Name: ______________________________
ID#: ______________________________
Instructor: ______________________________

Math 125
Exam 1
September 12, 2012

Turn off and put away your cell phone.
You may use a calculator, but no notes, books, or other assistance during this exam.
Read each question carefully, and answer each question completely.
Show all of your work; no credit will be given for unsupported answers.
Write your solutions clearly and legibly; no credit will be given for illegible solutions.
If any question is not clear, ask for clarification.

#	**Points**	**Score**
1	9	
2	7	
3	9	
4	6	
5	9	
Σ	40	

1. For each part below, find a function $f(x)$ satisfying the given properties, or explain why no such function can exist. Clearly circle your answer.

   (a) (3 points) A cubic polynomial with exactly two x-intercepts.

   In order for a cubic polynomial to have exactly, it needs to have a repeated root. For example, $f(x) = (x-1)^2(x)$ works.

   (b) (3 points) A rational function with horizontal asymptote at $y = 2$, vertical asymptotes at $x = -1$ and $x = 5$, and x-intercepts at $x = 0$ and $x = 1$.

   We need $x+1$ and $x-5$ in the denominator to have the vertical asymptotes, and copies of x and $x-1$ in the numerator for the x-intercepts. Last, the horizontal asymptote demands that the limit as $x \to \infty$ be equal to 2, so we adjust the coefficient accordingly:

$$f(x) = \frac{2x(x-1)}{(x+1)(x-5)}.$$

   (c) (3 points) A degree 2 polynomial satisfying the data in the table below.

x	f(x)
-8	-20
-1	2
3	-2
7	18

   This is not possible. Any polynomial is continuous, and therefore satisfies the Intermediate Value Theorem. From the table, this would imply that our function has 3 x-intercepts. However, a degree 2 polynomial can have at most 2 x-intercepts, so no such function can exist.

2. The number of hours of sunlight per day varies sinusoidally throughout the year. Let t be time measured in days starting with June 20 as $t = 0$ and let $D(t)$ give the number of hours of daylight as a function of t. Then $D(t)$ can be described as either a sine or cosine function. The longest day is June 20 with 14 hours of daylight, and the shortest day, half a year later, has 10 hours of daylight.

   (a) (3 points) Find a formula for $D(t)$. You should assume that a year has 365 days.

   Daylight varies from 10 to 14 hours, so the amplitude is 2 hours with a vertical shift of 12 hours. The period is one full year, or 365 days. Finally, we use cosine instead of sine, because the function starts at its maximum when $t = 0$. Thus,

$$D(t) = 12 + 2 \cdot \cos\left(\frac{2\pi t}{365}\right).$$

   (b) (4 points) How many days per year have at least 13 hours of sunlight? Your answer does not need to be a whole number.

   We need to find when $D(t) = 13$. Solving this yields

$$-\frac{1}{2} = \cos\left(\frac{2\pi t}{365}\right).$$

   Using arccos, this gives us $t \approx 61$ and $t \approx 304$. If we check the graph, we see that $D(t) \geq 13$ from time $t = 0$ until the first intercept at $t \approx 61$, and then from $t \approx 304$ until the end of the year. Thus, the total number of days is roughly 122. (Answers could vary by a day or so due to rounding).

3. Find each limit below, or show that the limit does not exist. You may use theorems and facts about limits from class *as long as you state them.* Show all your work.

   (a) (3 points) $\lim_{x\to 1} \frac{x^2-4x+3}{x^2-1}$.

   We solve this limit by factoring the numerator and denominator and cancelling like terms.

$$\lim_{x\to 1} \frac{x^2-4x+3}{x^2-1} = \lim_{x\to 1} \frac{(x-1)(x-3)}{(x-1)(x+1)} = \lim_{x\to 1} \frac{(x-3)}{(x+1)} = \frac{-2}{2} = -1.$$

   (b) (3 points) $\lim_{x\to 2} \frac{|x-2|}{x-2}$.

