Monotone Methods in Nonlinear Elliptic and Parabolic Boundary Value Problems

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1. Introduction. In this paper we discuss the use of monotone iteration schemes in the construction of solutions of nonlinear elliptic boundary value problems of the type

$$Lu + f(x, u) = 0 \quad \text{in} \quad D, \qquad (1.1)$$
$$Bu = g \quad \text{on} \quad \partial D$$

where L is a second order uniformly elliptic operator:

$$L = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x^i}, \qquad x = (x^1, \cdots, x^n)$$

and B is one of the boundary operators

$$Bu = u,$$

or

$$Bu = \frac{\partial u}{\partial \nu} + \beta(x)u, \qquad x \in \partial D.$$

Here $\partial/\partial\nu$ denotes the outward conormal derivative, and we assume $\beta \geqq 0$ everywhere on the boundary, ∂D.

The coefficients of L are assumed to be smooth (say Hölder continuous) and the matrix a_{ij} is to be uniformly positive definite on $\bar{D}$. Any undifferentiated terms in u may be grouped with the term $f(x, u)$, including linear terms. We assume the boundary of D is smooth, say $C^{2+\alpha}$, though this condition could be relaxed somewhat. $f(x, u)$ is assumed smooth throughout this paper, say f is C^1 in u and Hölder continuous in x.

Such nonlinear boundary value problems arise in numerous applications (see references) and in fact the present paper was motivated by D. S. Cohen's work [4] on problems arising in chemical reactor theory. See also [12], [13].

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In previous work various assumptions have been made on the term $f(x, u)$, such as monotonicity, positivity, convexity, concavity, or boundedness. These assumptions can be relaxed considerably (if not altogether) by using the iteration scheme found in [4], [12], and [13]. One of the contributions here will be to emphasize the importance of the idea of upper and lower solutions in these problems. A smooth function u_0 is said to be an *upper solution* of (1.1) if

$$Lu_0 + f(x, u_0) \leqq 0, \qquad Bu_0 \geqq g;$$

similarly, v_0 is called a *lower solution* of (1.1) if

$$Lv_0 + f(x, v_0) \geqq 0, \qquad Bv_0 \leqq g.$$

Upper and lower solutions have already been introduced in [12] and [18]. As an application of this technique we establish the validity of the singular perturbation arguments found in the opening remarks of Cohen's paper. Those arguments were produced on a formal basis as motivation for the author's investigation but we shall see here that the singular perturbation arguments do in fact yield good estimates of the actual solutions.

As a second application, we also investigate the stability of solutions obtained by monotone iteration methods when considered as equilibrium solutions of an associated time dependent evolution equation. It turns out that every solution obtained by this method is *a priori* stable! Furthermore, the upper and lower solutions provide an estimate of the extent of stability. These matters will be explained more carefully in §4.

We close the article with a number of examples showing how the technique of upper and lower solutions can be applied. We also give an extension to systems (§6) under certain conditions on the non-linear terms.

2. Construction of solutions by monotone methods. The theorem which follows was first proved by H. Amann [18]. We give a proof here for the sake of completeness. The final convergence arguments are based directly on the L_p estimates of Agmon, Douglis, and Nirenberg [20] for regular elliptic boundary value problems.

Theorem 2.1. *Let there exist two smooth functions* $u_0(x) \geqq v_0(x)$ *such that*

$$Lu_0 + f(x, u_0) \leqq 0, \qquad Bu_0 \geqq g$$

and

$$Lv_0 + f(x, v_0) \geqq 0, \qquad Bv_0 \leqq g.$$

Assume f is a smooth function on $\min v_0 \leqq u \leqq \max u_0$. *Then there exists a regular solution w of*

$$Lw + f(x, w) = 0, \qquad Bw = g$$

such that $v_0 \leqq w \leqq u_0$.

