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On the Total Variation of Solutions of Parabolic Equations

D. H. SATTINGER

1.

Let D be an open interval which may be finite or the entire x-axis, and, for $\tau > 0$, let $D_\tau = D \times (0, \tau)$. Let

$$L \equiv a(x,t)\frac{\partial^2}{\partial x^2} + b(x,t)\frac{\partial}{\partial x} - \frac{\partial}{\partial t}$$

be a uniformly parabolic operator on D_T. In this paper, under certain assumptions on a and b, a number of theorems are proved concerning the total variation of the solutions of $Lu = 0$.

In case $D = (-\infty, \infty)$ it is shown that the total variation of $u(x, t)$, as a function of x with fixed $t > 0$, is not greater than that of $u(x, 0)$. In the case of a boundary-initial value problem on a finite interval, the variation of the endpoint conditions must be added to that of the initial data. Let Γ_τ denote the sides and bottom of ∂D_τ; and, for $T > 0$ let u satisfy $Lu = 0$, $u = \psi$ on Γ_T. Then for $0 \leqq \tau \leqq T$ $\|u(\cdot, \tau)\|_V \leqq \operatorname{Var}\{\psi; \Gamma_\tau\}$, where $\|u(\cdot, \tau)\|_V$ is the total variation of $u(x, \tau)$ and $\operatorname{Var}\{\psi; \Gamma_\tau\}$ denotes the variation of ψ around Γ_τ.

If u is identically zero on the endpoints of D we can show further that $\|u(\cdot, t)\|_V$ decreases exponentially in time. Specifically, let there be positive constants μ and B such that $a \geqq \mu$ and $|b| \leqq B$ on D_T. Then there exist positive constants K and λ depending only on μ and B such that $\|u(\cdot, t)\|_V \leqq K e^{-\lambda t}\|u(\cdot, 0)\|_V$.

Theorems on the total variation of solutions of parabolic equations are important in the construction of solutions of non-linear first order equations by the method of "vanishing viscosity" [5]. For example a weak solution of the differential equation

$$u_t + (p(u))_x = 0 \tag{1}$$

can be constructed by considering the equation with a small viscosity term added, viz.,

$$u_t + (p(u))_x = \varepsilon u_{xx}\,. \tag{2}$$

Suppose that the existence of a regular solution u_ε of (2) satisfying the initial condition $u_\varepsilon(x, 0) = \varphi(x)$ can be established. Let φ be of finite total variation on $-\infty < x < \infty$. By the maximum principle and the theorems proved in this paper u_ε is *a priori* bounded and, for each fixed t, of uniformly bounded total variation as $\varepsilon \to 0$. Helly's theorem can be used to select a convergent subsequence whose limit is a weak solution of (1). For details the reader is referred to [5].

In § 2 the bounds on the total variation of solutions of $Lu=0$ are established under the assumption that a_x and b_x are Hölder continuous. In § 3 the theorem is proved for a boundary-initial value problem in the case where a and b are Hölder continuous on $\bar{D}_T$. In § 4 some topological properties of the positive and negative sets of solutions of a more general parabolic equation are established. These results are a generalization of theorems proved by Karlin [4] using the ideas of total positivity of the Green's function. They are used in § 5 to prove that the total variation decays exponentially when the endpoints are fixed at zero. In § 6 the total variation of the solutions of initial value problems with Neumann boundary conditions ($u_x=0$) at one endpoint is considered.

2.

Throughout this paper it is assumed that L is uniformly parabolic on D_T, thus that $a(x,t) \geqq \mu$ on D_T for some positive constant μ. For fixed $t>0$ the variation of u as a function of x on D is denoted by $\|u(\cdot,t)\|_V$, whether D be finite or infinite.

The maximum principle for parabolic equations on unbounded domains [3] is required in the proof of Theorem 1: *Let w be bounded below and continuous on $t \geqq 0$, and let w_x, w_{xx}, and w_t be continuous for $t>0$. Let*

$$\alpha w_{xx} + \beta w_x + \gamma w - w_t = 0$$

on $-\infty < x < \infty$, $t>0$, *where*

$$0 < \mu \leqq \alpha \leqq M(x^2+1), \qquad |\beta| \leqq M(|x|+1), \qquad |\gamma| \leqq M \tag{3}$$

for some positive constant M. If $w(x,0) \geqq 0$ then $w(x,t) \geqq 0$ for all $t \geqq 0$, $-\infty < x < \infty$.

1. Theorem. *Let $D=(-\infty,\infty)$, let a_x and b_x be Hölder continuous on D_T, and let there exist a positive constant M such that*

$$0 < \mu \leqq a \leqq M(x^2+1)\,; \qquad |a_x|, |b| \leqq M(|x|+1)\,; \qquad |b_x| \leqq M\,.$$

Let φ be of bounded variation, let $Lu=0$ in D_T, $u(x,0)=\varphi(x)$, and let $u \geqq -m$ for some positive constant m. Then $\|u(\cdot,t)\|_V \leqq \|\varphi\|_V$ for $0 \leqq t \leqq T$.

