: $e^x \to 0$ as $x \to -\infty$ and $e^x \to \infty$ as $x \to \infty$

Graph:



Continuous compound interest can be calculated by using the formula

$$A = Pe^{rt}$$

Write down the meaning of each value in the formula.

A: Amount at time t

P: Initial amount

r: Interest rate per year

t: Time in years

Warm-up

1. Use your calculator to approximate the following values rounded to four decimal places.

   (A) $e^{-3} \approx 0.0498$ (B) $12e^{0.16} \approx 14.0821$

Round only at the very end.

Class Notes and Examples

5.2.1 Each of the following functions were created using transformations of $f(x) = e^x$. Determine the transformations that were performed. List the domain, horizontal asymptote, range, and y-intercept. Graph the given function. (Note: we will discuss an algebraic method to determine the x-intercept later in chapter 5.)

(A) $J(x) = e^x + 2$

Transformation(s): Up 2

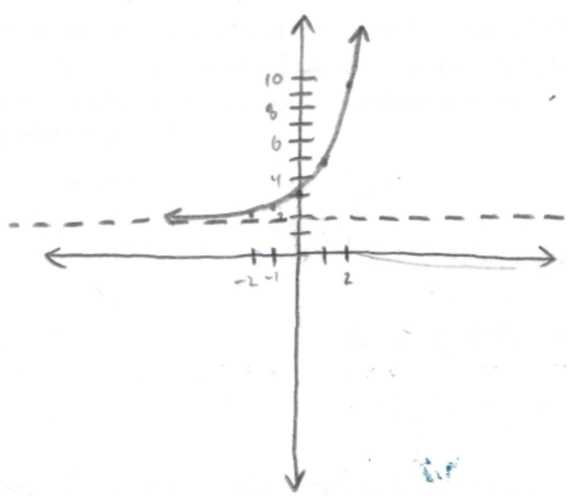
Domain: $(-\infty, \infty)$

Horizontal Asymptote: $y=2$

Range: $(2, \infty)$

y-intercept: $(0, 3)$ If $x=0$, $J(0) = e^0 + 2 = 1 + 2 = 3$.

Graph:



(B) $L(x) = -2 \cdot e^{x-3}$

Transformation(s): Right 3, Stretch vertically by a factor of 2, Reflect vertically (across the x-axis)

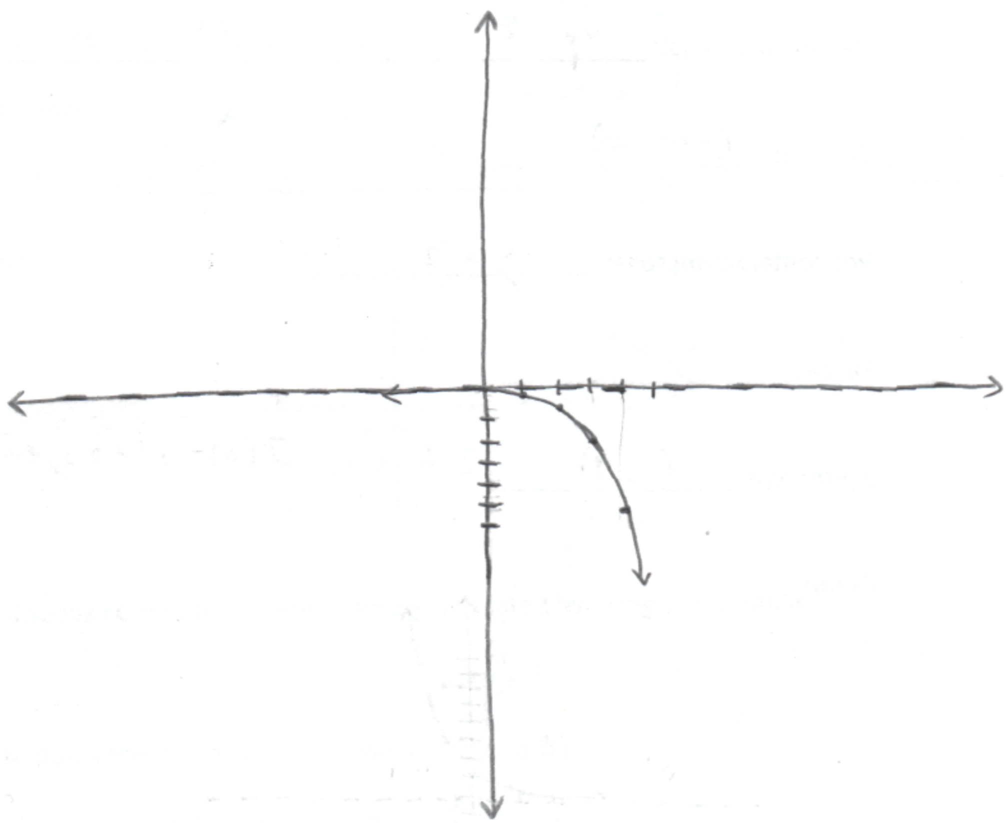
Domain: $(-\infty, \infty)$

Horizontal Asymptote: $y=0$

Range: $(-\infty, 0)$

y-intercept: $(0, -2e^{-3})$ If $x=0$, $L(0) = -2 \cdot e^{0-3} = -2e^{-3}$

Graph:



