Objectives

- Given a function represented by a graph, an equation, a table of values, or a verbal description, determine whether the function is one-to-one.
- Verify algebraically that two functions are inverse functions by finding the composition of the two functions.
- Given the graph of a one-to-one function, identify the domain and range of the inverse function, and sketch its graph.
- Given the domain and range of a one-to-one function, or given a one-to-one function for which you can identify the domain and range, determine the domain and range of the inverse function.
- Find the inverse function of a one-to-one function represented by a graph, an equation, or a table of values.

Preliminaries

Sketch the graph of each pair of functions on the same set of axes. Indicate domain, range, and intercepts.

1. $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$

2. $f(x) = 2x + 1$ and $g(x) = \frac{1}{2}(x - 1)$

Warm-up

1. If $f(x) = 3x + 1$ and $g(x) = \frac{x-1}{3}$, find $f(g(x))$ and $g(f(x))$.

2. Solve for y: $x = \frac{y+2}{5y}$

? What is meant by a one-to-one function? A function f is **one-to-one** if for any two values a and b in the domain of f, where $a \neq b$, $f(a) \neq f(b)$.

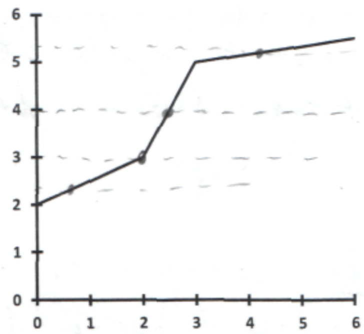
? In practice, this means that, in order for a function to be one-to-one, two different input values must yield different outputs.

? To determine whether a graph represents a one-to-one function, we use the horizontal line test.

? Any horizontal line can cross the graph of a one-to-one function at most how many times? One time.

3.6.1 Determine whether each function described below represents a one-to-one function.

(A) The function given by this graph:



One-to-one

Every horizontal line intersects the graph at most once.

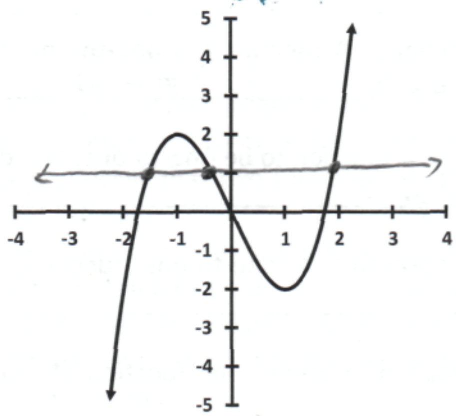
(B) $C(w)$ is the cost in cents of mailing a package that weighs w grams.

For certain weight ranges, the cost is fixed.
So the function is not one-to-one.

(C) g is the function that assigns to each UA student his/her NetID.

No, two $\underbrace{\text{students}}_{\text{input}}$ have the same $\underbrace{\text{NetID}}_{\text{output}}$.
So g is one-to-one.

(D) The function given by this graph:



The horizontal line shown intersects the graph more than once.

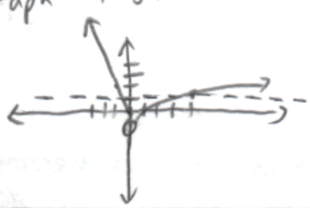
So the function is not one-to-one.

(E) S is the function that assigns to each UA student the last four digits of his/her social security number.

Not one-to-one.
Students may have the last 4 digits of their SSN's be the same even if their SSN's are different.

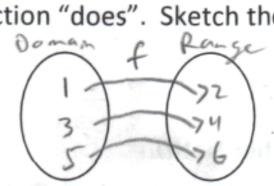
(F) $R(x) = \begin{cases} -2x & x \leq 0 \\ \sqrt{x} - 1 & x > 0 \end{cases}$

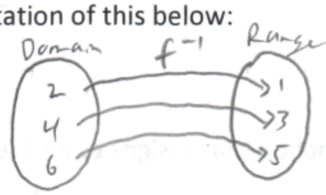
Graph first.



Not one-to-one.
The dashed horizontal line intersects the graph more than once.
Also note. $f(-1/2) = (-2)(-1/2) = 1 = 2-1 = \sqrt{4}-1 = f(4)$.
So $-1/2$ and 4 have the same output.

A one-to-one function has an **inverse function**. The inverse function of f is called "f inverse" and is denoted f^{-1}. Loosely speaking, an inverse function "undoes" what the function "does". Sketch the set representation of this below:





What does this tell us about the composition of a function and its inverse?

$f(f^{-1}(x)) = f^{-1}(f(x)) =$ ____ x ____.

? How do we find the inverse function for a one-to-one function algebraically?

Replace $f(x)$ by y
Solve for x
Replace x by $f^{-1}(y)$
If there is no context, replace y by x.

? How are the domain and range of the inverse function related to the domain and range of the inverse?

Domain of f is range of f^{-1}
Range of f is domain of f.

? How can you draw the graph of the inverse function, given only the graph of the original function?

Reflect it over the line $y=x$.
This is equivalent switching the x and y coordinates of each point.
For example, if $(1,2)$ is on the graph of f,
then $(2,1)$ is on the graph of f^{-1}.

3.6.2 Find the inverse of each one-to-one function.
(A) $f(x) = 7x^3 + 1$

Replace $f(x)$ by y.
Solve for x.

$$y = 7x^3 + 1$$
$$\underline{-1 \qquad -1}$$
$$\frac{y-1}{7} = \frac{7x^3}{7}$$
$$\sqrt[3]{\frac{y-1}{7}} = \sqrt[3]{x^3}$$
$$x = \sqrt[3]{\frac{y-1}{7}}$$

Replace x by $f^{-1}(y)$. $f^{-1}(y) = \sqrt[3]{\frac{y-1}{7}}$
If there is no context, we then replace y by x, because y is our new input and x usually denotes inputs.

$$f^{-1}(x) = \sqrt[3]{\frac{x-1}{7}}$$

(B) $f(x) = \frac{x-1}{2x+3}$

Replace $f(x)$ by y. $y = \frac{x-1}{2x+3}$

Solve for x.