   Note that when $x \geq 2$, then $x-2$ is positive, so $|x-2| = x-2$. Conversely, when $x \leq 2$, then $x-2$ is negative and $|x-2| = -(x-2)$. Then we can look at the one-sided limits and observe that the results to not agree. Hence, there is no limit.

$$\lim_{x\to 2+} \frac{|x-2|}{x-2} = \lim_{x\to 2+} \frac{x-2}{x-2} = \lim_{x\to 2+} 1 = 1$$

$$\lim_{x\to 2-} \frac{|x-2|}{x-2} = \lim_{x\to 2+} \frac{-(x-2)}{x-2} = \lim_{x\to 2+} -1 = -1$$

   (c) (3 points) $\lim_{x\to 1} \left(\ln(x) - e^{\arcsin(\frac{1}{2})-\frac{\pi}{6}x}\right)$.

   The key in this example is to note that as awful as the function looks, it is continuous at the point $x=1$. In class we noted that sums, products, differences, and compositions of continuous functions are still continuous, and all of the ingredients to this function (polynomials, natural logs, exponentials, and inverse trig functions) are all continuous. Thus, we can simply plug in $x=1$ and know that it will be equal to the limit. This gives

$$\ln(1) - e^{\arcsin(\frac{1}{2})-\frac{\pi}{6}} = 0 - e^{\frac{\pi}{6}-\frac{\pi}{6}} = -1.$$

4. (6 points) Consider the function $f(x) = 2x^2 - 16x + 34$. We could easily plot the graph of this function using a calculator. Instead, explain how to obtain the graph of $f(x)$ from that of $y = x^2$ by an explicit sequence of transformation (shifts, stretches, etc.). You should clearly list the transformations *in the correct order* and provide accomanying graphs after each subsequent transformation.

There are many possible answers here. I am just going to list one possible sequence of transformations, and leave off the graphing of the functions.

First transformation: Graph after first transformation:

Horizonal shift right by 4:

$(x-4)^2 = x^2 - 8x + 16$

Second transformation: Graph after two transformations:

Vertical shift up by 1:

$x^2 - 8x + 16 + 1 = x^2 - 8x + 17$

Third transformation: Graph after all transformations:

Vertical stretch by 2, making graph steeper:

$2(x^2 - 8x + 17) = f(x) = 2x^2 - 16x + 34.$

5. Frederick has decided to observe the growth of bacteria on a the surface of a loaf of bread, starting on the expiration date ($t = 0$). Let $A(t)$ be the surface area (in cm^2) of the bacteria colony after t days.

(a) (3 points) For the first 12 days, the surface area of the bacteria grows exponentially. During this time, the surface area doubles every 3 days. Write a formula for $A(t)$ for the domain $0 \le t \le 12$. Clearly label and explain any unspecified constants in your formula.

This is a doubling problem, so the base of our exponential will be 2. We are not given the initial surface area of the bacteria on the expiration date, so we simply call this A_0. Last, the surface area double every 3 days, so the exponent will be $t/3$. In this way, the function will go through one doubling cycle each time the exponent is equal to 1, that is, when $t = 3$. Combining all of this:

$$A(t) = A_0 \cdot (2)^{t/3},$$

(b) (3 points) During the time period from part (a), how long does it take for the surface area to triple?

The surface area will triple when it is equal to $3 \cdot A_0$. We solve for t:

$$\begin{aligned} 3 \cdot A_0 &= A_0 \cdot (2)^{t/3} \\ 3 &= (2)^{t/3} \\ \ln 3 &= (t/3) \cdot \ln(2) \\ t &= \frac{3\ln(3)}{\ln(2)} \approx 4.75 \end{aligned}$$

Thus it takes about 4.75 days for the surface area to triple.

(c) (3 points) After the first 12 days, the bacteria begins to die off. The surface area of the bacteria colony starts to decrease linearly at a rate of .2 cm^2/ day. By the 15^{th} day (when $t = 15$), the surface area has fallen to 7.4 cm^2. What was the initial surface area of the bacteria colony on the expiration date? (You may assume that $A(t)$ is continuous.)

Our function $A(t)$ is exponential for $0 \le t \le 12$ as given in part (a), and linear for $12 \le t \le 15$. We know that $A(15) = 7.4$, and that for$12 \le t \le 15$ the surface area decreases by a constant rate of .2 cm^2/ day. Thus, $A(12) = 8$.
Since $A(t)$ is continuous, we can use this result and the exponential formula from part (a) to say

$$8 = A(12) = A_0 \cdot (2)^{12/3} = 16 \cdot A_0.$$

Hence, the initial surface area A_0 is equal to .5 cm^2.