

Math 432 practice problems

Problem 1: Consider $H = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 1\}$ with the subspace topology. This is frequently called the hyperboloid of one sheet (picture drawn in class). Is H simply or multiply connected? Prove your answer.

Problem 2: If A is a countable subset of \mathbb{R}^2 , show that $\mathbb{R}^2 - A$ is path connected. Hint: How many lines pass through a point in \mathbb{R}^2 ?

Problem 3: Suppose X is a compact space and $f : X \rightarrow \mathbb{R}$ is continuous. Suppose also that $f(x) > 0$ for all $x \in X$. Prove that f is bounded away from zero, that is, show that there exists some $\epsilon > 0$ such that $f(x) \geq \epsilon$ for all $x \in X$. What if X is not compact?

Problem 4: For each of the parts below, identify a topological space with the specified topological invariants.

- (a) not compact, not hausdorff
- (b) compact, not hausdorff
- (c) hausdorff, compact, simply connected
- (d) hausdorff, compact, multiply connected
- (e) hausdorff, compact, not connected
- (f) hausdorff, simply connected, not compact
- (g) hausdorff, multiply connected, not compact
- (h) hausdorff, not connected, not compact

Problem 5: Let $f : [a, b] \rightarrow \mathbb{R}^n$.

- (a) Is the image $f([a, b]) \subset \mathbb{R}^n$ a topological manifold? Prove or provide an explicit counterexample.
- (b) If your answer to part (a) was 'no', under what additional conditions is the image a topological manifold? (This part would be too time consuming to put on the exam, but it is an excellent exercise).

Problem 6: Consider a function $f : X \rightarrow Y$ where Y is compact Hausdorff. Prove that f is continuous if and only if the graph of f , which is the set

$$G_f = \{a \times f(a) \mid a \in X\}$$

is closed in $X \times Y$.

Problem 7: If you haven't already, take a look at Exercise 10 in Professor Lee's notes.

Problem 8: Prove that the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is a topological manifold by explicitly creating an atlas.

Problem 9: Let $Y \subset X$; let X and Y be connected. Show that if A and B form a separation of $X - Y$, then $Y \cup A$ and $Y \cup B$ are connected.