Proof. We can assume $(\partial f/\partial u)(x, u)$ is bounded below for $x \varepsilon D$ and $\min v_0 \leqq u \leqq \max u_0$, so that

$$\frac{\partial f}{\partial u}(x, u) + \Omega u > 0$$

for all $x \varepsilon D$ and u in that interval, provided Ω is sufficiently large. We define the mapping T as follows: $v = Tu$ if

$$(L - \Omega)v = -[f(x, u) + \Omega u],$$
$$Bv = g.$$

T is completely continuous, since it takes C^α into $C^{2+\alpha}$ by the Schauder estimates for elliptic equations. Furthermore, it is monotone in the sense of Collatz [8] ($u \leqq v$ implies $Tu < Tv$) provided u and v are restricted to the set $\min v_0 \leqq u$, $v \leqq \max u_0$. In fact, if $u \leqq v$ then

$$(L - \Omega)Tu = -[f(x, u) + \Omega u],$$
$$BTu = g,$$
$$(L - \Omega)Tv = -[f(x, v) + \Omega v],$$
$$BTv = g.$$

Therefore

$$(L - \Omega)(Tv - Tu) = -[f(x, v) - f(x, u) + \Omega(v - u)], \qquad B(Tv - Tu) = 0. \tag{2.1}$$

Since $u \leqq v$ the quantity in brackets is non-negative. (Define $F(x, u) = f(x, u) + \Omega u$. Then $F_u > 0$ implies that F is increasing. The quantity in brackets is $F(x, v) - F(x, u)$.) So

$$(L - \Omega)w \leqq 0, \qquad Bw = 0$$

where $w = Tv - Tu$. By the strong maximum principle for elliptic operators, $w > 0$ in D (unless $w \equiv 0$ in which case $Tu = Tv$ and the right side of (2.1) is identically zero; but this happens only if $u \equiv v$, since F is strictly monotonic).

Now define $u_1 = Tu_0$ and $v_1 = Tv_0$. Let us show that $u_1 < u_0$ and $v_1 > v_0$ (strict inequalities in D). We have

$$(L - \Omega)u_1 = -[f(x, u_0) + \Omega u_0],$$
$$Bu_1 = g,$$

so

$$(L - \Omega)(u_1 - u_0) = -f(x, u_0) - \Omega u_0 - Lu_0 + \Omega u_0 = -[Lu_0 + f(x, u_0)] \geqq 0,$$

and

$$B(u_1 - u_0) = g - Bu_0 \leqq 0.$$

Therefore, by the strong maximum principle $u_1 < u_0$. (Assume $Lu_0 + f(x, u_0) \not\equiv 0$.) A similar argument shows that $v_1 > v_0$.

Since $u_1 < u_0$, $Tu_1 < Tu_0 = u_1$. Thus the sequence defined inductively by $u_1 = Tu_0$, $u_n = Tu_{n-1}$ is monotone decreasing. Similarly $v_n = Tv_{n-1}$, $v_1 = Tv_0$ defines a monotone increasing sequence. Furthermore, we have $v_n < u_n$ for all n:

$$v_0 < v_1 < v_2 < \cdots < v_n < \cdots < u_n < u_{n-1} < \cdots < u_1 < u_0 .$$

In fact, $v_0 < u_0$; suppose $v_{n-1} < u_{n-1}$. Then $u_n = Tu_{n-1} > Tv_{n-1} = v_n$, so the proof follows by induction.

Since the sequences $\{u_k\}$ and $\{v_k\}$ are monotone, the pointwise limits

$$\tilde{u}(x) = \lim_{k\to\infty} u_k(x) \quad \text{and} \quad \tilde{v}(x) = \lim_{k\to\infty} v_k(x)$$

both exist. The operator T is a composition of the nonlinear operation $u \to f(x, u) + \Omega u$ with the inversion of the linear, inhomogeneous elliptic boundary value problem $\varphi \to v$ defined by

$$(L - \Omega)v = \varphi,$$

$$Bv = g \quad \text{on} \quad \partial D.$$

For u bounded and $f(x, u)$ bounded on the range of u, the first operation takes bounded pointwise convergent into pointwise convergent sequences. The operation $\varphi \to v$ takes $L_p(D)$ continuously into the Sobolev space $W_{2,p}(D)$ for all p, $1 < p < \infty$ by the L_p estimates of Agmon, Douglis, and Nirenberg ([20], Theorem 15.2). Thus, since $u_k = Tu_{k-1}$ and since $\{u_k\}$ is a bounded, pointwise convergent sequence, it converges also in $W_{2,p}$. By the embedding lemma ([21], Theorem 3.6.6) $W_{2,p}$ is embedded continuously into $C_{1+\alpha}$ for $\alpha = 1 - n/p$, when $p > n$. Therefore $\{u_k\}$ converges in $C_{1+\alpha}$, and by the classical Schauder estimates for regular elliptic boundary value problems, $\{u_k\}$ then also converges in $C_{2+\alpha}$. We thus have

