CHAPTER 5

Section 5.1

Objectives

- Determine whether a function is exponential.
- Identify the characteristics of exponential functions of the form $f(x) = b^x$, including the domain, range, intercept, asymptote, end behavior, and general graphs.
- Determine the formula for an exponential function given its graph.
- Sketch the graph of exponential functions using transformations.
- Solve exponential equations by relating the bases.
- Solve compound interest application problems.
- Determine the present value of an investment.
- Solve exponential application problems

Preliminaries

An **exponential function** is a function of the form $f(x) =$ b^x, where x is any real number and b has the properties $b > 0$ and $b \neq 1$.

b is called the base of the exponential function.

List the properties of the exponential function $f(x) = b^x$, where $b > 0$ and $b \neq 1$.

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

y-intercept: $(0, 1)$ If $x = 0$, $f(0) = b^0 = 1$

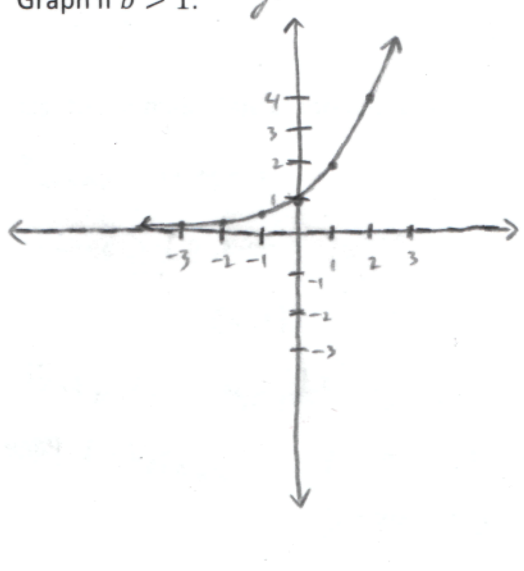
Asymptote: $y = 0$ It is a horizontal asymptote

End behavior if $b > 1$: $b^x \to 0$ as $x \to -\infty$ and $b^x \to \infty$ as $x \to \infty$

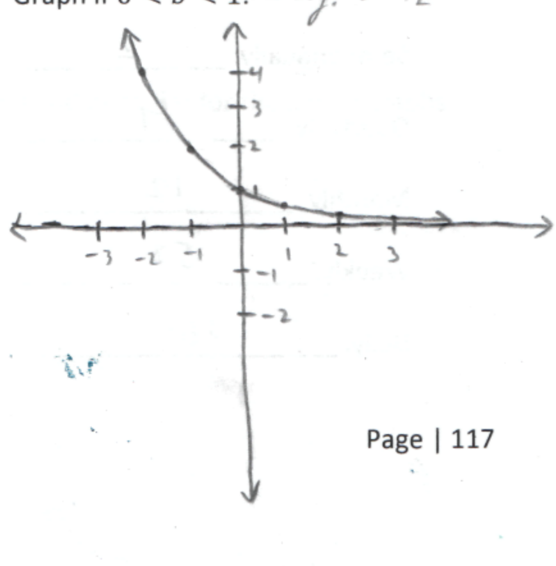
"$\to$" means "goes to" or "approaches"

End behavior if $0 < b < 1$: $b^x \to \infty$ as $x \to -\infty$ and $b^x \to 0$ as $x \to \infty$

Graph if $b > 1$: e.g. $b = 2$



Graph if $0 < b < 1$: e.g. $b = 1/2$



Periodic compound interest can be calculated by using the formula

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Write down the meaning of each value in the formula.

