Section 4.6

Objectives

- Identify whether a function represented by an equation is a rational function.
- Determine the domain of a rational function represented by an equation or a graph.
- Given a rational function in the form $\frac{f(x)}{g(x)}$, use polynomial division to rewrite in the form $q(x) + \frac{r(x)}{d(x)}$, where $q(x)$ is the quotient, and $r(x)$ is the remainder when $f(x)$ is divided by $g(x)$.
- Given a rational function represented by an equation, determine the equation of any vertical asymptote for the function.
- Given a rational function represented by an equation, determine whether the function has a horizontal asymptote, a slant asymptote, or neither.
- Given a rational function represented by an equation, find the equation of the horizontal asymptote, if it exists.
- Given a rational function represented by an equation, find the equation of the slant asymptote, if it exists, by using polynomial long division.
- Use the horizontal or slant asymptote of a rational function to determine the end behavior of the function.
- Given a rational function represented by an equation, determine whether the graph of a rational function has a hole, and find the x-value at which the hole in the graph occurs.
- Given the graph of a rational function, identify and use all important features of the graph to find a possible equation for the function.

Preliminaries

Complete the following definitions:

A rational function is a function of the form $f(x) = \frac{g(x)}{h(x)}$. where $g(x)$ and $h(x)$ are polynomials.

To find the *y*-intercept of a function, set x equal to zero.

To find the *x*-intercept(s) of a function, Set y or f(x) equal to zero.

Warm-up

1. Find the domain of each of the following. Verify by graphing.

(A) $f(x) = \frac{x^2-2x+1}{x-2}$

(B) $g(x) = \frac{2}{x^2+1}$

? How do we find the domain of a rational function?

Find where the denominator is zero.

Remove those points from $(-\infty, \infty)$.

? How do we find the zero(s) of a rational function?

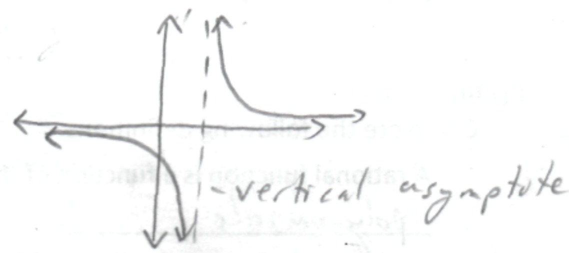
Find where the numerator is zero.

Check that the points are in the domain

? What is a vertical asymptote, and when will a rational function have one?

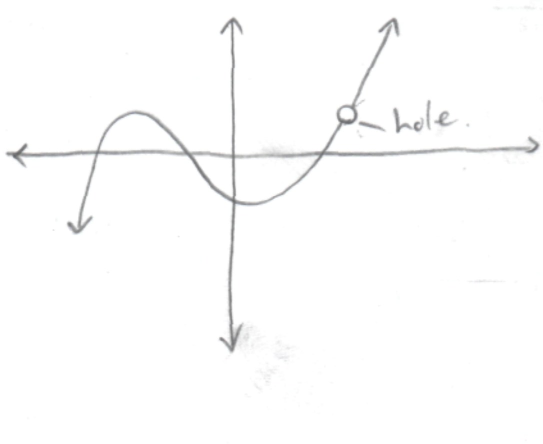
A vertical asymptote is a vertical line $x=a$ such that as x approaches "a", $f(x)$ approaches $\pm\infty$.

A rational function will have a vertical asymptote $x=a$ if the denominator is zero when $x=a$ but the numerator is not zero.



? What is meant by a removable discontinuity? When will a rational function have one?

A removable discontinuity is a hole. A point in the graph that is not in the domain, but that when filled in makes the function continuous.



Cancel common factors before finding vertical asymptotes.
Common factors produce holes.