Proof. We may assume that $\varphi(x)$ is continuously differentiable. The more general case can be treated by approximating the initial data by smooth functions as follows. Let ω be a non-negative C^∞ function with support in $|x| \leqq 1$ for which $\int \omega\, dx = 1$. Let $\omega_h = \omega(x \mid h)/h$ and set

$$\varphi_h = \omega_h * \varphi = \int_{-\infty}^{\infty} \omega_h(x-y)\,\varphi(y)\,dy\,.$$

Now $\|\varphi_h\|_V \leqq \|\varphi\|_V$. In fact, if $-\infty < x_1 < \cdots < x_n < \infty$, then

$$\begin{aligned}\sum_{i=1}^{n} |\varphi_h(x_{i+1}) - \varphi_h(x_i)| &= \sum_{i=1}^{n} \left| \int_{-\infty}^{\infty} \omega_h(y) \{\varphi(x_{i+1}-y) - \varphi(x_i - y)\}\, dy \right| \\ &\leqq \int_{-\infty}^{\infty} \omega_h(y) \sum_{i=1}^{n} |\varphi(x_{i+1}-y) - \varphi(x_i-y)|\, dy \\ &\leqq \|\varphi\|_V \int_{-\infty}^{\infty} \omega_h\, dy = \|\varphi\|_V .\end{aligned}$$

Since this is true for any partition $x_1 < \cdots < x_n$ it follows that $\|\varphi_h\|_V \leqq \|\varphi\|_V$. Now let u_h be the solution of the initial value problem $Lu_h = 0$, $u_h(x, 0) = \varphi_h(x)$. The representation

$$u_h(x, t) = \int_{-\infty}^{\infty} Z(x, t; \xi, 0)\, \varphi_h(\xi)\, d\xi$$

holds, where Z is the fundamental solution of L. Now as h tends to zero, $\lim \varphi_h = \varphi$ [mean p]; so by the dominated convergence theorem $\lim u_h(x, t) = u(x, t)$ [unif.] on compact subsets of D_T. If Theorem 1 is true for u_h then $\|u_h(\cdot, t)\|_V \leqq \|\varphi_h\|_V \leqq \|\varphi\|_V$; and letting h tend to zero we have $\|u(\cdot, t)\|_V \leqq \|\varphi\|_V$. This shows that Theorem 1 holds even for discontinuous initial data.

Now let us prove the theorem for smooth non-decreasing initial data. Let $u(x, 0) = \varphi(x)$, where $\varphi'(x)$ is non-negative and continuous on $(-\infty, \infty)$. The derivative $w = u_x$ must satisfy the initial value problem

$$\begin{aligned} a w_{xx} + (a_x + b) w_x + b_x w - w_t &= 0 \\ w(x, 0) &= \varphi'(x) . \end{aligned} \tag{4}$$

Let w be the (unique) bounded solution of (4). Since the coefficients in (4) satisfy the conditions (3), $w(x, t) \geqq 0$ in D_T. The solution of $Lu = 0$, $u(x, 0) = \varphi(x)$ must therefore be non-decreasing in x for fixed $t > 0$. Moreover, since u is bounded and a and b satisfy (3), the maximum principle applies to u and

$$\inf_x \varphi \leqq \inf_x u(x, t) \leqq \sup_x u(x, t) \leqq \sup_x \varphi .$$

Since u and φ are monotonic it follows that

$$\begin{aligned} \|u(\cdot, t)\|_V &= \sup_x u(x, t) - \inf_x u(x, t) \\ &\leqq \sup_x \varphi - \inf_x \varphi = \|\varphi\|_V . \end{aligned}$$

The general case can now be treated by decomposing φ into the difference of its positive and negative variations. Then $\varphi = \varphi^+ - \varphi^-$ and $\|\varphi\|_V = \|\varphi^+\|_V + \|\varphi^-\|_V$. By the principle of superposition the solution of the initial value problem can be decomposed into $u = u^+ - u^-$, where $Lu^\pm = 0$ and $u^\pm(x, 0)$

$= \varphi^{\pm}(x)$. From the previous result we have

$$\begin{aligned}\|u(\cdot,\tau)\|_V &\leqq \|u^+(\cdot,\tau)\|_V + \|u^-(\cdot,\tau)\|_V \\ &\leqq \|\varphi^+\|_V + \|\varphi^-\|_V = \|\varphi\|_V .\end{aligned}$$

This completes the proof of Theorem 1.

We now prove the theorem in the case of a boundary-initial value problem. The variation of the boundary data around Γ_τ is denoted by $\operatorname{Var}\{\psi;\Gamma_\tau\}$.

2. Theorem. *Let a, b, a_x, and b_x be Hölder continuous on $\bar{D}_T$ and let ψ be of bounded variation and continuous around Γ_T. Then the solution of $Lu=0$, $u=\psi$ on Γ_T satisfies $\|u(\cdot,\tau)\|_V \leqq \operatorname{Var}\{\psi;\Gamma_\tau\}$ for $0\leqq\tau\leqq T$.*

Proof. Since ψ is of bounded variation on Γ_T it can be decomposed into the difference of its positive and negative variations, ψ^+ and ψ^-, computed, say, as Γ_T is traversed in a counter clockwise direction. The solution of the boundary value problem may then be written $u=u^+-u^-$, where $Lu^{\pm}=0$ in D_T and $u^{\pm}=\psi^{\pm}$ on Γ_T. Let us show that for fixed τ the functions $u^{\pm}(x,\tau)$ are nondecreasing in x. It follows that

$$\begin{aligned}\|u(\cdot,\tau)\|_V &\leqq \|u^+(\cdot,\tau)\|_V + \|u^-(\cdot,\tau)\|_V \\ &= (u^+(l,\tau)-u^+(0,\tau)) + (u^-(l,\tau)-u^-(0,\tau)) \\ &= (\psi^+(l,\tau)-\psi^+(0,\tau)) + (\psi^-(l,\tau)-\psi^-(0,\tau)) \\ &= \operatorname{Var}\{\psi;\Gamma_\tau\} .\end{aligned}$$

Suppose, then, that $Lu=0$ and $u=\psi$ on Γ_T, where ψ is continuous and nondecreasing as Γ_T is traversed in a counter-clockwise direction. First assume that ψ has an extension Ψ onto $\bar{D}_T$ such that $\Psi\in C^{3+\alpha}(\bar{D}_T)$ and $\Psi=\psi$ on Γ_T. (The notation $C^{3+\alpha}(\bar{D}_T)$ denotes the class of all functions whose derivatives up to third order exist and are Hölder continuous on $\bar{D}_T$ with exponent α.)

Differentiating the equation $Lu=0$ with respect to x we obtain a parabolic equation for u_x with Hölder continuous coefficients. The function u_x is continuous on $\bar{D}_T$ since ψ has a smooth extension into the interior of D_T.