A: Amount at time t

P: Initial amount

r: Interest rate per year (Write as a decimal not a percent)

n: Number of times compounded per year

t: Time in years

Warm-up

1. Solve the following equations.

(A) $2^x = 8$

$2^x = 2^3$

$x = 3$

(B) $3^x = \frac{1}{9}$

$3^x = 3^{-2}$

$x = -2$

(C) $4^x = 1$

$4^x = 4^0$

$x = 0$

2. How many times per year would interest be calculated in each of the situations described below?

Annually: 1

Semi-annually: 2

Quarterly: 4

Monthly: 12

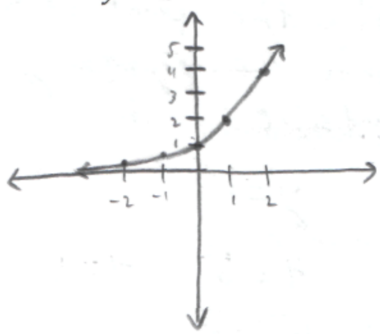
Weekly: 52

Daily: 365

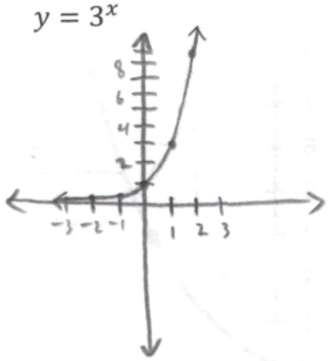
Class Notes and Examples

5.1.1 Sketch the graphs of the following exponential functions.

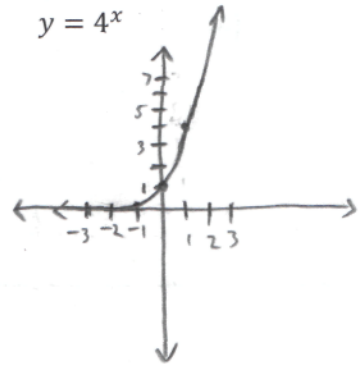
$y = 2^x$



$y = 3^x$



$y = 4^x$



List the domain, range, intercept, asymptote, and end behavior for the above graphs.

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

y-intercept: $(0, 1)$

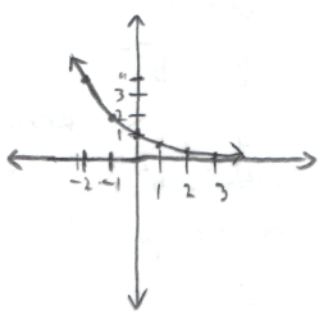
Horizontal asymptote: $y = 0$

End behavior: $y \to 0$ as $x \to -\infty$

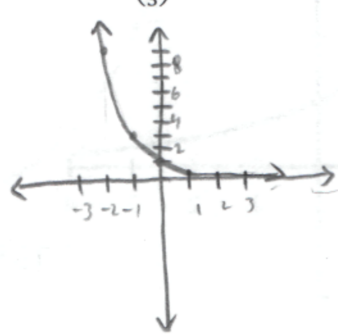
$y \to \infty$ as $x \to \infty$

5.1.2 Sketch the graphs of the following exponential functions.

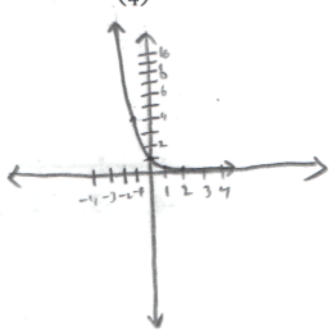
$y = \left(\frac{1}{2}\right)^x$



$y = \left(\frac{1}{3}\right)^x$



$y = \left(\frac{1}{4}\right)^x$



List the domain, range, intercept, asymptote, and end behavior for the above graphs.