$$\tilde{u} = \lim_{k\to\infty} u_k = \lim_{k\to\infty} Tu_{k-1} = T \lim_{k\to\infty} u_{k-1} = T\tilde{u}$$

and similarly for $\tilde{v}$, by the continuity of T. Thus $\tilde{u}$ and $\tilde{v}$ are fixed points of T, and furthermore, they are of class $C_{2+\alpha}(D)$ for $0 < \alpha < 1$. They are therefore regular solutions of the elliptic boundary value problem (1.1). This completes the proof of Theorem 2.1.

Corollary 2.2. *The solutions $\tilde{u}$ and $\tilde{v}$ constructed in the proof of Theorem* 2.1 *are maximal and minimal solutions in the region $u_0 \leqq u \leqq u_0$; that is, if w is any solution of* (1.1) *such that $v_0 \leqq w \leqq u_0$, then $\tilde{u} \leqq w \leqq \tilde{v}$.*

Proof. We have $w = Tw$, $u_1 = Tu_0$; since $w \leqq u_0$, $Tw < Tu_0$, or $w < u_1$. By induction, $w \leqq u_n$ for all n; hence $w \leqq \tilde{u}(x)$. Similarly, $w \geqq \tilde{v}(x)$.

3. Initial value problems. The same monotone methods of §2 apply naturally to parabolic problems, since these too admit a maximum principle. The procedure is as follows. Let D be a region in $\mathbf{R}^n$ and let $\Gamma_T = D \times (0, T)$. This time the boundary operator has the form

$$u = g \quad \text{on} \quad \partial\Gamma_T$$

or

$$\frac{\partial u}{\partial \nu} + \beta u = g(x, t) \quad \text{on} \quad \partial D \times (0, T),$$

$$u(x, 0) = g(x) \quad \text{on} \quad D \cap \{t = 0\}.$$

A function $u_0(x, t)$ is an upper solution on Γ_T if

$$(3.1) \qquad Lu_0 + f(x, u_0) - \frac{\partial u_0}{\partial t} < 0 \quad \text{in} \quad \Gamma_T ,$$

$$Bu_0 \geqq g \quad \text{on} \quad \partial\Gamma_T .$$

A lower solution is defined by reversing the inequalities in (3.1). Given upper and lower solutions $u_0(x, t)$ and $v_0(x, t)$, with $v_0 \leqq u_0$ on Γ_T we choose Ω so large that $f_u + \Omega > 0$ on the region $(x, t) \in \Gamma_T$, $\min v_0 \leqq u \leqq \max u_0$. Then define u_1 by

$$(3.2) \qquad Lu_1 - \Omega u_1 - \frac{\partial u_1}{\partial t} = -[f(x, u_0) + \Omega u_0],$$

$$Bu_1 = g \quad \text{on} \quad \partial\Gamma_T .$$

By the maximum principle for parabolic equations it is easily seen that $u_1(x, t) < u_0(x, t)$ in Γ_T . The mapping $u_0(x, t) \to u_1(x, t)$ is denoted by $u_1 = \mathcal{G}u_0$. $\mathcal{G}$ again is a monotone operator in the sense of Collatz, the monotone arguments go through exactly as before, and we obtain