4.6.1 Find the domain of each rational function, determine the vertical asymptotes and holes, and find the zeros.

(A) $p(x) = \frac{3x-2}{x+1}$

Domain: $(-\infty, -1) \cup (-1, \infty)$

$$p(x) = \frac{3\left(x - \frac{2}{3}\right)}{x+1}$$

There are no holes.

Zeros: $x = 2/3$

vertical asymptotes: $x = -1$

(B) $R(x) = \frac{2x+1}{x^2-16}$

Domain: $(-\infty, -4) \cup (-4, 4) \cup (4, \infty)$

$$R(x) = \frac{2\left(x + 1/2\right)}{(x-4)(x+4)}$$

There are no holes.

Zeros: $x = -1/2$

Vertical asymptotes: $x = 4$ and $x = -4$

(C) $L(x) = \frac{x^2-5x+1}{x+1}$

Domain: $(-\infty, -1) \cup (-1, \infty)$

$x^2 - 5x + 1 = 0$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{5 \pm \sqrt{21}}{2}$$

The numerator and denominator are not zero for the same value of x.
Thus, there are no holes.

Zeros: $x = \frac{5 \pm \sqrt{21}}{2}$

vertical asymptotes: $x = -1$

(D) $F(x) = \frac{x^2-1}{x^2-x-2}$

Domain: $(-\infty, -1) \cup (-1, 2) \cup (2, \infty)$

$$F(x) = \frac{(x-1)(x+1)}{(x-2)(x+1)}$$

There is a hole at $x = -1$

Zeros: $x = 1$

vertical asymptotes: $x = 2$

? How do you determine the long-term behavior of a rational function?

The long determine behavior is determined by the quotient $q(x)$ after division.

For large positive and negative x, the rational function is similar to the function $y = q(x)$.

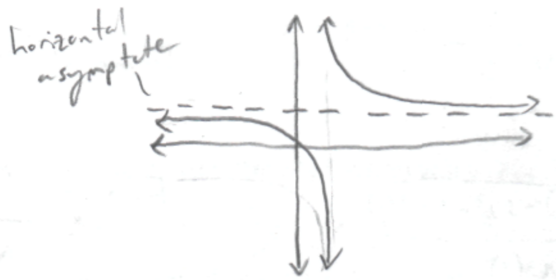
If (degree numerator < degree denominator), the horizontal asymptote is $y=0$.

If (degree numerator = degree denominator), the horizontal asymptote is $y=a$, where "a" is the ratio of leading coefficient of the numerator to the leading coefficient of the denominator

If (degree numerator = degree denominator +1), divide to find the slant asymptote.

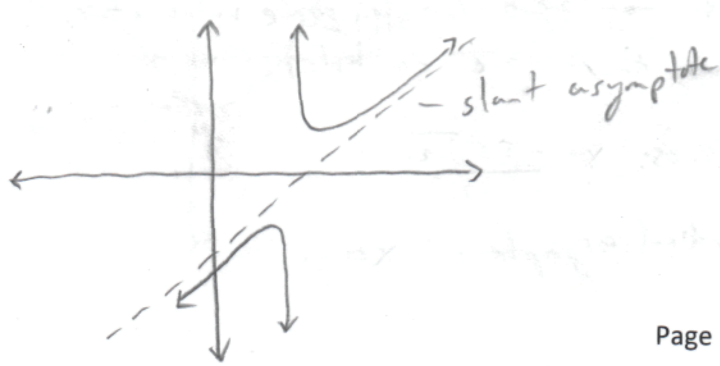
? What is a horizontal asymptote, and when will a rational function have one?

A horizontal asymptote is a horizontal line $y=b$ that the function will approach for large positive x and/or large negative x.



? What is a slant asymptote, and when will a rational function have one?

A slant asymptote is a line $y=mx+b$ $(m \neq 0)$ that the function approaches for large positive x and/or large negative x.



4.6.2 Use polynomial division to rewrite in the form $q(x) + \frac{r(x)}{d(x)}$. Find the horizontal or slant asymptote, if one exists.

