Objectives

- Identify polynomial functions.
- Sketch graphs of power functions.
- Determine the end behavior of polynomials from the leading term property.
- Given the graph of a polynomial, determine the possible degree of the polynomial, the constant coefficient, and sign of the leading coefficient.
- Determine the intercepts of the graph of a polynomial function.
- Given a polynomial function in factored form, determine the zeros and their multiplicities.
- Sketch the graph of a polynomial function.
- Determine a possible equation of a polynomial function given its graph.

Preliminaries

A **polynomial function** is a function of the form:

$$f(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$$

where $a_0, a_1, a_2, \ldots, a_n$ are real numbers and n is a non-negative integer.

The **degree** of the polynomial is n. if $a_n \neq 0$.

$a_0, a_1, a_2, \cdots, a_n$ are called the coefficients.

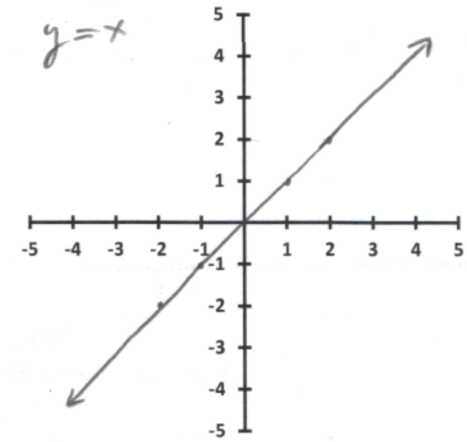
a_n is called the leading coefficient.

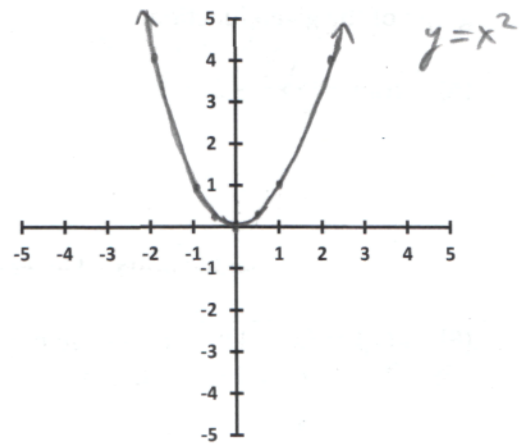
$a_n x^n$ is called the leading term.

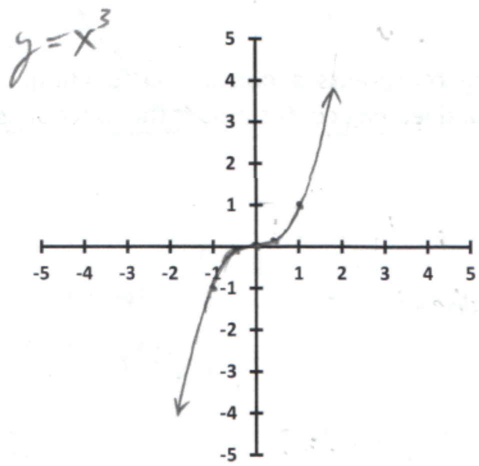
a_0 is called the constant coefficient. (y-intercept)

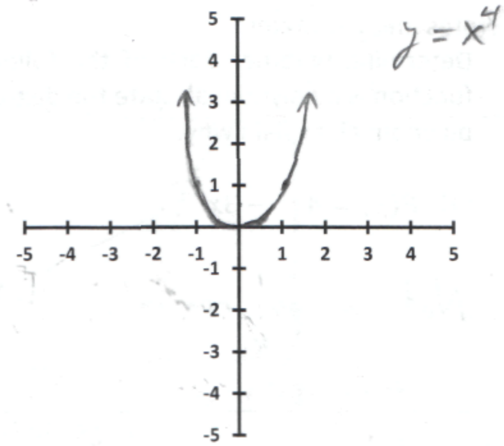
The domain of every polynomial function is $(-\infty, \infty)$.

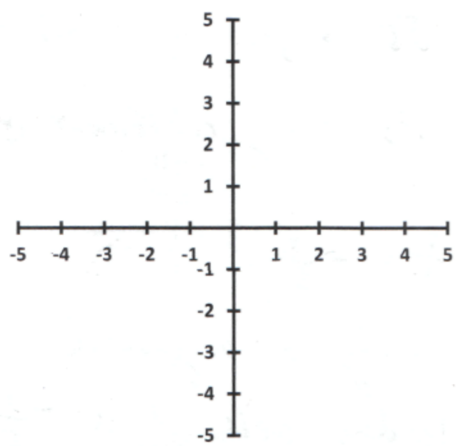
Sketch the graphs of $y = x^n$ for $n = 1, 2, 3, 4,$ and 5.