Since ψ is increasing around Γ_T, $w(x,0)=\psi'(x)\geqq 0$ on $(0,l)$ and therefore, by continuity, on $[0,l]$. Moreover, the minimum and maximum of u on $\bar{D}_\tau$ must occur at the points $(0,\tau)$ and (l,τ) respectively, and so we must have $u_x(0,\tau)\geqq 0$ and $u_x(l,\tau)\geqq 0$ for $0<\tau\leqq T$. Therefore $u_x\geqq 0$ everywhere in $\bar{D}_T$, which was to be proved.

Now assume ψ is only continuous on Γ_T and approximate ψ by a sequence $\{\psi_n\}$ with the following properties: (i) $\lim\psi_n=\psi$ [unif. on Γ_T]; (ii) ψ_n has a smooth extension into $\bar{D}_T$; (iii) ψ_n is nondecreasing around Γ_T.

Such a sequence may be constructed as follows. For each integer n, sufficiently large, let $\hat{\psi}_n$ be the function identically equal to ψ everywhere except on intervals of length n^{-1} at the corners of D_T, where it takes on the values of ψ at those corners. Then $\hat{\psi}_n$ is nondecreasing and identically constant in a neighborhood of the corners of Γ_T. Let $\hat{\psi}_{n,h}$ be the C^∞ function obtained by smoothing $\hat{\psi}_n$ on Γ_T. For a sufficiently small value of the smoothing parameter h (say

$h \leqq 1/2n$), $\hat{\varphi}_{n,h}$ will be identically constant in a neighborhood of the corners of Γ_T. Such a function has a smooth extension into $\bar{D}_T$. Furthermore, $\hat{\varphi}_{n,h}$ is nondecreasing since smoothing preserves monotonicity; and $\lim \psi_n = \psi$ [unif.] on Γ_T. For the sequence $\{\psi_n\}$ we take, therefore, $\psi_n = \hat{\varphi}_{n,h}$, where $h = 1/2n$.

Now let u_n satisfy $Lu = 0$, $u_n = \psi_n$ on Γ_T. By the maximum principle, $\lim u_n = u$ [unif.] on $\bar{D}_T$. Since each u_n is monotonic in x,

$$0 \leqq \lim_n [u_n(x, \tau) - u_n(y, \tau)] = u(x, \tau) - u(y, \tau)$$

for all $0 < x < y < l$ and $0 \leqq \tau \leqq T$. Thus u is monotonic in x for fixed τ, and the proof of Theorem 2 is complete.

3.

Theorem 2 can be extended to the case where a and b are merely Hölder continuous by approximating them uniformly by smooth coefficients. Schauder-type *a priori* estimates, stated below, are needed to show that the solutions corresponding to the smoothed coefficients converge to the solution of the given equation.

For points $P = (x, t)$ and $Q = (\xi, \tau)$ set

$$d(P, Q) = ((x - \xi)^2 + |t - \tau|)^{\frac{1}{2}},$$

and let

$$\overline{|u|}_\alpha = |u|_0 + \bar{H}_\alpha(u),$$
$$\overline{|u|}_{2+\alpha} = \overline{|u|}_\alpha + \overline{|u_x|}_\alpha + \overline{|u_{xx}|}_\alpha + \overline{|u_t|}_\alpha,$$

where

$$|u|_0 = \sup_{P \in D_T} |u(P)|,$$
$$\bar{H}_\alpha(u) = \sup_{P, Q \in D_T} \frac{|u(P) - u(Q)|}{d(P, Q)^\alpha}.$$

Here α is a number between zero and one.

For a function ψ defined on Γ_T we say that $\psi \in \bar{C}_{2+\alpha}$ if ψ has an extension Ψ into D_T for which $\overline{|u|}_{2+\alpha} < +\infty$. We then set $\overline{|\Psi|}_{2+\alpha} = \inf \overline{|\Psi|}_{2+\alpha}$, where the infimum is taken over all extensions of ψ.

The *boundary estimates* state [1]: *Let $\overline{|a|}_\alpha \leqq K$, $\overline{|b|}_\alpha \leqq K$ and let $a \geqq \mu$ on D_T. Then there exists a constant K_1, depending only on K and μ, such that $\overline{|u|}_{2+\alpha} \leqq K_1 \overline{|\psi|}_{2+\alpha}$, where u is the solution of $Lu = 0$, $u = \psi$ on Γ_T.*

Using this fact we can now prove

3. Theorem. *Let $\overline{|a|}_\alpha$ and $\overline{|b|}_\alpha$ be finite and let $Lu = 0$, $u = \psi$ on Γ_T, where ψ is continuous and of bounded variation around Γ_T. Then $\|u(\cdot, \tau)\|_V \leqq \operatorname{Var}\{\psi; \Gamma_\tau\}$ for $0 \leqq \tau \leqq T$.*

Proof. If $\psi \notin \bar{C}_{2+\alpha}$ then ψ can be approximated uniformly on Γ_T by a sequence $\{\psi_n\}$ with the following properties:

(i) $\operatorname{Var}\{\psi_n;\Gamma_\tau\} \leqq \operatorname{Var}\{\psi;\Gamma_\tau\}$; (ii) $\psi_n \in \overline{C}_{2+\alpha}$. The sequence $\{\psi_n\}$ can be constructed in the same way as the corresponding sequence was constructed in Theorem 2. If the $\{\hat{\psi}_n\}$ are defined as before, then $\operatorname{Var}\{\hat{\psi}_n;\tau\} \leqq \operatorname{Var}\{\psi_n;\tau\}$; and the $\{\psi_n\}$ are obtained by smoothing the $\{\hat{\psi}_n\}$. The process of smoothing preserves total variation. If u_n satisfies $Lu_n = 0$, $u_n = \psi_n$ on Γ_T, then u_n tends to u uniformly on $\overline{D}_T$; and we have $\|u(\cdot,\tau)\|_V \leqq \lim \|u_n(\cdot,\tau)\|_V \leqq \operatorname{Var}\{\psi_n;\tau\} \leqq \operatorname{Var}\{\psi;\tau\}$.