Theorem 3.1. Let there exist an upper solution $u_0(x, t)$ and a lower solution $v_0(x, t)$ in Γ_T :

$$Lu_0 + f(x, u_0) - \frac{\partial u_0}{\partial t} \leqq 0,$$

$$Bu_0 \geqq g \textit{ on } \partial\Gamma_T ,$$

$$Lv_0 + f(x, v_0) - \frac{\partial v_0}{\partial t} \geqq 0,$$

$$Bv_0 \leqq g \textit{ on } \partial\Gamma_T$$

with $v_0 \leqq u_0$. Define sequences u_n and v_n inductively by $u_{n+1} = \mathcal{G}u_n$, $v_{n+1} = \mathcal{G}v_n$. If Ω is chosen large enough so that

$$\frac{\partial f}{\partial u}(x, u) + \Omega > 0 \quad on \quad \min_{\Gamma_T} v_0 < u < \max_{\Gamma_T} u_0 \,,$$

then the sequences $\{u_n\}$ *and* $\{v_n\}$ *are monotone decreasing and increasing respectively. As* n *tends to infinity they both tend to a unique fixed point* $u = \mathcal{G}u$, *which is a strong solution of*

$$Lu + f(x, u) - \frac{\partial u}{\partial t} = 0, \qquad Bu = g \quad on \quad \partial\Gamma_T \,. \tag{3.3}$$

Let us outline the proof of Theorem 3.1 for the second boundary condition. The page and section numbers which follow refer to [19]. Given the linear inhomogeneous boundary value problem

$$\left(L - \Omega - \frac{\partial}{\partial t}\right)u = f, \qquad \frac{\partial u}{\partial \nu} + \beta u = g, \qquad u(x\ \ 0) = \psi$$

a solution may be constructed of the form

$$u(x, t) = \int_D G(x, \xi, t)\psi(\xi)\, d\xi + \int_0^t \int_D G(x, \xi, t - \tau)f(\xi, \tau)\, d\xi\, d\tau$$

$$+ \int_0^t \int_S G(x, \xi, t - \tau)\varphi(\xi, \tau)\, d\xi\, d\tau$$

where the third integral is a single layer potential, S is the boundary of D and G is the fundamental solution. See §15, pp. 395–406. The solution of this problem leads to a Volterra integral equation for φ with a weak singularity. This equation may be solved by successive approximations.

Given the sequence of iterates constructed by the monotone procedure described above it is easily seen that the limiting function (the iterations are bounded and converge monotonically) has a representation of the form above with continuous φ and f replaced by $-[f(\xi, u(\xi, \tau)) + \Omega u(\xi, \tau)]$.

To get the regularity we proceed by a ladder argument using estimates (13.1), p. 376 on the fundamental solution. First one can show that u is continuous on $\bar{D} \times [0, T]$. The first and third terms are regular for x in the interior of D and $t > 0$ so we need only investigate the second term. We first show that u is Lipschitz continuous in x. This follows from the fact that u is bounded and measurable as follows: We put

$$V_h(x, t) = \int_0^{t-h} \int_D G(x, \xi, t - \tau)[f(\xi, u) + \Omega u]\, d\xi\, d\tau,$$

differentiate with respect to x_i under the integral sign, apply (13.1) on p. 376 for $r = 0$, $s = 1$ and obtain a uniform estimate for $|\partial V_h/\partial x_i|$ as $h \to 0$. This proves V is Lipschitz continuous in x. Next one uses the Lipschitz continuity in x to prove Lipschitz continuity in t by similar arguments. The arguments proceed from here along the usual lines for obtaining the Schauder estimates. The boundary conditions are assumed by u in the sense (15.9), p. 404 [19].

In the case of the Dirichlet problem ($u = g$ on $\partial D \times (0, T)$) we can argue as follows. The operator $\mathcal{G}$ described above takes bounded measurable functions into continuous functions on $\bar{D} \times [0, \tau]$. This follows from the fact that the singularity of the Green's function is weak (see 16.16, p. 412 [19]). Furthermore, by applying the Lebesgue dominated convergence theorem, we see that $\mathcal{G}$ takes bounded pointwise convergent sequences into convergent sequences. Therefore $u = \lim_{n\to\infty} u_n = \lim_{n\to\infty} \mathcal{G}u_{n-1} = \mathcal{G}u$. Since u is bounded and measurable, $u = \mathcal{G}u$ is continuous on $\bar{D} \times [0, T]$. The interior regularity of u follows by the same kind of ladder argument as above, this time using the estimates in Theorem 16.3, p. 413, [19]. The following corollary is immediate:

Corollary 3.2. *Let $u_0(x)$ and $v_0(x)$ be upper and lower solutions of the elliptic problem*

$$Lu + f(x, u) = 0, \qquad Bu = g(x) \quad on \quad \partial D$$

with $v_0(x) \leqq u_0(x)$. Then for any continuous $\varphi(x)$ with $v_0(x) \leqq \varphi(x) \leqq u_0(x)$ we obtain a global regular solution of the initial value problem with initial data φ, and the solution $\varphi(x, t)$ satisfies $v_0(x) \leqq \varphi(x, t) \leqq u_0(x)$ for all $t > 0$. Note here that g is time independent.