(A) $p(x) = \frac{3x-2}{x+1}$

$$\begin{array}{r} 3 \\ x+1 \overline{\smash{)}\,3x-2} \\ -\,3x+3 \\ \hline -5 \end{array}$$

$p(x) = 3 + \frac{-5}{x+1}$

The horizontal asymptote is $y=3$

(B) $R(x) = \frac{2x+1}{x^2-16}$

$$\begin{array}{r} 0 \\ x^2-16 \overline{\smash{)}\,2x+1} \\ -\,0 \\ \hline 2x+1 \end{array}$$

$R(x) = 0 + \frac{2x+1}{x^2-16}$

The horizontal asymptote is $y=0$

(C) $L(x) = \frac{x^2-5x+1}{x+1}$

$$\begin{array}{r} x-6 \\ x+1 \overline{\smash{)}\,x^2-5x+1} \\ -\,x^2+x \\ \hline -6x+1 \\ -6x-6 \\ \hline 7 \end{array}$$

$L(x) = x-6 + \frac{7}{x+1}$

The slant asymptote is $y=x-6$

(D) $F(x) = \frac{x^2-1}{x^2-x-2}$

$$\begin{array}{r} 1 \\ x^2-x-2 \overline{\smash{)}\,x^2-1} \\ -\,x^2-x-2 \\ \hline x+1 \end{array}$$

$F(x) = 1 + \frac{x+1}{x^2-x-2}$

The horizontal asymptote is $y=1$

? What strategies can you use to graph a rational function? What important characteristics will you need to find in order to sketch an accurate graph?

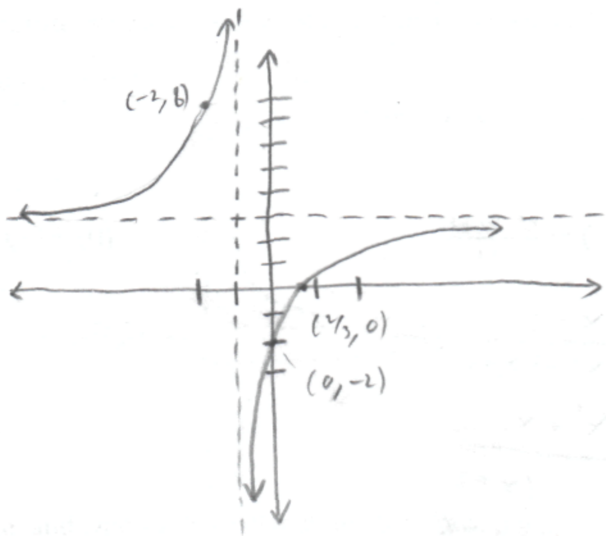
First, sketch the asymptotes.
Second, plot points to the left and right of all vertical asymptotes.
Third, connect points with a curve that approaches the asymptotes.
Fourth, remove holes.

4.6.3 Use the results from questions 4.6.1 and 4.6.2 to write the domain, holes/vertical asymptotes, zeros, and horizontal/slant asymptote. Sketch a graph of the rational function. Verify your graph using your calculator.

(A) $p(x) = \frac{3x-2}{x+1}$

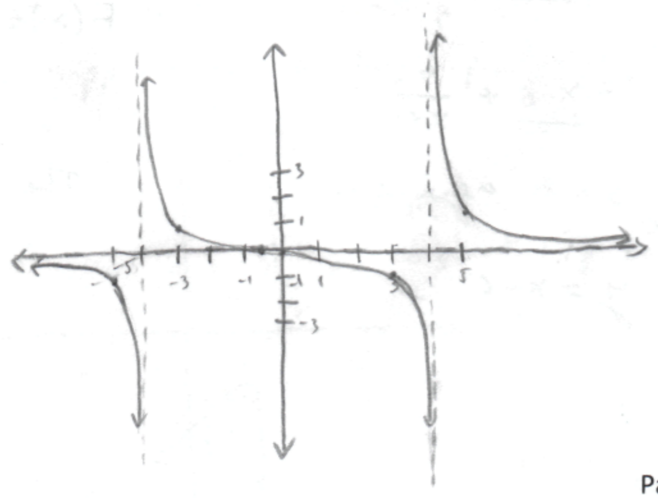
Holes: None
Zeros: $x = 2/3$
V.A: $x = -1$
H.A: $y = 3$

Find points on both sides of the vertical asymptote.