Warm-up

1. Determine the transformations that are performed on a base function and sketch a graph of the given function.

   (A) $a(x) = -x^2 + 5$

   (B) $c(x) = (x-4)^3 - 1$

Class Notes and Examples

4.3.1 Determine whether each of the following represents a polynomial function. If the function is a polynomial, state the degree and leading coefficient. If the function is not a polynomial, explain why.

(A) $P(x) = 4x^2 - 3x^{-1}$

Not a polynomial. Exponents must be non-negative integers $(0, 1, 2, 3, \ldots)$

(B) $Q(x) = \sqrt{5}x^4 - 4x^3 + 6$

Yes it is a polynomial. $\sqrt{5}$ is just a number.

Its degree is 4 and its leading coefficient is $\sqrt{5}$

(C) $R(x) = \frac{7+2x^2-3x^4}{5} = -\frac{3}{5}x^4 + \frac{2}{5}x^2 + \frac{7}{5}$

After dividing by 5, it is clearly a polynomial

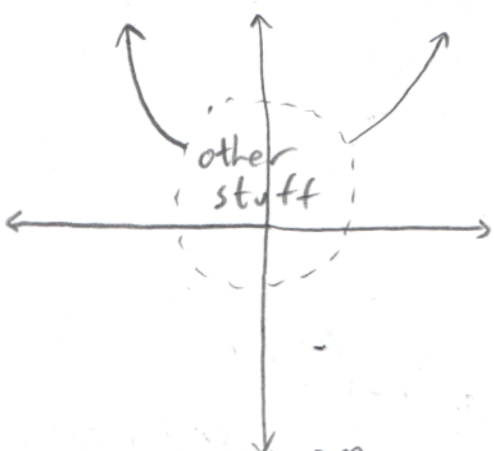
Its degree is 4 and its leading coefficient is $-\frac{3}{5}$.

❓ How can you determine the end behavior of a polynomial function of the form $f(x) = \underbrace{a_n x^n} + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$?

By the leading term. You note whether the degree, n, is even or odd and whether the leading coefficient, a_n, is positive or negative.

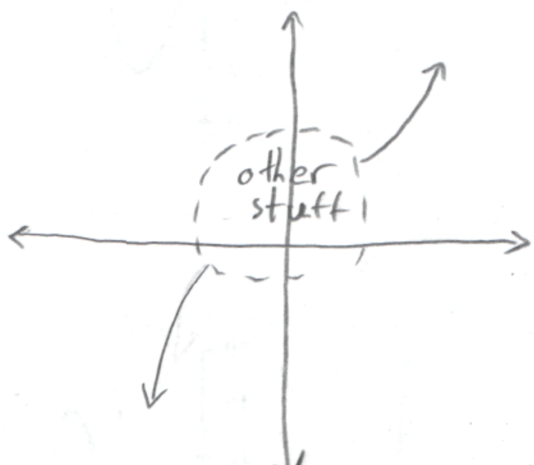
Sketch the four possible end behaviors for a polynomial function below.

Even degree
Positive leading coefficient e.g. $y = x^2$



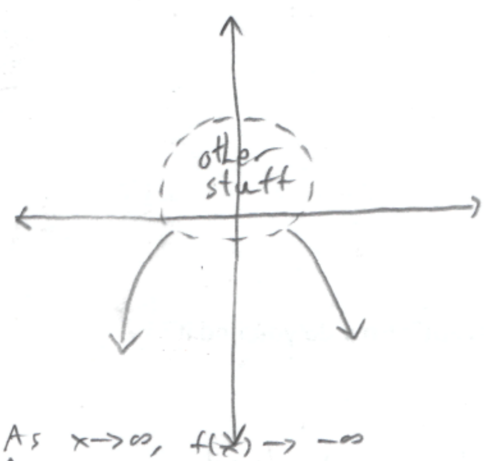
As $x \to \infty$, $f(x) \to \infty$
As $x \to -\infty$, $f(x) \to \infty$