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

y-intercept: $(0, 1)$

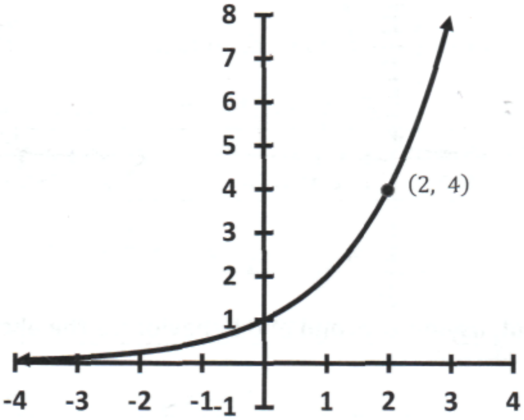
Horizontal asymptote: $y = 0$

End behavior: $y \to \infty$ as $x \to -\infty$

$y \to 0$ as $x \to \infty$

5.1.3 Determine a formula of the form $y = b^x$ for each of the exponential functions graphed below.

(A)



Plug the point (2, 4) into the equation $y=b^x$ and solve for b.

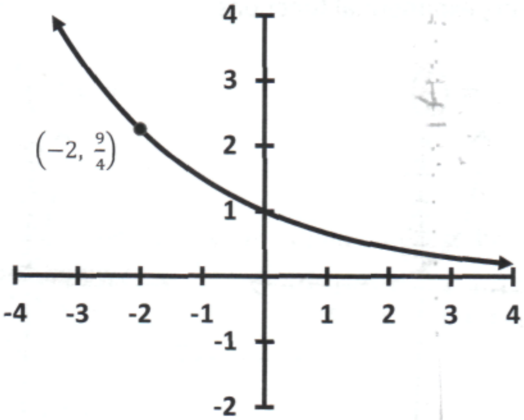
$$y=b^x$$
$$4=b^2$$
$$\sqrt{4}=\sqrt{b^2}$$
$$\pm 2=b$$

But $b>0$, So $b=2$

Plug 2 in for b in the equation $y=b^x$.

$$\underline{y=2^x}.$$

(B)



Plug the point $(-2, 9/4)$ into the equation $y=b^x$ and solve for b.

$$y=b^x$$
$$9/4=b^{-2}$$
$$\frac{9}{4}=\frac{1}{b^2}$$

Recall: $x^{-n}=\frac{1}{x^n}$

e.g. $10^{-2}=\frac{1}{10^2}$

Take the reciprocal of both sides.

$$\frac{4}{9}=b^2$$
$$\sqrt{\frac{4}{9}}=\sqrt{b^2}$$
$$\pm\frac{2}{3}=b$$

But $b>0$. So $b=\frac{2}{3}$.

Plug 2/3 in for b in the equation $y=b^x$.

$$\underline{y=(2/3)^x}.$$

5.1.4 Each of the following functions were created using transformations of a base exponential function. State the base function and the transformations that were performed. List the domain, horizontal asymptote, range, and y-intercept. Graph the given function.