$(2x+3)y = x-1$

$2xy + 3y = x - 1$
$-x \quad -x$

$2xy - x + 3y = -1$
$-3y \quad -3y$

$2xy - x = -3y - 1$

Factor out an x.

$\frac{x(2y-1)}{2y-1} = \frac{-3y-1}{2y-1}$

$x = \frac{-3y-1}{2y-1}$

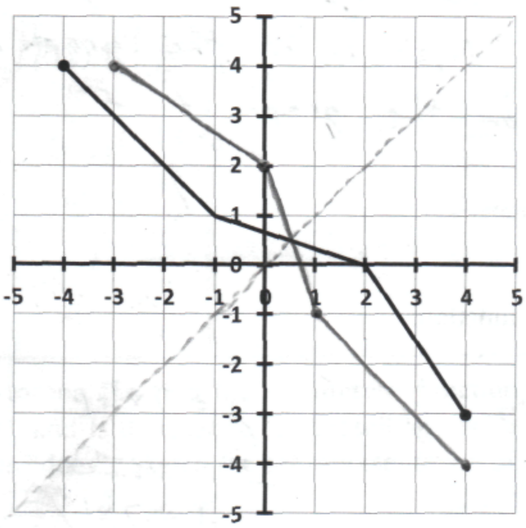
Replace x by $f^{-1}(y)$

$f^{-1}(y) = \frac{-3y-1}{2y-1}$

Finally replace y by x.

$f^{-1}(x) = \frac{-3x-1}{2x-1}$

(C) The function f whose graph is shown below.



Flip x and y values.
The point $(4,-3)$ goes to $(-3,4)$
Note the symmetry about the dotted line $y = x$.

(D) The function f described by the set of ordered pairs below:

n	3	−4	1	2
$f(n)$	2	6	−1	0

Flip inputs and outputs.

n	2	6	−1	0
$f^{-1}(n)$	3	−4	1	2

3.6.3 For each of the following functions, determine the value of $f^{-1}(2)$.

(A) $f(x) = 7x^3 + 1$

For what x-value is $f(x) = 2$?

$2 = 7x^3 + 1$. Solve for x. $x = \sqrt[3]{1/7}$. $f(\sqrt[3]{1/7}) = 2$.

So $f^{-1}(2) = \sqrt[3]{1/7}$

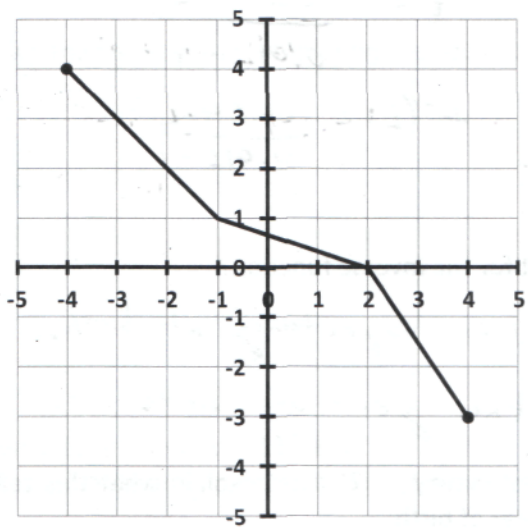
(B) $f(x) = \frac{x-1}{2x+3}$

For what x-value is $f(x) = 2$?

$2 = \frac{x-1}{2x+3}$. Solve for x. $x = -\frac{7}{3}$. $f(-7/3) = 2$.

So $f^{-1}(2) = -7/3$.

(C) The function f whose graph is shown below.



For what x-value is $f(x) = 2$?

For $x = -2$, $f(-2) = 2$

So $f^{-1}(2) = -2$.

(D) The function f described by the set of ordered pairs below:

n	3	-4	1	2
$f(n)$	2	6	-1	0

For what n-value is $f(n) = 2$?

For $n = 3$, $f(3) = 2$.

So $f^{-1}(2) = 3$

3.6.4 The life expectancy, L, of a child (at birth) can be modeled approximately by the formula $L = f(t) = \frac{t+66.94}{0.01t+1}$, where t is the year of birth, with $t = 0$ corresponding to the year 1950.

(A) Explain how you know that this function is one-to-one, and find the inverse function.

If we graph the function, we can see it passes the vertical line test.

Replace $f(t)$ by L. $\quad L = \frac{t+66.94}{0.01t+1}$

Solve for t. $\quad L(0.01t+1) = t + 66.94$

$$0.01Lt + L = t + 66.94$$
$$-L-t \quad -L-t$$
$$0.01Lt - t = 66.94 - L$$

Factor out t $\quad t(0.01L-1) = 66.94 - L$

$$t = \frac{66.94 - L}{0.01L - 1}$$

Replace t by $f^{-1}(L)$. $\quad f^{-1}(L) = \frac{66.94-L}{0.01L-1}$

(B) What are the input and output for the inverse function?

The input is L, life expectancy at birth.

The output is t, the year of birth.

(C) Use the inverse function to estimate $f^{-1}(70)$ and explain what this tells you in terms of life expectancy and year of birth.

$f^{-1}(70) = 10.2$

$L = 70$ years life expectancy.

$t = f^{-1}(70) = 10.2$ is the year of birth.

This tells you that people who had a life expectancy of 70 years were born 10.2 years after 1950. That is, they were born 0.2 years into the year 1960.