Odd degree
Positive leading coefficient e.g. $y = x^3$



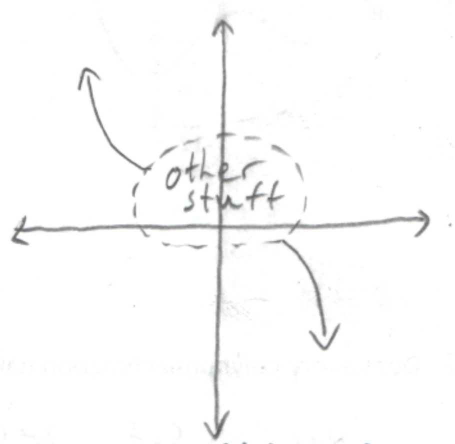
As $x \to \infty$, $f(x) \to \infty$
As $x \to -\infty$, $f(x) \to -\infty$

Even degree
Negative leading coefficient e.g. $y = -x^2$



As $x \to \infty$, $f(x) \to -\infty$
As $x \to -\infty$, $f(x) \to -\infty$

Odd degree
Negative leading coefficient e.g. $y = -x^3$



As $x \to \infty$, $f(x) \to -\infty$
As $x \to -\infty$ $f(x) \to \infty$

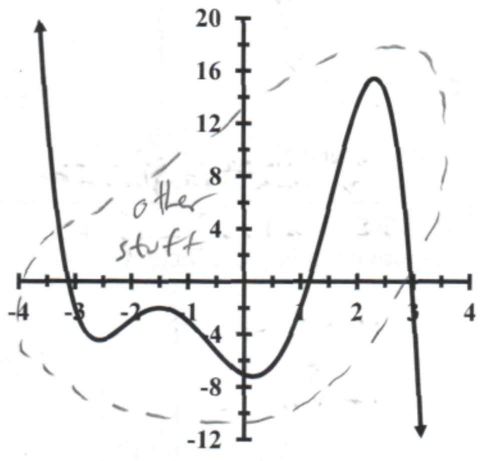
The "other stuff" consists of smooth continuous bumps.

4.3.3 Consider the following graphs of polynomial functions. Determine whether the leading coefficient is positive or negative and whether the degree is even or odd.

(A)

The degree is even
The leading coefficient is positive

(B)



The degree is odd.
The leading coefficient is negative.

❓ Does every polynomial function have a y-intercept? How do you find it?

Yes. Set $x=0$.
The y-intercept is just the constant term.

? How do you find the zeros of a polynomial function?

Factor (possibly by grouping) and see where each factor equals zero.

4.3.4 Determine the intercepts of each polynomial by factoring.

(A) $f(x) = x^3 - 5x^2 - 4x + 20$

x-intercepts: Set f(x) equal to zero. Factor by grouping.

$x^3 - 5x^2 - 4x + 20 = 0$

$x^2(x-5) - 4(x-5) = 0$

$(x^2-4)(x-5) = 0$

$(x+2)(x-2)(x-5) = 0$

$x = -2, 2,$ or 5

The x-intercepts are $(-2,0)$, $(2,0)$, and $(5,0)$.

y-intercepts: set x equal to zero. $f(0) = 0^3 - 5(0)^2 - 4(0) + 20 = 20$

The y-intercept is $(0,20)$.

(B) $N(t) = t^4 - 18t^2 + 81$

x-intercepts: Set N(t) equal to zero. Factor

$t^4 - 18t^2 + 81 = 0$

$(t^2-9)(t^2-9) = 0$

$(t+3)(t-3)(t+3)(t-3) = 0$

$(t+3)^2(t-3)^2 = 0$

$t = -3$ or 3

The x-intercepts are $(-3,0)$ and $(3,0)$

y-intercept: Set x equal to 0. $N(0) = 0^4 - 18(0)^2 + 81 = 81$

The y-intercept is $(0,81)$.

? What is meant by the **multiplicity** of a zero?

The number of times the factor corresponding to the zero appears in the factorization of the polynomial.