Therefore let us assume that $\psi \in \overline{C}_{2+\alpha}$. Let families $\{a_h\}$ and $\{b_h\}$ be constructed such that

(i) $\lim a_h = a$, $\lim b_h = b$ [unif.] on $\overline{D}_T$.

(ii) $\overline{|b_h|}_\alpha \leqq \overline{|b|}_\alpha$, $\overline{|a_h|}_\alpha \leqq \overline{|a|}_\alpha$, $a_h \geqq \mu$.

(iii) $a_h, b_h \in C^\infty$.

For example, to construct $\{a_h\}$, first extend a to all of $\mathbb{R}^2$ as follows. For $P \notin D_T$ let a assume the value it attains at the boundary point of D_T nearest to P. Then for any $P, Q \in \mathbb{R}^2$

$$\left|\frac{a(P)-a(Q)}{d(P,Q)^\alpha}\right| \leqq \overline{H}_\alpha(a). \tag{5}$$

Now let ω be a nonnegative C^∞ kernel on $\mathbb{R}^2$ with support in $|P| \leqq 1$ such that $\iint \omega(P')\,dP' = 1$. For each $h > 0$ set

$$a_h(P) = \iint\limits_{\mathbb{R}^2} \omega_h(|P-P'|)\,a(P')\,dP'.$$

It is easily verified that $a_h \geqq \mu$ everywhere. Furthermore

$$\begin{aligned}\frac{a_h(P)-a_h(Q)}{d(P,Q)^\alpha} &= \iint\limits_{\mathbb{R}^2} \omega_h(|P'|)\,\frac{a(P-P')-a(Q-P')}{d(P,Q)^\alpha}\,dP' \\ &= \iint\limits_{\mathbb{R}^2} \omega_h(|P'|)\,\frac{a(P-P')-a(Q-P')}{d(P-P',Q-P')^\alpha}\,dP'.\end{aligned}$$

Hence by (5) $\overline{|a_h|}_\alpha \leqq \overline{|a|}_\alpha$. The sequence $\{b_h\}$ is constructed in the same manner.

Now let u_h be the solution of $L_h u_h = 0$, $u_h = \psi$ on Γ_T. By the *a priori* estimates $\overline{|u_h|}_{2+\alpha} \leqq K_1 \overline{|\psi|}_{2+\alpha}$ for all h; and so by the Arzéla-Ascoli theorem there is a sequence $\{u_n\}$ such that $\{u_n\}$, $\{u_{n,x}\}$, $\{u_{n,xx}\}$, $\{u_{n,t}\}$ converge uniformly on $\overline{D}_T$. Let $u = \lim u_n$; then the derivatives of u_n converge to the corresponding derivatives of u, and $Lu = 0$, $u = \psi$ on Γ_T. Hence the sequence $\{u_n\}$ converges to the (unique) solution of the boundary value problem. Theorem 3 now follows by the usual limiting argument.

Remark. We might point out that, since the solution of the problem $Lu = 0$, $u = \psi$ on Γ_T is unique, every sequence of the family $\{u_h\}$ contains a subsequence which converges uniformly to the same limit. Therefore the family $\{u_h\}$ itself converges uniformly to u as h tends to zero.

4.

In this section some theorems are proved concerning the positive and negative sets of the solutions of a parabolic equation. Given a function u defined on D_τ the positive and negative sets of u are defined to be

$$U^+ = \{P : P \in D_\tau, u(P) > 0\},$$
$$U^- = \{P : P \in D_\tau, u(P) < 0\}.$$

A *component* of U^+ or U^- is a maximal open connected subset of U^+ or U^-. We have

4. Theorem. *Let u be a solution of the parabolic equation*

$$\alpha u_{xx} + \beta u_x + \gamma u - u_t = 0 \tag{6}$$

on the domain D_T which is continuous on $\bar{D}_T$. Let α, β, and γ be continuous and let $\alpha(x,t) \geqq \mu > 0$ on D_T. Suppose that on Γ_τ there are precisely $n(m)$ disjoint intervals where u is positive (negative). Then $U^+(U^-)$ has at most $\underline{n}(m)$ components in D_τ, and the closure of each component must intersect Γ_τ in at least one interval.

Proof. Let I be a maximal interval on Γ_τ where u is positive and suppose that two open connected subsets F_1 and F_2 of U^+ intersect Γ_τ in disjoint open intervals I_1 and I_2 contained in I. Since u is continuous there is an open neighborhood G in D_τ whose closure in $\bar{D}_\tau$ contains I. Since G must contain points in both F_1 and F_2, these must belong to the same open connected component of U^+. Thus at most one component of U^+ intersects each of the n open intervals on Γ_τ where u is positive. The same result holds for the components of U^-. Therefore it suffices to show that every component of U^+ or U^- must intersect Γ_τ in an interval.

We can assume that $\gamma \leqq 0$ on D_τ. Otherwise we make the transformation $u = e^{\lambda t} v$, where λ is some number such that $\gamma \leqq \lambda$ on D_T. The positive and negative sets of u are the same as those of v, and v satisfies a parabolic equation similar to (6) with $\gamma \leqq 0$.

Now let F be a component of U^+ in D_τ. Since u is continuous it must assume a (positive) maximum on $\bar{F}$. On F we have

$$\alpha u_{xx} + \beta u_x - u_t = -\gamma u \geqq 0,$$

so u cannot attain its maximum at an interior point of F or along the line $\{t = \tau\}$. On the other hand, by continuity, u is zero at any boundary point of F which is interior to D_τ. Therefore F must have a boundary point Q on Γ_τ at which u is positive; and by continuity u must be positive in an interval of Γ_τ about Q. This completes the proof of Theorem 4.