(A) $H(x) = 2^{x+3}$

Base function: $h(x) = 2^x$

Transformation(s): Left 3

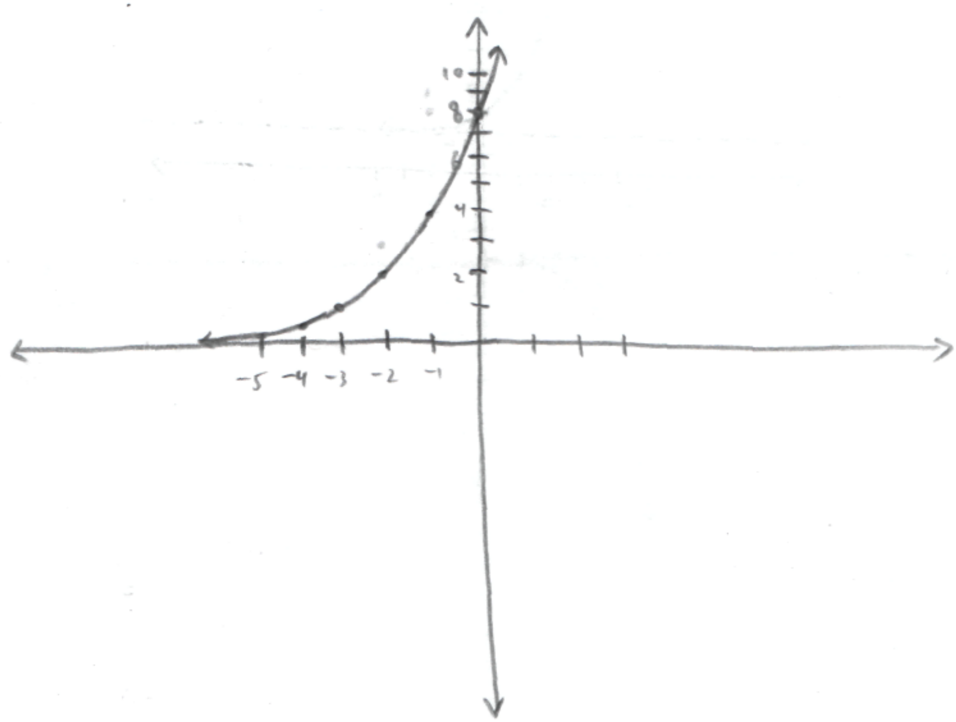
Domain: $(-\infty, \infty)$

Horizontal Asymptote: $y = 0$

Range: $(0, \infty)$

y-intercept: $(0, 8)$ If $x=0$, $H(0) = 2^{0+3} = 2^3 = 8$.

Graph:



(B) $J(x) = \left(\frac{1}{3}\right)^x + 2$

Base function: $j(x) = \left(\frac{1}{3}\right)^x$

Transformation(s): Shift up 2.

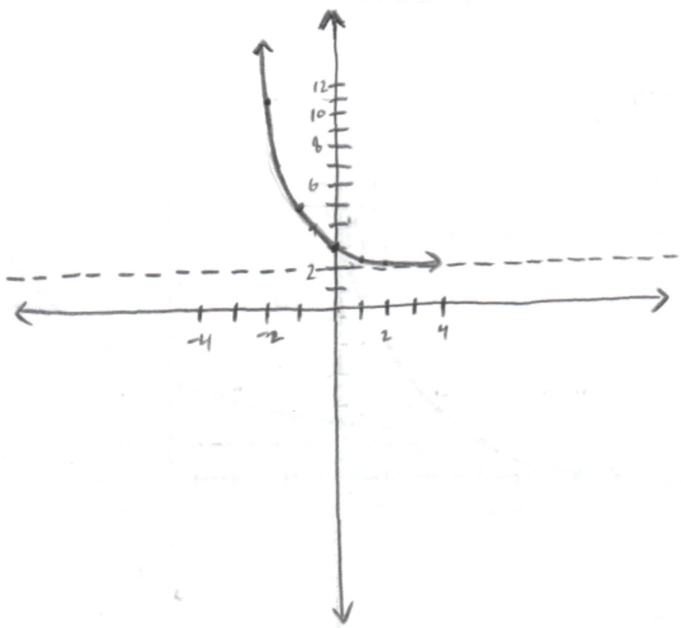
Domain: $(-\infty, \infty)$

Horizontal Asymptote: $y = 2$

Range: $(2, \infty)$

y-intercept: $(0, 3)$ If $x=0$, $J(0) = \left(\frac{1}{3}\right)^0 + 2 = 1 + 2 = 3$.

Graph:



(C) $L(x) = -2 \cdot 4^{x-3}$

Base function: $L(x) = 4^x$

Transformation(s): Right 3, Stretch vertically by a factor of 2, Reflect vertically (across the x-axis)