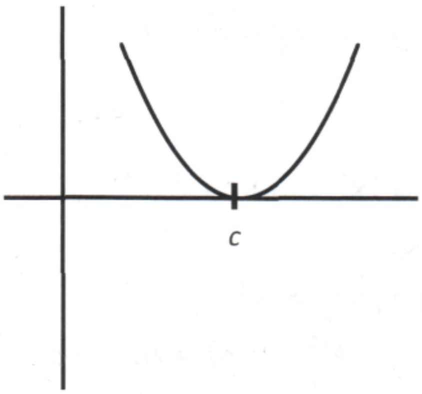
$$f(x) = (x-1)^2(x+1)(x+2)^3 = \underbrace{(x-1)(x-1)}_{2\ \text{twice}}\underbrace{(x+1)}_{\text{once}}\underbrace{(x+2)(x+2)(x+2)}_{\text{thrice}}$$

Zeros	Multiplicity
1	2
-1	1
-2	3

If c is a real zero of a polynomial function f, then $x = c$ is a **zero of multiplicity** $\boldsymbol{k}$ if the factor $(x-c)$ appears k times in the factorization of f.

Note: There are two general cases for the shape of the graph of a polynomial near a zero $x = c$.

When $x = c$ is a zero with **even multiplicity**, then the shape of the graph near $x = c$ is either:



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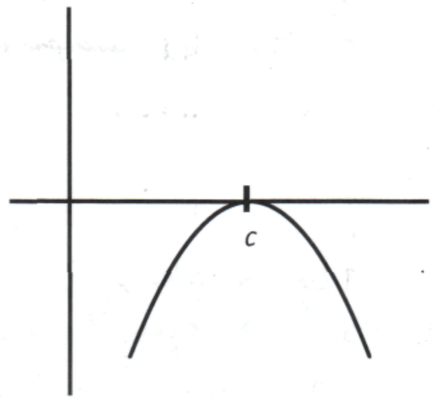
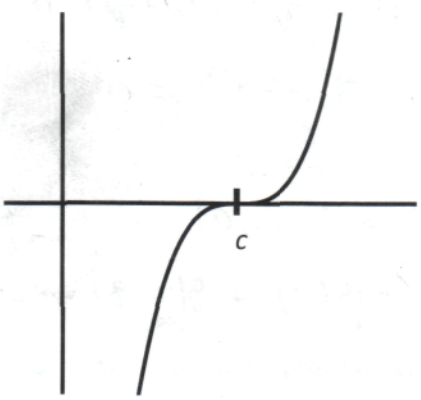


Even multiplicity: Bounces when hits axis

Higher multiplicity: Flatter when hits axis

(4, 6, 8, ...) When $x = c$ is a zero with **odd multiplicity**, then the shape of the graph near $x = c$ is either:



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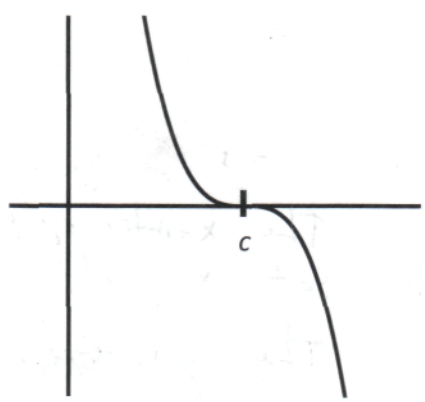
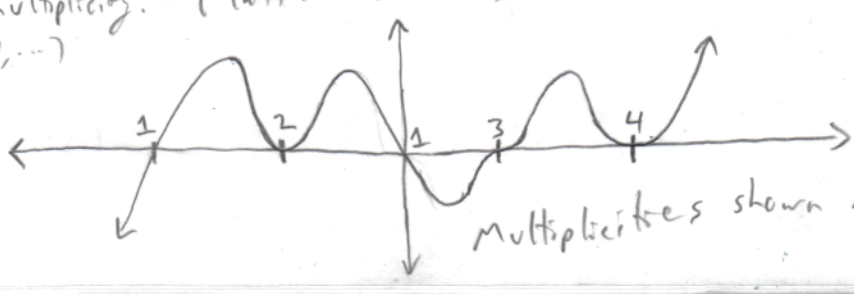


Odd multiplicity: Crosses when hits axis.

Higher multiplicity: Flatter when hits axis.

(3, 5, 7, ...)



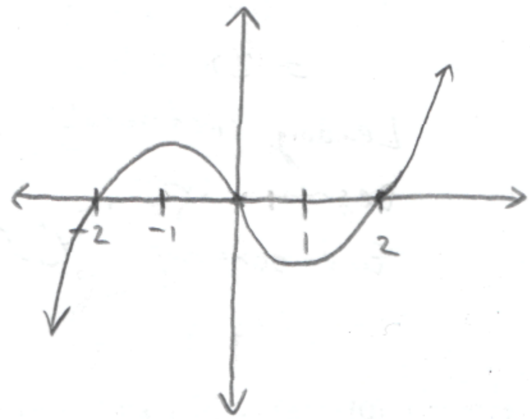
Multiplicities shown next to zeros.