The following corollaries are immediate.

5. Corollary. *Under the assumptions of Theorem 4 the number of sign changes of $u(x,\tau)$ on $(0,l)$, where u is a solution of* (6), *is not greater than the number of sign changes of u on Γ_τ.*

Karlin [4] has proved theorems along the lines of Corollary 5 for parabolic equations on an unbounded domain $(-\infty < x < \infty)$.

6. Corollary. *If a component G of U^+ or U^- in D_τ intersects the horizontal line $\{t=\tau\}$ in two disjoint intervals then the solution of* (6) *is identically zero in the region between G and the line $\{t=\tau\}$.*

5.

In this section we consider the initial value problem

$$Lu=0, \qquad u(x,0)=\varphi(x), \qquad 0<x<l \tag{7a}$$

with the boundary conditions

$$u(0,t)=u(l,t)=0\,. \tag{7b}$$

In this case the solution and its total variation decay exponentially. We have

7. Theorem. *Let u satisfy* (7a), (7b)*; and let $\overline{|a|}_\alpha$ and $\overline{|b|}_\alpha$ be finite, $a(x,t) \geqq \mu$ on D_T, and $|b(x,t)| \leqq B$ on D_T. Then there are constants $K>0$ and $\lambda>0$ depending only on μ, B, and l such that $\|u(\cdot,t)\|_V \leqq Ke^{-\lambda t}\|u(\cdot,0)\|_V$.*

The following lemma is needed in the proof of Theorem 7.

8. Lemma. *Let u satisfy the boundary-initial value problem*

$$Lu=0, \qquad 0\leqq x \leqq X, \qquad 0 \leqq t \leqq T,$$

$$u(x,0)=u_0(x),$$

$$u(0,t)=u(X,t)=0\,.$$

Let $a \geqq \mu > 0$ and $|b| \leqq B$ on $D_T=(0,X)\times(0,T)$. Then there are positive constants K and λ depending only on μ, B, and X such that $\|u(\cdot,t)\|_\infty \leqq Ke^{-\lambda t}\|u(\cdot,0)\|_\infty$.

The notation $\|u(\cdot,t)\|_\infty$ denotes the supremum of $u(x,t)$, for fixed t, on $[0,X]$. The coefficients need only be continuous in Lemma 8.

Proof of Lemma 8. Let $\vartheta(x)=1-\beta e^{\alpha x}$ where α and β are constants to be suitably chosen, and set $u=ve^{-\lambda t}\vartheta$. A simple calculation shows that

$$\begin{aligned} &av_{xx}+\left(2a\frac{\vartheta_x}{\vartheta}+b\right)v_x \\ &\quad+\frac{1}{\vartheta}(a\vartheta_{xx}+b\vartheta_x+\lambda\vartheta)v-v_t=0\,. \end{aligned} \tag{8}$$

Let us show that it is possible to choose positive constants α, β, and λ in such a way that

$$a\vartheta_{xx}+b\vartheta_x+\lambda\vartheta \leqq 0, \qquad \vartheta>0 \quad \text{on} \quad \bar{D}_T\,. \tag{9}$$

We have

$$\begin{aligned} a\vartheta_{xx}+b\vartheta_x+\lambda\vartheta &= \alpha\beta e^{\alpha x}\left[-a\alpha-b+\frac{\lambda}{\alpha\beta}(e^{-\alpha x}-\beta)\right] \\ &\leqq \alpha\beta e^{\alpha x}\left[-\alpha\mu+B+\frac{\lambda}{\alpha\beta}(e^{-\alpha x}-\beta)\right]. \end{aligned}$$

Now choose α so large that $-\alpha\mu+B\leqq -1/2$ and β small enough that $1-\beta e^{\alpha x}>0$ on $[0, X]$. Taking $\lambda=\alpha\beta/2$ we see that (9) is satisfied.

Now v satisfies the parabolic Eq. (8), so by the maximum principle $\|v(\cdot, t)\|_\infty \leqq \|v(\cdot, 0)\|_\infty$. Therefore

$$\begin{aligned} \|u(\cdot, t)\|_\infty = \|e^{-\lambda t}\vartheta v\|_\infty &\leqq e^{-\lambda t}\|v(\cdot, t)\|_\infty \\ &\leqq e^{-\lambda t}\|v(\cdot, 0)\|_\infty \leqq e^{-\lambda t}\|u_0/\vartheta\|_\infty \\ &\leqq K e^{-\lambda t}\|u_0\|_\infty, \end{aligned}$$

where $K=\|1/\vartheta\|_\infty$, and Lemma 8 is proved.

Interior estimates of Schauder type are also required in the proof of Theorem 7. For each $P\in D_T$ define

$$d_P=\inf_{Q\in\Gamma_T} d(P, Q), \qquad d=d_{PQ}=\min\{d_P, d_Q\}.$$

The interior norms are then

$$|u|_\alpha = |u|_0 + H_\alpha(u),$$

$$|u|_{2+\alpha} = |u|_\alpha + |du_x|_\alpha + |d^2 u_{xx}|_\alpha + |d^2 u_t|_\alpha,$$

where

$$|d^m u|_0 = \sup_{P\in D_T} d_P^m |u(P)|,$$

$$H_\alpha(d^m u) = \sup_{P, Q\in D_T} d_{PQ}^{m+\alpha}\frac{|u(P)-u(Q)|}{d(P, Q)^\alpha},$$

$$|d^m u|_\alpha = |d^m u|_0 + H_\alpha(d^m u).$$

The *interior estimates* state [1]: *Let* $Lu=0$ *on* D_T, *where* $|a|_\alpha\leqq K$, $|db|_\alpha\leqq K$, *and* $a\geqq\mu>0$. *There is a constant* K_1 *depending only on* K *and* μ *such that* $|u|_{2+\alpha}\leqq K_1|u|_0$.

