Analysis Qualifying Exam Notes

Dominated Convergence Theorem: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions such that $f_n \to f$ a.e. and such that for some $g \in L^1$, $|f_n| \leq g$ a.e. for all n. Then $\lim_{n\to\infty} \int f_n = \int f$.

Fatou's Lemma: If $\{f_n\}_{n=1}^{\infty}$ is a sequence in L^+, then $\int(\liminf f_n) \leq \liminf \int f_n$.

Theorem: Let f be a bounded real-valued function on $[a, b]$.
f is Riemann integrable iff $\{x \in [a, b] : f \text{ is discontinuous at } x\}$ has Lebesgue measure zero.

Theorem: Let f be a bounded real-valued function on $[a, b]$.
If f is Riemann integrable, then f is Lebesgue measurable and $\int_a^b f(x)dx = \int_{[a,b]} f dm$.

Convergence theorems:

1. Convergence in L^1 implies convergence in measure.
2. If $\{f_n\}_{n=1}^{\infty}$ converges to f in L^1, then there is a subsequence $\{f_{n_j}\}_{j=1}^{\infty}$ that converges to f a.e.

Definition of Regularity: A Borel measure μ is said to be regular if for all Borel sets E

1. $\mu(E) = \inf\{\mu(U) : U \supseteq E, U \text{ is open.}\}$
2. $\mu(E) = \sup\{\mu(K) : K \subseteq E, K \text{ is compact.}\}$

Lebesgue measure is regular.

Fubini-Tonelli theorem: Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measures spaces.

Tonelli: If $f \in L^+(X \times Y)$, then $\int f d(\mu \times \nu) = \int \int f d\mu d\nu = \int \int f d\nu d\mu$

Fubini: If $f \in L^1(\mu \times \nu)$, then $\int f d(\mu \times \nu) = \int \int f d\mu d\nu = \int \int f d\nu d\mu$

Hölder's inequality: Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f, g are measurable functions, then $\|fg\|_1 \leq \|f\|_p \|g\|_q$.

Generalized Hölder's inequality: Let $1 \leq p_1, ..., p_n \leq \infty$ and $\sum_{j=1}^{n} \frac{1}{p_j} = \frac{1}{r} \leq 1$. If $f_1, ..., f_n$ are measurable, then $\| \prod_{j=1}^{n} f_j \|_r \leq \prod_{j=1}^{n} \|f_j\|_{p_j}$

Jensen's inequality: Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) = 1$.
If $g : X \to (a, b)$ is integrable and F is convex on (a, b), then $F(\int g d\mu) \leq \int (F \circ g) d\mu$

Minowski's inequality for integrals: Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measures spaces, and let f be an $(\mathcal{M} \otimes \mathcal{N})$-measurable function on $X \times Y$. Let $1 \leq p < \infty$.
If $f \geq 0$ or $f(\cdot, y) \in L^p(\mu)$ for a.e. y, then $\| \int f(x, y) d\nu(y) \|_p \leq \int \|f(x, y)\|_p d\nu(y)$.

That is, $[\int |\int f(x,y)d\nu(y)|^p d\mu(x)]^{1/p} \leq \int [\int |f(x,y)|^p d\mu(x)]^{1/p} d\nu(y)$

Young's inequality: If $f \in L^1$ and $g \in L^p$ where $1 \leq p \leq \infty$, then $f * g$ exists a.e., $f * g \in L^p$, and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.

Chebyshev's inequality: If $f \in L^p$ $(0 < p < \infty)$ and $\alpha > 0$, then $\alpha^p \mu(\{x : |f(x)| > \alpha\}) \leq \|f\|_p^p$

FTC for Lebesgue integrals: If $F : [a,b] \to \mathbb{C}$, then the following are equivalent:

1. F is absolutely continuous on $[a,b]$.
2. $F(x) - F(a) = \int_a^x f(t)dt$ for some $f \in L^1([a,b])$.
3. F is differentiable a.e. on $[a,b]$, $F' \in L^1([a,b])$, and $F(x) - F(a) = \int_a^x F'(t)dt$.

FTC for Riemann integrals:

1. If f is continuous on $[a,b]$ and $F(x) = \int_a^x f(t)dt$, then $F'(x) = f(x)$.
2. If $F'(x) = f(x)$ on $[a,b]$ and f is Riemann integrable, then $F(x) - F(a) = \int_a^x f(t)dt$.