? How can we graph a polynomial function by hand?

4.3.5 Determine the zeros and their multiplicities for the following polynomial functions. Using the information about end behavior, zeros, and multiplicities, sketch a graph of each by hand.

(A) $M(x) = x^3 - 4x$ $= x(x^2-4) = x(x+2)(x-2)$

Zeros	Multiplicity
0	1
-2	1
2	1



$M(x) = x^3 - 4x$

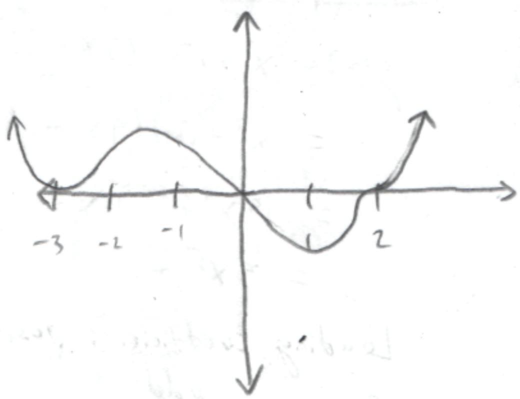
Leading coefficient: positive.

Degree: odd.

End Behavior:

(B) $R(x) = x(x+3)^2(x-2)^3$ $= (x-0)(x+3)^2(x-2)^3$

Zeros	Multiplicity
0	1
-3	2
2	3



◌ means something of a small degree

$R(x) = x(x+◌)^2(x-◌)^3$

$= x(x^2+◌)(x^3-◌)$

$= x^6 + ◌$

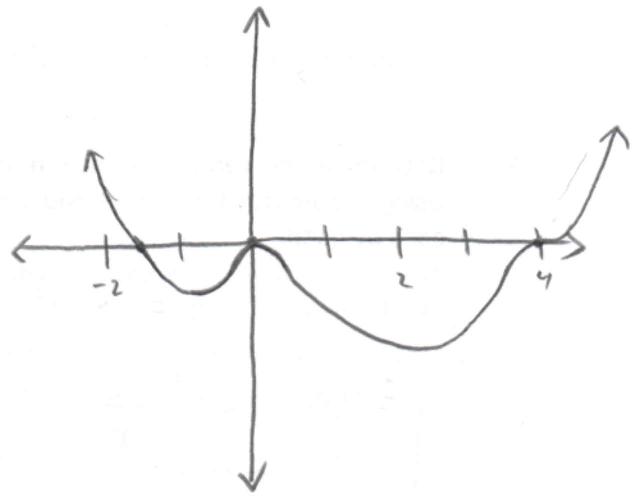
Leading coefficient: positive

Degree: even

End Behavior:

(C) $f(x) = x^2(2x+3)(x-4)^3 = 2\,(x-0)^2\left(x+\frac{3}{2}\right)(x-4)^3$

Zeros	Multiplicity
0	2
$-3/2$	1
4	3



$f(x) = x^2\,(2x+\bigcirc)\,(x-\bigcirc)^3$

$= (2x^3+\bigcirc)\,(x^3-\bigcirc)$

$= 2x^6 + \bigcirc$

Leading coefficient: positive

Degree: even

End behavior:

(D) $g(x) = (x^2+1)(3x-5)^2(x+1) = (x^2+1)\left(3\left(x-5/3\right)\right)^2(x+1)$

$= (x^2+1)\;3^2\,(x-5/3)^2\,(x+1)$

$= 9(x^2+1)\,(x-5/3)^2\,(x+1)$

Zeros	Multiplicity
$5/3$	2
-1	1

$g(x) = (x^2+\bigcirc)(3x-\bigcirc)^2(x+\bigcirc)$

$= (x^2+\bigcirc)(9x^2+\bigcirc)(x+\bigcirc)$

$= (9x^4+\bigcirc)(x+\bigcirc)$

$= 9x^5+\bigcirc$

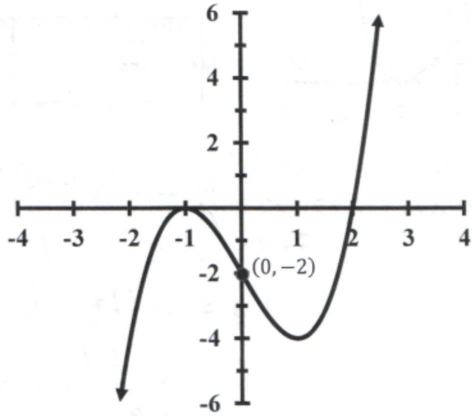
Leading coefficient: positive

Degree: odd

End behavior:

4.3.7 Determine a possible equation for the polynomial functions graphed below. Verify by graphing.

(A)



Zeros	Mult
−1	2
2	1

$y = a(x+1)^2(x-2)$

Then plug in a point and solve for a.