The interior estimates are used to prove the following approximation theorem.

9. Lemma. *Let* u *satisfy* (7a), (7b) *and let* $\{u^h\}$ *satisfy the corresponding problem with* a *and* b *replaced by* $\{a_h\}$ *and* $\{b_h\}$. *Suppose that* (i) $\lim a_h=a$, $\lim b_h=b$ *on* D_T; (ii) $|a_h|_\alpha$, $|db_h|_\alpha$, *and* $|b_h|_0$ *are bounded as* $h\to 0$; (iii) $a_h\geqq\mu>0$ *on* D_T *for all* h. *Then for each fixed* $t>0$ $\lim u^h(x, t)=u(x, t)$ [unif.] *on* $[0, l]$.

Proof. Let $|b_h|_0\leqq B$ for all h. Let us show that

$$\int_0^l (u_x^h(x, t))^2\, dx \leqq K \tag{10}$$

for some constant K independent of h. Choose a number κ such that $B \leqq 2\kappa a_h$. This is possible since $a_h \geqq \mu$.

Letting $z = \kappa x$ the differential equation becomes

$$\kappa^2 a_h u^h_{zz} + \kappa b_h u^h_z - u^h_t = 0, \qquad 0 \leqq z \leqq \kappa l.$$

Multiplying by u^h_{zz} and integrating we get

$$-\int_0^{\kappa l} u^h_t u^h_{zz}\,dz + \int_0^{\kappa l} \kappa^2 a_h (u^h_{zz})^2\,dz + \kappa \int_0^{\kappa l} b_h u^h_z u^h_{zz}\,dz = 0.$$

Integrating the first term by parts and applying the boundary conditions (7b) we obtain

$$\frac{\partial}{\partial t}\frac{1}{2}\int_0^{\kappa l} (u^h_z)^2\,dz + \kappa^2 \int_0^{\kappa l} a_h (u^h_{zz})^2\,dz \leqq \frac{\kappa B}{2}\int_0^{\kappa l} \{(u^h_z)^2 + (u^h_{zz})^2\}\,dz.$$

Hence

$$\frac{\partial}{\partial t}\frac{1}{2}\int_0^{\kappa l} (u^h_z)^2\,dz \leqq \frac{\kappa B}{2}\int_0^{\kappa l} (u^h_z)^2\,dz + \int_0^{\kappa l}\left(\frac{\kappa B}{2} - \kappa^2 a_h\right)(u^h_{zz})^2\,dz \leqq \frac{\kappa B}{2}\int_0^{\kappa l} (u^h_z)^2\,dz.$$

This implies that

$$\left.\int_0^{\kappa l} (u^h_z)^2\,dz\right|_t \leqq \left.\int_0^{\kappa l} (u^h_z)^2\,dz\right|_{t=0} e^{\kappa B t},$$

and therefore that

$$\int_0^{l} (u^h_x)^2\,dx \leqq \int_0^{l} (\varphi')^2\,dx\, e^{\kappa B t}.$$

Thus (10) is established.

Now by the maximum principle $\{u^h\}$ is a uniformly bounded family of functions on $\bar{D}_T$. By the interior estimates $\{u^h\}$, $\{u^h_x\}$, $\{u^h_{xx}\}$, and $\{u^h_t\}$ are uniformly bounded and equicontinuous on compact subsets of D_T. Therefore there is a sequence $\{u^n\}$ which converges, along with its derivatives $\{u_x^n\}$, $\{u_{xx}^n\}$, and $\{u_t^n\}$, uniformly on compact subsets of D_T. If $\tilde{u}$ denotes the limit of u^n, then u_x^n, u_{xx}^n, and u_t^n converge to the corresponding derivatives of $\tilde{u}$, and $L\tilde{u}=0$ in D_T.

On the other hand, by (10) the family $\{u^n\}$ is uniformly bounded and equicontinuous, for fixed $t>0$, on $[0, l]$. Therefore $\lim u^n(x,t)=\tilde{u}(x,t)$ [unif.] on D for fixed $t>0$.

It follows that $\tilde{u}=u$, the unique solution of (7a), (7b). Finally, by the uniqueness of the solution of (7a), (7b), every infinite subclass of $\{u^h\}$ contains a sequence which converges to u. Therefore the entire family converges to u uniformly on $[0, l]$ for each fixed $t>0$, and Lemma 9 is established.

Proof of Theorem 7. We can assume that a_x and b_x are Hölder continuous on $\bar{D}_T$ and that ψ, equal to zero on the sides and to φ on the bottom of Γ_T, has a smooth extension into the interior of D_T. These restrictions can be removed by the usual approximation arguments. We write $\varphi=\varphi^+-\varphi^-$, where $\varphi^\pm(0)=0$, $\varphi^\pm$ are nondecreasing on $[0, l]$, and $\|\varphi\|_V=\|\varphi^+\|_V+\|\varphi^-\|_V=\varphi^+(l)+\varphi^-(l)=2\varphi^+(l)$.

Now define a^*, b^*, and φ^* as follows:

$$a^*(x,t)=\begin{cases} a(x,t) & 0\leqq x\leqq l \\ a(2l-x,t) & l\leqq x\leqq 2l, \end{cases}$$

$$b^*(x,t)=\begin{cases} b(x,t) & 0\leqq x\leqq l \\ -b(2l-x,t) & l\leqq x\leqq 2l, \end{cases}$$

$$\varphi^*(x)=\begin{cases} \varphi^+(x) & 0\leqq x\leqq l \\ \varphi^-(2l-x) & l\leqq x\leqq 2l. \end{cases}$$

We can assume, without difficulty, that φ^* has a smooth extension into $[0, 2l]\times[0, T]$. In addition, assume for the time being that $a_x(l,t)=b(l,t)=0$, so that a^*, b^*, a_x^*, and b_x^* are Hölder continuous on $0\leqq x\leqq 2l$, $0\leqq t\leqq T$. Later this restriction will be dropped.