(B) $R(x) = \frac{2x+1}{x^2-16}$

Holes: None
Zeros: $x = -1/2$
V.A: $x = 4, x = -4$
H.A: $y = 0$



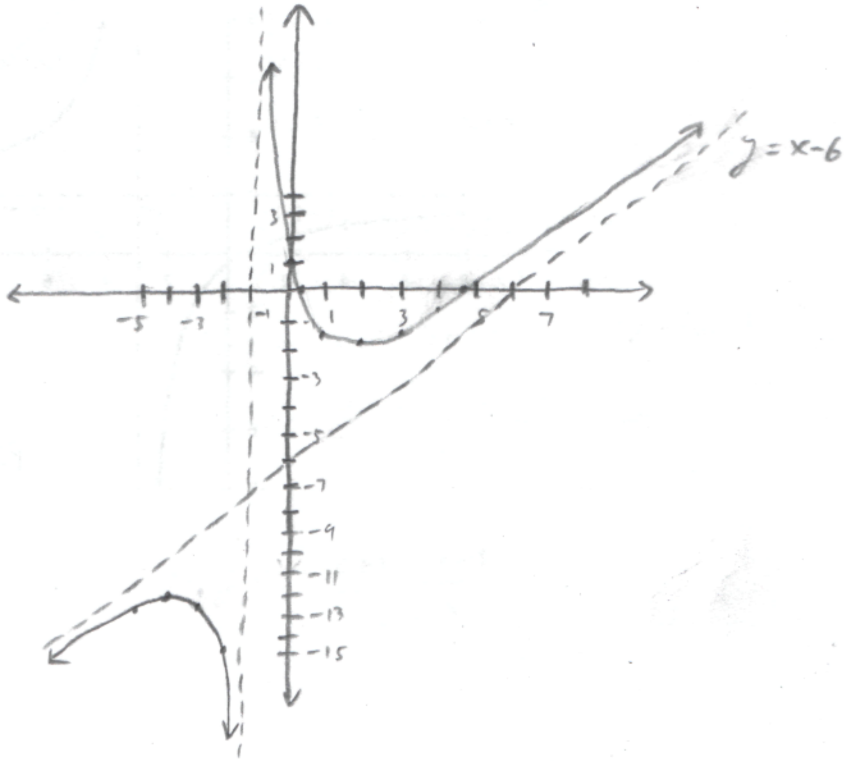
(C) $L(x) = \frac{x^2-5x+1}{x+1}$

Holes: None

Zeros: $x = \frac{5 \pm \sqrt{21}}{2}$

V.A.: $x = -1$

S.A.: $y = x - 6$



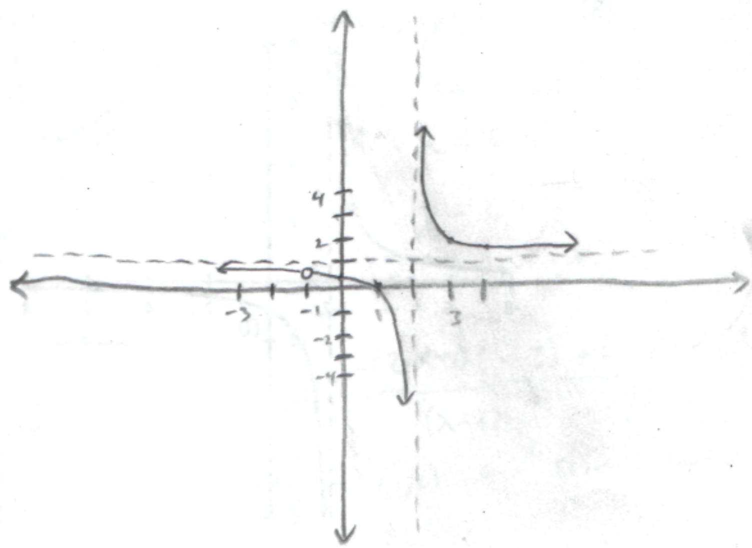
(D) $F(x) = \frac{x^2-1}{x^2-x-2}$

Holes: $x = -1$

Zeros: $x = 1$

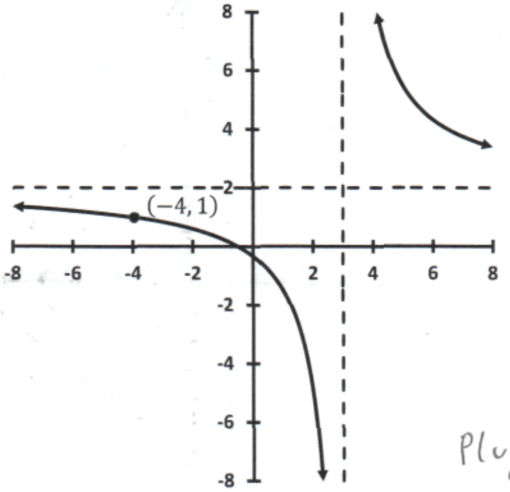
V.A.: $x = 2$

H.A.: $y = 1$