$-2 = a(0+1)^2(0-2)$

$-2 = a(1)^2(-2)$

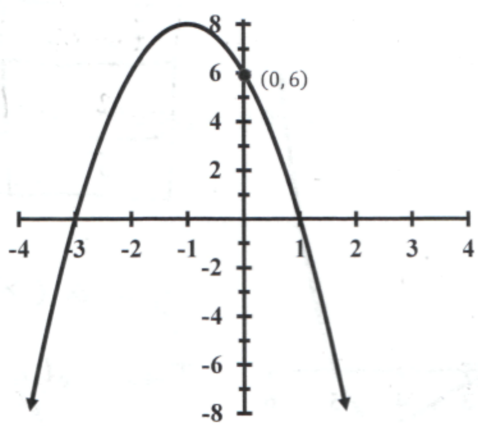
$\frac{-2}{-2} = \frac{-2a}{-2}$

$1 = a$

$y = 1(x+1)^2(x-2)$

$y = (x+1)^2(x-2)$

(B)



Zeros	Multiplicity
−3	1
1	1

$y = a(x+3)(x-1)$

Then plug in a point and solve for a.

$6 = a(0+3)(0-1)$

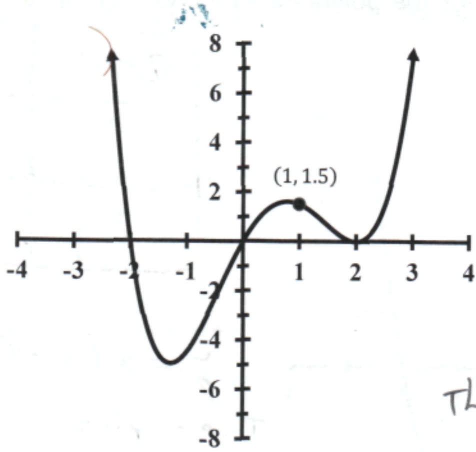
$6 = a(3)(-1)$

$\frac{6}{-3} = \frac{-3a}{-3}$

$-2 = a$

$y = -2(x+3)(x-1)$

(C)



Zeros	Multiplicity
-2	1
0	1
2	2

$y = a(x+2)(x-0)(x-2)^2$

Then plug in a point and solve for a

$1.5 = a(1+2)(1-0)(1-2)^2$

$1.5 = a(3)(1)(-1)^2$

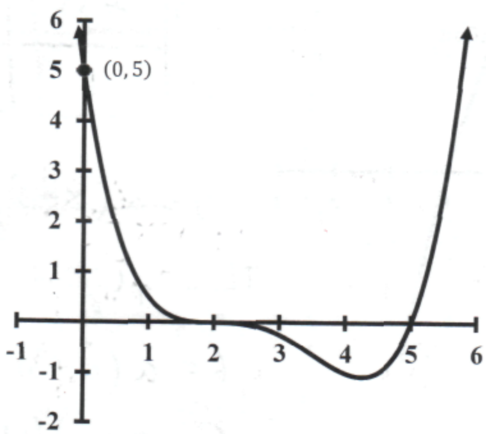
$\frac{1.5}{3} = \frac{3a}{3}$

$1/2 = a$

$y = \frac{1}{2}(x+2)(x-0)(x-2)^2$

$y = \frac{1}{2}(x+2)x(x-2)^2$

(D)



Zeros	Multiplicity
2	3
5	1

$y = a(x-2)^3(x-5)$

Then plug in a point and solve for a.

$5 = a(0-2)^3(0-5)$

$5 = a(-2)^3(-5)$

$5 = a(-8)(-5)$

$\frac{5}{40} = \frac{40a}{40}$

$\frac{1}{8} = a$

$y = \frac{1}{8}(x-2)^3(x-5)$