We now want to establish certain monotonicity properties of solutions of the initial value problem. If $u_0(x)$ is an upper solution of the elliptic problem, then, as we have seen, it can be made the starting point of a monotone decreasing sequence of iterates. Here we shall also see that when $u_0(x)$ is taken as initial data for the initial value problem, the corresponding solution $u(x, t)$ is monotone decreasing in time.

One can see this as follows. Let

$$Lu_0 + f(x, u_0) < 0, \qquad Bu_0 \geqq g(x) \quad \text{on} \quad \partial D$$

and suppose $u(x, t)$ satisfies (3.3). Then from the proof of Theorem 3.1 we see that $u(x, t) \leqq u_0(x)$. Let $h > 0$ and define

$$w_h(x, t) = \frac{u(x, t + h) - u(x, t)}{h}.$$

Then $w_h(x, 0) = (u(x, h) - u(x, 0))/h \leqq 0$ and w_h satisfies the problem

$$Lw_h + \xi_h w_h - \frac{\partial w_h}{\partial t} = 0,$$

$$w_h(x, 0) \leqq 0,$$

$$Bw_h = 0 \quad \text{on} \quad \partial D$$

where

$$\xi_h(x, t) = \int_0^1 f_u(x, su(x, t + h) + (1 - s)u(x, t))\, ds.$$

By the maximum principle, $w_h(x, t) \leqq 0$ for all $t > 0$. Since u is a regular solution of (3.3) $u_t = \lim_{h \to 0+} w_h(x, t) \leqq 0$. Thus we have:

Theorem 3.3. *Every smooth* ($C^{2+\alpha}$) *upper solution gives rise to monotonically nonincreasing solutions of* (3.3) *when taken as initial data, while every smooth lower solution gives rise to nondecreasing solutions.*

Arguments similar to those used here have been used in [14]. The concept of upper and lower solutions is introduced, implicitly, in that the authors actually are constructing upper and lower solutions in specific cases. The monotonicity of solutions with such initial data is proved.

For the next theorem it will be convenient to remove the inhomogeneity from the boundary conditions. Note, however, that this results in no loss in generality. For, suppose we are given the boundary value problem

$$Lu + f(x, u) = 0, \qquad Bu = g. \tag{3.6}$$

Putting $u = v + \psi$ where $L\psi = 0$, $B\psi = g$, we get the boundary value problem

$$Lv + f(x, v + \psi) = 0, \qquad Bv = 0 \tag{3.7}$$

for v. Now suppose u_0 is an upper solution for (3.6). That is $Lu_0 + f(x, u_0) \leqq 0$, $Bu_0 \geqq g$. Then putting $v_0 = u_0 - \psi$ we get $Lv_0 + f(x, v_0 + \psi) \leqq 0$, $Bv_0 \geqq 0$; thus v_0 is an upper solution for (3.7).

We now show how the concept of upper and lower solutions can be weakened to correspond more to the classical notion of super and sub-harmonic functions in potential theory. Associated with the operator L is the adjoint operator L^*. The domain of L^* is defined to be

$$\mathfrak{D}\{L^*\} = \{\varphi : L^*\varphi \in L_2(D) \text{ and there exists } \varphi^* \text{ such that}$$
$$(Lu, \varphi) = (u, \varphi^*) \text{ for all } u \text{ in } \mathfrak{D}(L)\}.$$

If $\varphi \in \mathfrak{D}(L^*)$, then we write $\varphi^* = L^*\varphi$.