4.6.4 Find the equation of each rational function graphed below.

(A)



Holes: None.

Zeros: Unclear where is

V.A: $x=3$

H.A: $y=2$

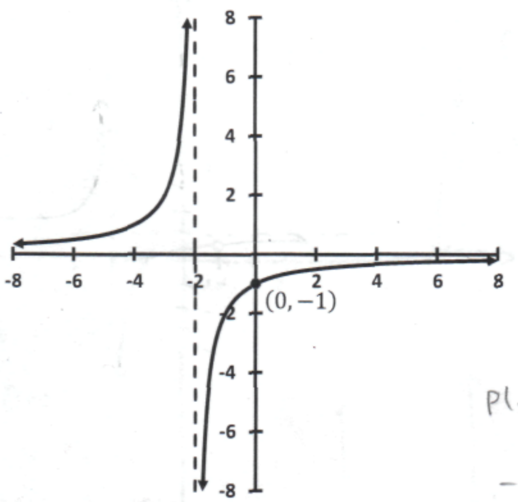
$$y = \frac{2x+b}{(x-3)}$$

Plug in point to solve for b.

$$1 = \frac{2(-4)+b}{(-4-3)} \rightarrow b=1$$

$$y = \frac{2x+1}{x-3}$$

(B)



Holes: None

Zeros: None

V.A: $x=-2$

H.A: $y=0$

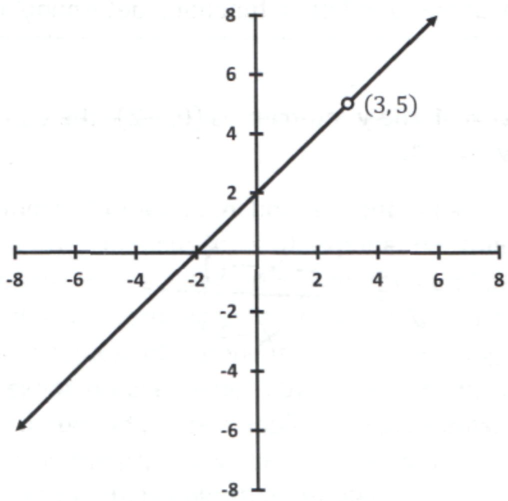
$$y = \frac{b}{(x+2)}$$

Plug in point to solve for b.