5.2.2 Solve the following equations. Check your answers in the original equation.

(A) $\sqrt[3]{e^5} = e^{x-1}$

(B) $\frac{e^{2x}}{e^3} = e^{x+1}$

$(e^5)^{1/3} = e^{x-1}$ Recall $\sqrt[3]{x} = x^{1/3}$

$e^{5/3} = e^{x-1}$ Recall $(x^a)^b = x^{ab}$

$5/3 = x-1$
$+1 \quad +1$

$8/3 = x$

$x = 8/3$

$e^{2x-3} = e^{x+1}$ Recall $\frac{x^b}{x^a} = x^{b-a}$

$2x-3 = x+1$
$-x \quad -x$

e.g. $\frac{e^5}{e^2} = e^{5-2} = e^3$

$x-3 = 1$
$+3 \quad +3$

$x = 4$

5.2.3 Dmitry invests \$3200 in a savings account that earns 4.6% interest compounded continuously. How much money would Dmitry have in the account after 3.5 years?

$A = Pe^{rt}$

$A = 3{,}200\, e^{0.046 \times 3.5}$

$A \approx 3758.9919$

Dmitry will have \$3,758.99 in his account after 3.5 years.

5.2.4 Anna has a choice between two investment options for a \$1000 gift she received. The first option earns 7.8% interest compounded continuously. The second option earns 7.9% interest compounded semi-annually. Which option would yield the greatest amount of money after 5 years?

Option 1 $A = Pe^{rt}$

$A = 1{,}000\, e^{0.078 \times 5}$

$A \approx 1476.980794$

\$1476.98

Option 2 $A = P(1+\frac{r}{n})^{nt}$

$A = 1{,}000\left(1 + \frac{0.079}{2}\right)^{2 \cdot 5}$

$A \approx 1{,}473.143103$

\$1,473.14

option 1 would yield the greatest amount of money after five years.

5.2.5 Arturo wants to have $15,000 in 6 years, so he will place money into a savings account that pays 3.7 % interest compounded continuously. How much should Arturo invest now to have $15,000 in 6 years? Check your answer.

$A = Pe^{rt}$

$15,000 = Pe^{0.037 \times 6}$

$15,000 \approx P \times 1.248571378$ Don't round until the end

$P \approx \frac{15,000}{1.248571378}$

$P \approx 12,013.73047$

Arturo should invest $12,013.73 now.

? What is the general exponential growth model?

$P(t) = P_0 e^{kt}$ — time. rate

Population at time t. Initial Population

5.2.6 The population of a city can be measured by $P(t) = 12{,}500e^{0.02t}$, where t represents time in years after 1985.

(A) What was the population in 1985?

$t = 0.$

$P(0) = 12,500\, e^{0.02(0)} = 12,500$ The population was 12,500

(B) What was the population in 2000?

$t = 15$ $P(15) = 12,500\, e^{0.02(15)} \approx 16,873.235$ round

The population in 2000 was 16,873

(C) What does the model predict the population to be in the year 2020?

$t = 35$ $P(35) = 12,500\, e^{0.02 \times 35} \approx 25171.908$

The population in 2000 was 25,172.

5.2.7 An invasive beetle was discovered in a small Pacific island 15 years ago. It is estimated that there are 12,400 beetles on the island now, with a relative growth rate of 16%.

(A) How many beetles were initially discovered 15 years ago?

$P(t) = P_0 e^{kt}$

$t = 15 \quad P(15) = 12,400$

$12,400 = P_0 e^{0.16 \times 15}$

$12,400 \approx P_0 \times 11.02317638$

$P_0 \approx \dfrac{12,400}{11.02317638}$

$P_0 \approx 1124.902$

Don't round until the very end.

1,125 beetles were initially discovered.

(B) How many beetles will there be after another 15 years?

Now let $t=0$ correspond to now when there are 12,400 beetles on the island,

$t = 15 \quad P(15) = 12,400 e^{0.16 \times 15}$

$P(15) \approx 136,687.38$

There will be 136,687 beetles in another 15 years.