Let u^* satisfy

$$a^* u_{xx}^* + b^* u_x^* - u_t^* = 0, \tag{11a}$$

$$u^*(x,0)=\varphi^*(x), \tag{11b}$$

$$u^*(0,t)=u^*(2l,t)=0, \tag{11c}$$

and set $u(x,t)=u^*(x,t)-u^*(2l-x,t)$ for $0\leqq x\leqq l$. Then $u(x,t)$ so constructed is the solution of (7a), (7b).

Let us show that there is a point $x^*=x^*(\tau)$ such that $0<x^*<2l$ and $u^*(x,\tau)$ is nondecreasing for $x\leqq x^*$, nonincreasing for $x\geqq x^*$. Differentiating (11a) with respect to x we obtain a parabolic equation for $w=u_x^*$. Since $u^*(x,t)\geqq 0$ for $0\leqq t\leqq T$, $w(0,t)\geqq 0$ and $w(2l,t)\leqq 0$. Furthermore $w\geqq 0$ on $[0, l]$, and $w\leqq 0$ on $[l, 2l]$. Therefore by Corollary 5 w has at most one sign change on $t=\tau$ for $0\leqq\tau\leqq T$, and the existence of the point $x^*(\tau)$ is established.

Now if $x^*(t) < l$ then

$$\begin{aligned}\|u(\cdot,t)\|_V &\leqq \|u^*(\cdot,t)\|_V + \|u^*(2l-x,t)\|_V \\ &= [u^*(x^*,t) + (u^*(x^*,t) - u^*(l,t))] \\ &\qquad + [u^*(l,t)] \\ &= 2u^*(x^*,t).\end{aligned}$$

The same result holds if $l \leqq x^*(t) \leqq 2l$. On the other hand, by Lemma 8 there are positive constants K and λ such that

$$\begin{aligned}\|u(\cdot,t)\|_V &\leqq 2u^*(x^*,t) \leqq 2Ke^{-\lambda t}\sup_x |u^*(x,0)| \\ &= 2Ke^{-\lambda t}\varphi^+(l) = Ke^{-\lambda t}\|\varphi\|_V.\end{aligned}$$

Theorem 7 has now been established under the restrictions $a_x(l,t) = b(l,t) = 0$. To remove these restrictions it suffices to approximate a and b by families of functions $\{a_h\}$ and $\{b_h\}$ which satisfy properties (i)–(iii) of Lemma 9 and the conditions $a_{h,x}(l,t) = b_h(l,t) = 0$.

For each $h > 0$ let ζ_h be a C^∞ function such that $\zeta_h \equiv 0$ for $l-h \leqq x \leqq l$, $\zeta_h \equiv 1$ for $0 \leqq x \leqq l-2h$, and $|\zeta_h'| \leqq C/h$ for some constant C independent of h. Then $|d\zeta_h|_\alpha$ is bounded as h tends to zero. Letting $b_h = \zeta_h b$ we see that $b_h(l,t) = 0$, $|b_h(x,t)| \leqq B$ on D_T, and $|db_h|_\alpha \leqq |\bar{b}|_\alpha\, |d\zeta_h|_\alpha$ is bounded as h tends to zero.

For $(x,t) \in D_T$ let

$$\hat{a}_h = \begin{cases} a(x,t) & 0 \leqq x \leqq l-2h \\ a(l-2h,t) & l-2h \leqq x \leqq l \end{cases}$$

and extend $\hat{a}_h$ to all of $\mathbb{R}^2$ as in the proof of Theorem 3. Then let

$$a_h(P) = \iint_{\mathbb{R}^2} \omega_h(|P'|)\,\hat{a}_h(P-P')\,dP'$$

where ω_h is the smoothing kernel on $\mathbb{R}^2$. It is easily verified that $a_h \geqq \mu$, $\partial a_h/\partial_x \equiv 0$ for $l-h \leqq x \leqq l$ and that $|a_h|_\alpha$ is uniformly bounded as $h \to 0$.

The coefficients $\{a_h\}$ and $\{b_h\}$ have been constructed as required, and so the proof of Theorem 7 is complete.

6.

No heat flows into or out of the domain at an endpoint where a Neumann condition ($u_x = 0$) is satisfied. In such a case equalization still takes place, and we should expect the variation to be nonincreasing. This conjecture is in fact correct, and in this section we prove a number of theorems on the total variation of solutions of problems involving Neumann boundary conditions. We first prove

10. Lemma. *Let u satisfy $Lu = 0$ in D_T, $u(x,0) = \varphi(x)$, and the boundary conditions*

$$u_x(0,t) = u(l,t) = 0 \tag{12a}$$

or

$$u(0,t)=u_x(l,t)=0\,. \tag{12b}$$

Let a and b be continuous, $a \geqq \mu$ and $|b| \leqq B$ on $\bar{D}_T$. Then there are positive constants K and λ depending only on μ, B, and l such that $\|u(\cdot,t)\|_\infty \leqq K e^{-\lambda t}\|u(\cdot,0)\|_\infty$.

Proof. Consider the case (12a) and let $\vartheta = 1-\beta e^{\alpha x}$ be the function constructed in the proof of Lemma 8. Setting $u=\vartheta e^{-\lambda t} v$ we see that v satisfies the parabolic Eq. (8) and the boundary conditions

$$\begin{gathered} v(l,t)=0 \\ \vartheta'(0)\,v(0,t)+\vartheta(0)\,v_x(0,t)=0\,. \end{gathered} \tag{13}$$

Now v cannot attain a positive maximum at a boundary point $(0,\tau)$. In fact, at such a point $v_x(0,\tau)\leqq 0$ and $v(0,\tau)>0$, while $\vartheta'(0)<0$ and $\vartheta(0)>0$, contradicting the second condition in (13). Applying the same argument to $-v$ we see that v cannot attain a negative minimum on the left-hand endpoint. This together with the first condition in (13) and the strong maximum principle shows that $\|v(\cdot,t)\|_\infty < \|v(\cdot,0)\|_\infty$. The rest of the proof of Lemma 10 parallels that of Lemma 8. The case (12b) is dealt with by employing the function $\vartheta(x)=1-\beta e^{-\alpha x}$ and proceeding as above.