$$-1 = \frac{b}{0+2} \rightarrow b=-2$$

$$y = \frac{-2}{x+2}$$

(C)



Holes: at $x=3$

Zeros: $x=-2$

V.A: None

S.A: $y=x+2$

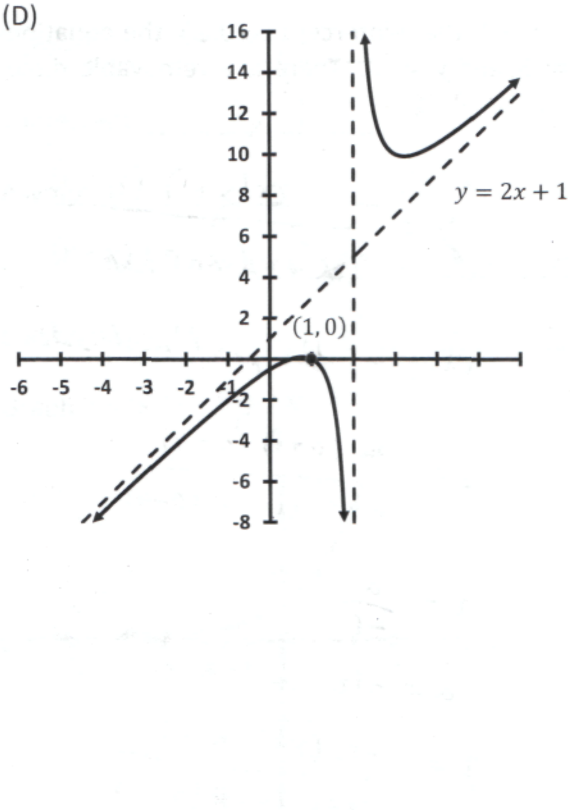
$$y=\frac{(x+2)(x-3)+b}{(x-3)}$$

Plug in point $(0,2)$ and solve for b

$$2=\frac{(0+2)(0-3)+b}{(0-3)} \rightarrow b=0$$

$$y=\frac{(x+2)(x-3)}{x-3}$$

(D)



Holes: None

Zeros: $x=1$

V.A: $x=2$

S.A: $y=2x+1$

$$y=\frac{(2x+1)(x-2)+b}{(x-2)}$$

Plug in $(1,0)$ and solve for b.

$$0=\frac{(2(1)+1)(1-2)+b}{(1-2)} \rightarrow b=3$$

$$y=\frac{(2x+1)(x-2)+3}{x-2}=2x+1+\frac{3}{x-2}$$

4.6.5 Given the following information about a rational function, determine its equation. Check your solutions.

(A) The zero of the function is $x = 4$, the y-intercept is $(0, -2)$, the equations of the asymptotes are $x = 2$ and $y = -1$.

Holes: None

Zeros: $x=4$

V.A: $x=2$

H.A: $y=-1$

Point: $(0,-2)$

$$y = \frac{-(x-4)}{(x-2)}$$

(B) The zero of the function is $x = -1$, the y-intercept is $(0, 2)$, the equations of the asymptotes are $x = -2, x = 3$, and $y = 0$. There is a removable discontinuity (hole) at $x = 4$.

Holes: $x=4$

Zero: $x=-1$

V.A: $x=-2$, $x=3$

H.A: $y=0$

Point: $(0,2)$

$$y = \frac{a(x+1)(x-4)}{(x+2)(x-3)(x-4)}$$

Plug in the point and solve for a

$$2 = \frac{a(0+1)(0-4)}{(0+2)(0-3)(0-4)}$$

$$2 = \frac{a}{-6}$$

$$a = -12$$

$$y = \frac{-12(x+1)(x-4)}{(x+2)(x-3)(x-4)}$$