Definition. *We say that u_0 is a weak lower solution if u_0 is bounded and measurable on D and*

$$\iint_D u_0 L^*\varphi + f(x, u_0)\varphi \, dx \geqq 0 \tag{3.8}$$

for all $\varphi > 0$, $\varphi \in \mathfrak{D}(L^*)$.

We shall prove:

Theorem 3.4. *Let u_0 be a weak lower* (*upper*) *solution and let $u(x, t)$ satisfy the initial value problem*

$$Lu + f(x, u) - \frac{\partial u}{\partial t} = 0, \qquad u(x, 0) = u_0(x), \qquad Bu|_{\partial D} = 0 \tag{3.9}$$

in the region $D \times (0, T)$. Then $\partial u/\partial t \geqq 0$ ($\leqq 0$) in $D \times (0, T)$.

We first need to prove Theorem 3.4 for the linear case:

Lemma 3.5. *Let $u_0 \in L_2(D)$ and suppose $(u_0, L^*\varphi) \geqq 0$ $(\leqq 0)$ for all $\varphi \geqq 0 \in \mathfrak{D}(L^*)$. Then the solution of the initial boundary value problem $Lu - u_t = 0$, $Bu|_{\partial D} = 0$, $u(x, 0) = u_0(x)$ is monotone nondecreasing (nonincreasing) for $t > 0$.*

Proof. If the boundary is smooth, the operator L is closed and generates a semi-group e^{tL} on $L_2(D)$. It is not hard to see that L^* also generates a semi-group and that $e^{tL^*} = (e^{tL})^*$. We thus have, for any $\varphi \in \mathfrak{D}(L^*)$, $\varphi > 0$,

$$(u_t, \varphi) = (Le^{tL}u_0, \varphi) = (u_0, (e^{tL})^*L^*\varphi) = (u_0, e^{tL^*}L^*\varphi) = (u_0, L^*e^{tL^*}\varphi).$$

Now since $\varphi \in \mathfrak{D}(L^*)$ and $\varphi > 0$, $e^{tL^*}\varphi \in \mathfrak{D}(L^*)$ and is also positive. (The operator e^{tL} is the integral operator whose kernel is the Green's function $G(x, y, t) > 0$; the kernel of $(e^{tL})^* = e^{tL^*}$ is $G(y, x, t)$.) Therefore, from our assumption on u_0, $(u_t, \varphi) \geqq 0$ for any $\varphi \in \mathfrak{D}(L^*)$, $\varphi \geqq 0$. Since u_t is a continuous function for $t > 0$, $u_t \geqq 0$.

Going back to the proof of Theorem 3.4, we have to solve $Lu + f(x, u) - u_t = 0$, $u(x, 0) = u_0(x)$. First we solve

$$Lu_1 - \Omega u_1 - \frac{\partial u_1}{\partial t} = -[f(x, u_0) + \Omega u_0],$$

$$Bu_1|_{\partial D} = 0, \qquad u_1(x, 0) = u_0(x)$$

for a suitable $\Omega > 0$. (Ω is to be chosen so that $f_u(x, u) + \Omega > 0$ for $\min_{\Gamma_T} u(x, t) \leqq u \leqq u_0(x)$, where $u(x, t)$ is the solution of (3.9).) Let $\psi = Tu_0$; that is,

$$(L - \Omega)\psi = -[f(x, u_0) + \Omega u_0],$$

$$B\psi|_{\partial D} = 0.$$

Then setting $u_1 = v_1 + \psi$ we get for v_1 the equation

$$Lv_1 - \Omega v_1 - \frac{\partial v_1}{\partial t} = 0,$$

$$Bv_1(x, t) = Bu_1(x, t) - B\psi(x) = 0 \quad \text{on} \quad \partial D \times \{t > 0\},$$

$$v_1(x, 0) = u_0(x) - \psi(x).$$

For any $\varphi \in \mathfrak{D}(L^*)$, $\varphi > 0$, we have

$$\begin{aligned}\iint_D v_1(x, 0)(L^* - \Omega)\varphi\, dx &= \iint_D u_0(x)(L^* - \Omega)\varphi - (Tu_0)(L^* - \Omega)\varphi\, dx \\ &= \iint_D u_0(L^* - \Omega)\varphi + (f(x, u_0) + \Omega u_0)\varphi\, dx \\ &= \iint_D u_0L^*\varphi + f(x, u_0)\varphi\, dx < 0\end{aligned}$$