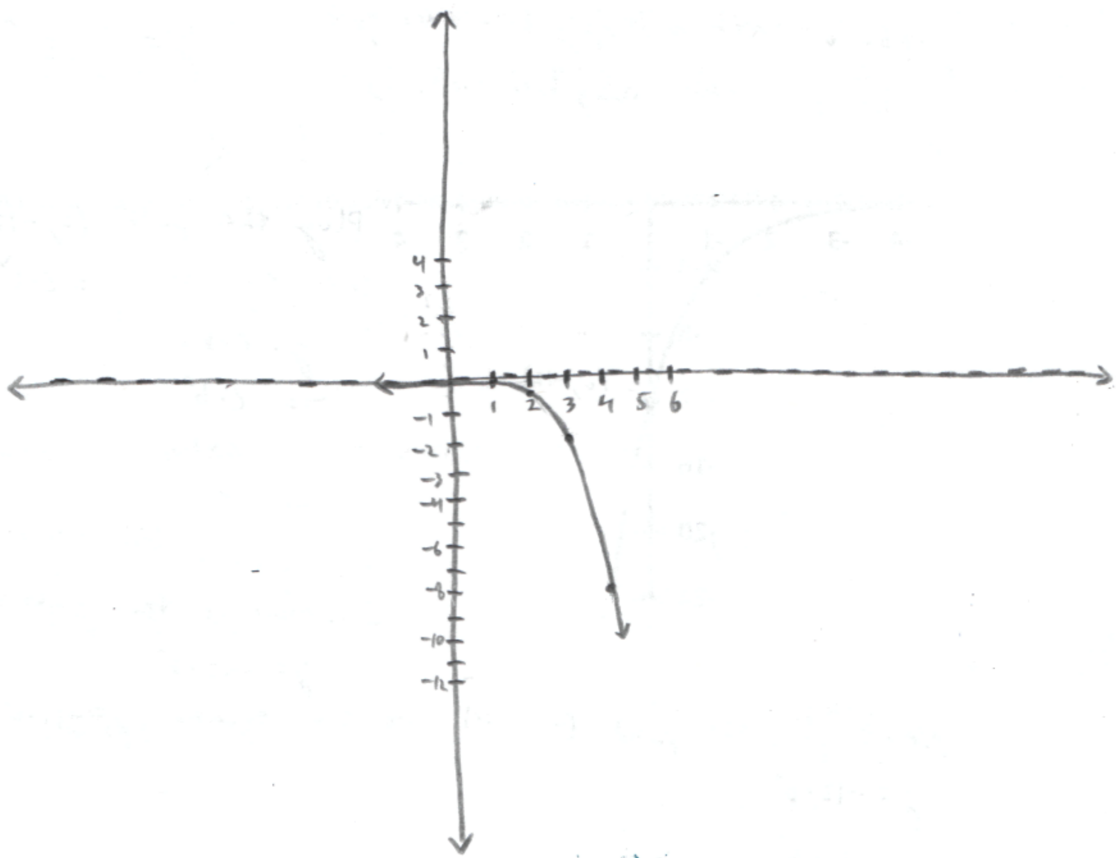
Domain: $(-\infty, \infty)$

Horizontal Asymptote: $y=0$

Range: $(-\infty, 0)$

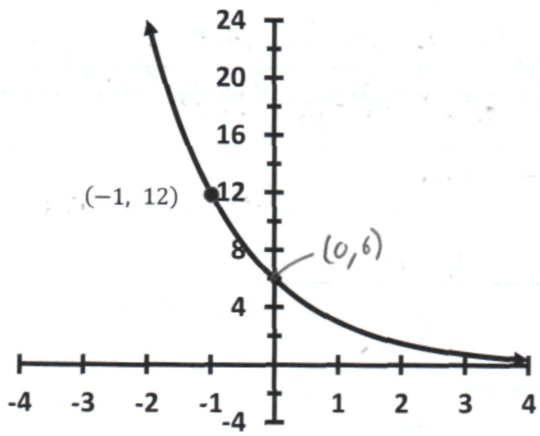
y-intercept: $(0, -\frac{1}{32})$ If $x=0$, $L(0) = -2 \cdot 4^{0-3} = -2 \cdot \frac{1}{4^3} = -\frac{2}{64} = -\frac{1}{32}$

Graph:



5.1.5 Determine a formula of the form $y = C \cdot b^x$ for the exponential functions graphed below. Check by graphing on your calculator.