Lebesgue-Radon-Nikodym theorem: Let ν be a σ-finite signed measure and μ a σ-finite measure on $(X, \mathcal{M})$. Then there exist unique σ-finite signed measures λ, ρ on $(X,)$ such that $\nu = \lambda + \rho$ with $\lambda \perp \mu$ and $\rho \ll \mu$. Moreover, there exists a measurable function $f : X \to \mathbb{R}$ unique μ-a.e. such that $\rho = f\mu$.

Lebesgue differentiation theorem: If $f \in L^1_{loc}$ and $\{E_r\}_{r>0}$ is a family that shrinks nicely to x, then $\lim_{r \to 0^+} \frac{\int_{E_r} f(y)dy}{m(E_r)} = f(x)$ a.e.

Proposition: If $f \in L^1(m)$ then $F(x) = \int_{-\infty}^x f(t)dt$ is NBV, absolutely continuous, and $f = F'$ a.e. If F is NBV and absolutely continuous, then $F' \in L^1(m)$ and $F(x) = \int_{-\infty}^x F'(t)dt$.

Arszela-Ascoli theorem: Let X be a compact Hausdorff space. If $\mathcal{F} \subseteq C(X)$ is pointwise bounded and equicontinuous, then the closure of $\mathcal{F}$ in the uniform metric is compact.

The Stone-Weierstrass theorem: Let X be a compact Hausdorff space. If B is a subalgebra of $C(X, \mathbb{R})$ that separates points and if there is no $x_0 \in X$ such that $f(x_0) = 0$ for all $f \in B$, then B is dense in $C(X, \mathbb{R})$.

The Uniform Boundedness Principle: Let X be a Banach space, Y be a normed vector space, and A be a subset of $L(X,Y)$, the set of all bounded linear functions from X to Y. If $\sup\{\|Tx\| : T \in A\} < \infty$ for all $x \in X$, then $\sup\{\|T\| : T \in A\} < \infty$.

Definitions for Fourier Analysis:

1. Convolution: $f * g(x) = \int f(x-y)g(y)dy$
2. Fourier Transform: $\hat{f}(w) = \int f(x)e^{-2\pi iwx}dx$
3. Fourier Series: $f = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi inx} = \sum_{n\in\mathbb{Z}} \big(\int f(x)e^{-2\pi inx}dx\big)e^{2\pi inx}$.
   (The equality is in the L^2 norm. That is, the Fourier series converges to f in L^2.)
4. The inverse of a Fourier Transform: $\check{f}(x) = \hat{f}(-x) = \int f(w)e^{2\pi iwx}dx$

Theorems for Fourier Analysis:

1. Convolutions associate and commute
2. $\widehat{f * g} = \hat{f}\hat{g}$
3. The Fourier inversion theorem: If $f \in L^1$ and $\hat{f} \in L^1$, then f agrees a.e. with a continuous function f_0, and $f_0 = \check{\hat{f}} = \hat{\check{f}}$.

Riemann-Lebesgue lemma: If $f \in L^1(\mathbb{R}^n)$, then $\hat{f} \in C_0(\mathbb{R}^n)$.

Parseval's Identity: If $\{U_\alpha\}_{\alpha\in A}$ is an orthonormal basis of a Hilbert space $\mathcal{H}$, then for all $x \in \mathcal{H}$, $\|x\|^2 = \sum_{\alpha\in A} |\langle x, u_\alpha\rangle|^2$. (If $\mathcal{H} = L^2([a,b], dx)$, then $\{e^{2\pi inx}\}_{n\in\mathbb{Z}}$ is a possible basis.)

Theorem: Let $\phi \in L^1(\mathbb{R}, m)$, $\int \phi(x)dx = a$, and $\phi_t(x) = \frac{1}{t}\phi(\frac{x}{t})$.
If $f \in L^p$ $(1 \leq p < \infty)$, then $f * \phi_t \to af$ in L^p as $t \to 0$.

Theorem: Let p, q be conjugate exponents. If $\phi \in (L^p)^*$ there exists a $g \in L^q$ such that $\phi(f) = \int fg$ for all $f \in L^p$. Hence, L^q is isometrically isomorphic to $(L^p)^*$. The same holds for $p = 1$ provided μ is σ-finite.

Corollary: For $1 < p < \infty$, L^p is reflexive.

Theorem: For $1 \leq p < \infty$, L^p is a Banach space.

Theorem:

1. A reflexive Banach space is separable if and only if its dual is separable.
2. If the dual of a normed vector space is separable, then so is the space itself.
   The converse is not true: For example, the space l^1 is separable, but its dual is l^∞ is not.