It follows immediately from Lemma 10 that the solution of the initial value problem with boundary conditions (12a) or (12b) is unique.

If u attains its maximum in $\bar{D}_\tau$ at the boundary point (l,τ') then clearly $u_x(l,\tau')\geqq 0$. The *maximum principle at the boundary* [2, 6] shows that if $\tau'<\tau$ then we actually have $u_x(l,\tau')>0$. Similarly, if u attains its maximum on $\bar{D}_\tau$ at $(0,\tau')$, where $\tau'<\tau$, then $u_x(0,\tau')<0$. Analogous statements hold if u attains its minimum on the lateral boundaries of D_τ. This fact is used in the proof of Lemma 11.

11. Lemma. *Let u satisfy the boundary-initial value problem of Lemma 10. Let the coefficients a and b be continuous on $\bar{D}_T$ and let $a\geqq\mu$ there. Then $\|u(\cdot,\tau)\|_\infty \leqq \|\varphi\|_\infty$.*

Proof. Let us prove Lemma 11 for the case of boundary conditions (12a), the proof of the case (12b) being similar. The positive maximum of u on $\bar{D}_\tau$ cannot occur at an interior point or along the line segment $\{t=\tau, 0<x<l\}$ by the strong maximum principle. It cannot occur on the right boundary since $u=0$ there. Moreover, it cannot occur on $\{x=0, 0<t<\tau\}$ by the maximum principle at the boundary.

Suppose that the maximum of u on $\bar{D}_\tau$ occurs at $(0,\tau)$. If $\tau'>\tau$ then the maximum of u on $\bar{D}_{\tau'}$ must, by the same reasoning as above, occur at $(0,\tau')$. Thus $u(0,\tau')\geqq u(0,\tau)$ and $u(0,\tau)$ is non-decreasing for $t\geqq\tau$. Let us show, using Lemma 10, that this cannot happen.

Extend a and b into $t\geqq T$ so that a and b are Hölder continuous in $D_\infty = (0,l)\times(0,\infty)$. (For example, define $a(x,t)=a(x,T)$ for $t\geqq T$, etc.) Since the solution of the initial value problem is unique, u has a unique extension to all of D_∞ defined as the solution of the boundary-initial value problem on D_∞ with the coefficients a and b extended as above. By Lemma 10 u tends to zero uniformly as $t \to \infty$, while by the above argument, $u(0,t)$ is nondecreasing for $t \geqq \tau$. This contradiction shows that u cannot attain its maximum in $\bar{D}_\tau$ at the point $(0,\tau)$. Thus u attains its (strict) maximum in $\bar{D}_\tau$ on the initial interval $\{t=0, 0 \leqq x \leqq l\}$, and $\|u(\cdot,t)\|_\infty < \|u(\cdot,0)\|_\infty$.

We can now prove the following theorem on the total variation of solutions of initial value problems with Neumann boundary conditions.

12. Theorem. *Let* $Lu=0$ *on* D_T, $u(x,0)=\varphi(x)$, *where* $|a|_\alpha$ *and* $|b|_\alpha$ *are finite and* φ *is continuous and of bounded variation on* $[0,l]$. *Let* u *satisfy the boundary conditions* (12a) *or* (12b), *and assume* $\varphi(l)=0$ *in case* (12a) *or* $\varphi(0)=0$ *in case* (12b). *Then* $\|u(\cdot,t)\|_V \leqq \|\varphi\|_V$ *and there exist positive constants* K *and* λ *such that* $\|u(\cdot,t)\|_V \leqq K e^{-\lambda t}\|\varphi\|_V$.

Proof. As usual it suffices to prove the theorem under the additional restrictions

(i) φ is nondecreasing on $[0,l]$.

(ii) There is a smooth function Ψ on $\bar{D}_T$ which satisfies $\Psi(x,0)=\varphi(x)$ and the boundary conditions (12a) or (12b).

(iii) a, b, a_x, and b_x are Hölder continuous on $\bar{D}_T$.

The restriction (iii) can be eliminated by a straightforward modification of the approximation arguments of Lemma 9 and Theorem 7.

Let us assume, then, that φ is nondecreasing on $[0,l]$. In case (12b) $\varphi(0)=0$ so $\varphi(x) \geqq 0$. It follows that $u \geqq 0$ on $\bar{D}_T$ and therefore that $u_x(0,t) \geqq 0$. Since $u_x(x,0) \geqq 0$ and $u_x(l,t)=0$ in case (12b) we see that $u_x \geqq 0$ on $\bar{D}_T$ and $u(x,t)$ is nondecreasing on $0 \leqq x \leqq l$ for each fixed $t>0$. By Lemma 11 we have

$$\|u(\cdot,t)\|_V = u(l,t) - u(0,t) = u(l,t) < \varphi(l) = \varphi(l) - \varphi(0) = \|\varphi\|_V .$$

The case (12a) is treated similarly.

To prove the second conclusion of Theorem 12 in the case (12b) we note that, from Lemma 10, there are constants K and λ such that

$$\begin{aligned}\|u(\cdot,t)\|_V &= u(l,t) - u(0,t) = \|u(\cdot,t)\|_\infty \\ &\leqq K e^{-\lambda t}\|u(\cdot,0)\|_\infty = K e^{-\lambda t}\varphi(l) \\ &= K e^{-\lambda t}\|\varphi\|_V .\end{aligned}$$

This completes the proof of Theorem 12.

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Prof. Dr. D. H. Sattinger
University of California
Department of Mathematics
405 Hilgard Avenue
Los Angeles, California 90024, USA

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