(A)



Plug the point (0,6) into the equation $y=C\cdot b^x$.

$y=C\cdot b^x$

$6=C\cdot b^0$

$6=C\cdot 1$

$6=C$

So $C=6$.

C is always the y-intercept.

$y=6\cdot b^x$

Next plug the point (−1, 12) into the equation $y=6\cdot b^x$ and solve for b.

$y=6\cdot b^x$

$12=6\cdot b^{-1}$

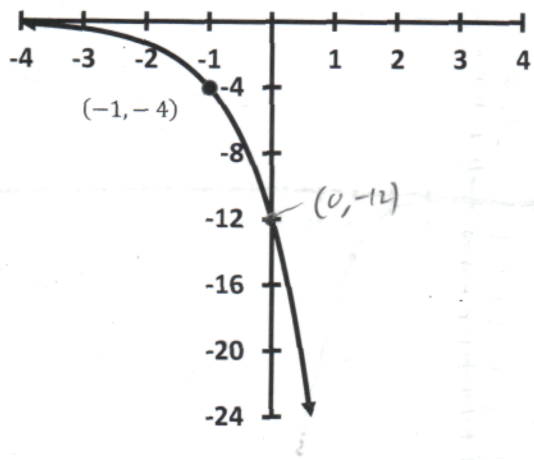
$12=\frac{6}{b}$ Recall: $b^{-1}=\frac{1}{b}$

$12b=6$ after multiplying both sides by b.

$b=\frac{1}{2}$ after dividing both sides by 12.

$y=6\cdot\left(\frac{1}{2}\right)^x$

(B)



Plug the point (0, −12) into the equation $y=C\cdot b^x$

$y=C\cdot b^x$

$-12=C\cdot b^0$

$-12=C\cdot 1$

$-12=C$

So $C=-12$.

C is always the y-intercept

$y=-12\cdot b^x$

Next plug the point (−1, −4) into the equation $y=-12\cdot b^x$ and solve for b.

$y=-12\cdot b^x$

$-4=-12\cdot b^{-1}$

$-4=\frac{-12}{b}$ Recall $b^{-1}=\frac{1}{b}$

$-4b=-12$ after multiplying both sides by b

$b=3$ after dividing both sides by −4.

$y=-12\cdot 3^x$

5.1.6 Determine a function of the form $y = C \cdot b^x$ that passes through the points $(1, 12)$ and $(3, 192)$. Check your answer.

Plug the points $(1,12)$ and $(3,192)$ into $y=C\cdot b^x$ and solve for C and b.

$12 = C\cdot b^1$ $\quad 192 = C\cdot b^3$ $\quad$ Use substitution.

Solve for C in the equation $12=Cb$. $C=\frac{12}{b}$.

Substitute $\frac{12}{b}$ for C in the equation $192=C\cdot b^3$. $192=\frac{12}{b}\cdot b^3 = 12b^2$.

Solve for b. But $b>0$! So $\underline{b=4}$. Plug 4 in for b in the equation $12=C\cdot b$ and solve for C. $12=4C$. So $\underline{C=3}$.

$\frac{192}{12}=\frac{12b^2}{12}$

$16=b^2$

$b=\pm 4$.

Therefore, $y=3\cdot 4^x$

5.1.7 The table below gives values for a function of the form $y = C \cdot b^x$. Determine the values of C and b.

x	1	2	3	4
y	8	12	18	27

Choose two points. I choose $(1,8)$ and $(2,12)$. Plug them into $y=C\cdot b^x$ and solve for C and b. $8=C\cdot b^1=Cb$ $\quad 12=C\cdot b^2=Cb^2$

Solve for C in the equation $8=Cb$. $C=\frac{8}{b}$. Substitute $\frac{8}{b}$ for C in the other equation. $12=\frac{8}{b}b^2=8b$. $b=\frac{3}{2}$. $C=\frac{8}{b}=\frac{8}{(3/2)}=8\cdot\frac{2}{3}=16/3$

$y=\frac{16}{3}\cdot\left(\frac{3}{2}\right)^x$

? What strategy can you use to solve an exponential equation of the form $b^u = b^v$?

Simply set $u=v$.

5.1.8 Solve the following equations by rewriting in the form $b^u = b^v$.

(A) $8^{-x} = 64$

$8^{-x}=8^2$

$-x=2$

$x=-2$

(B) $27^x = \left(\frac{1}{3}\right)^{2x+1}$

$(3^3)^x = (3^{-1})^{2x+1}$ $\quad$ Recall $(3^a)^b = 3^{ab}$

$3^{3x} = 3^{-1(2x+1)}$

$3x = -1(2x+1)$

$3x = -2x-1$

$+2x \quad +2x$

$5x=-1$

$x=-\frac{1}{5}$

5.1.9 Eric has started a new weightlifting routine. The amount of weight, in pounds, he is able to lift at the end of t weeks can be modeled by the following function.

$$w(t) = 260 - 140(2.6)^{-0.2t}$$

(A) How much was Eric able to lift at the start of his weightlifting routine?

$t=0$ is the start of Eric's weightlifting routine,

$w(0) = 260 - 140(2.6)^{-0.2(0)} = 260 - 140(2.6)^0 = 260 - 140 \cdot 1 = 120$

Eric was able to lift 120 pounds at the start of his weightlifting routine.

(B) How much will Eric be able to lift at the end of 10 weeks? (Round to the nearest pound.)

$w(10) = 260 - 140(2.6)^{-0.2 \cdot 10} = 260 - 140(2.6)^{-2} \approx 239.2899408$

To the nearest pound, Eric will be able to lift 239 pounds at the end of 10 weeks.

5.1.10 Suppose Melissa invests \$9400 into a high-yield savings account that pays 5.7% interest compounded quarterly. Her brother, Billy, invests \$10,200 into a different account that pays 4.8% compounded monthly. If no other investments are made, who will have more money in their account at the end of 10 years? How much more money will that person have?

$A = P(1+\frac{r}{n})^{nt}$

For Melissa,

$A = 9{,}400\left(1+\frac{0.057}{4}\right)^{4t}$

For Billy,

$A = 10{,}200\left(1+\frac{0.048}{12}\right)^{12t}$

At the end of ten years,

$A = 9{,}400\left(1+\frac{0.057}{4}\right)^{4 \cdot 10}$

$\approx 16{,}554.97$

$A = 10{,}200\left(1+\frac{0.048}{12}\right)^{12 \cdot 10}$

$\approx 16{,}468.18$

Melissa will have more money in her account at the end of ten years.

She will have $86.79 = 16{,}554.97 - 16{,}468.18$ dollars more than Billy.

5.1.12 Phillip wants to have \$10,000 in 6 years, so he will place money into a savings account that pays 3.2% interest compounded weekly. How much should Phillip invest now to have \$10,000 in 6 years? Check your answer.

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

Compounded weekly means $n=52$

3.2% interest means $r=0.032$

The time we are considering is 6 years from the initial investment. So $t=6$.

Phillip wants \$10,000 when $t=6$.

So the amount, A, Phillip wants at time $t=6$ is 10,000.

$$10{,}000 = P\left(1+\frac{0.032}{52}\right)^{52\cdot 6}$$

Now solve for P.

$$10{,}000 = P \times 1.211598967$$

Don't round until the end.

$$P = \frac{10{,}000}{1.211598967}$$

$$P = 8253.556065$$

We are dealing with money, so we have to round up to the nearest cent.

$$P = 8253.56$$

Phillip should invest \$8253.56 now to have \$10,000 in 6 years.