since $\psi = Tu_0 \in \mathfrak{D}(L)$ (that is, $B\psi = 0$ on ∂D). Therefore by Lemma 3.5 applied to the operator $(L - \Omega)$,

$$\frac{\partial u_1}{\partial t} = \frac{\partial v_1}{\partial t} \leqq 0.$$

Now let u be the actual solution and put $u = u_1 + v$. We get for v the equation

$$Lv + f(x, u_1 + v) - \frac{\partial v}{\partial t} = f(x, u_0) + \Omega u_0 - \Omega u_1 ,$$

$$Bv|_{\partial D} = 0, \qquad v(x, 0) = 0.$$

As before we form the time differences

$$w_h(x, t) = \frac{v(x, t + h) - v(x, t)}{h}, \qquad u_{1,h}(x, t) = \frac{u_1(x, t + h) - u_1(x, t)}{h},$$

$$\xi_h(x, t) = \int_0^1 f_u(x, s(u_1(x, t + h) + v(x, t + h)) + (1 - s)(u_1(x, t) + v(x, t))) \, ds.$$

We get

$$Lw_h + \xi_h w_h - \frac{\partial w_h}{\partial t} = -[\xi_h + \Omega]u_{1,h} .$$

For $h = 0$, $\xi_0(x, t) = f_u(x, u_1 + v)$, and so for an appropriate choice (originally) of Ω we will have $\xi_h + \Omega > 0$. Since $\partial u_1/\partial t \leqq 0$, $u_{1,h} \leqq 0$ so the right side above is nonnegative. Furthermore, $Bw_h = 0$ on ∂S and

$$w_h(x, 0) = \frac{(u(x, h) - u_1(x, h)) - (u(x, 0) - u_1(x, 0))}{h} = \frac{u(x, h) - u_1(x, h)}{h} \leqq 0.$$

This last inequality follows from the fact that the iterates decrease monotonically to u: $u_0(x) \geqq u_1(x, t) \geqq \cdots \geqq u(x, t)$ for $t \geqq 0$. Now the maximum principle applied to w_h shows that $w_h(x, t) \leqq 0$; hence in the limit as $h \to 0$, $\partial v/\partial t \leqq 0$. This shows $\partial u/\partial t \leqq 0$. Q.E.D.

Finally, let us prove

Theorem 3.6. *Let u_0 and v_0 be lower and upper solutions of the elliptic equation with $u_0(x) \leqq v_0(x)$. Let u and v be solutions of the initial value problem with initial data u_0 and v_0 respectively, and $Bu = 0$. Then $u(x, t) \uparrow \bar{u}(x)$ and $v(x, t) \downarrow \bar{v}(x)$, $\bar{u} \leqq \bar{v}$ and $\bar{v}$ and $\bar{u}$ are regular stationary solutions of the elliptic boundary value problem.*

Proof. By the comparison theorem for parabolic equations ([15], p. 187), we have $u_0 \leqq u(x, t) \leqq v(x, t) \leqq v_0(x)$ for all $t \geqq 0$. By Theorem 3.4 we have seen that $v_t \leqq 0$ and $u_t \geqq 0$ so v is nonincreasing and u is nondecreasing. Therefore the pointwise limits

$$\tilde{u}(x) = \lim_{t\to\infty} u(x, t)$$

and

$$\tilde{v}(x) = \lim_{t\to\infty} v(x, t)$$

exist, and $\tilde{u}(x) \leqq \tilde{v}(x)$. It will suffice to prove that $\tilde{u}$ is a strong solution of the stationary equation.

For all $\varphi \in \mathfrak{D}(L^*)$ and all $t > 0$ we have

$$\int_D u_t\varphi \, dx = \int_D \{Lu\varphi + f(x, u)\varphi\} \, dx = \int_D \{uL^*\varphi + f(x, u)\varphi\} \, dx.$$

Operating on both sides with $T^{-1} \int_0^T dt$ we have

$$\int_D \frac{u(x, T) - u(x, 0)}{T} \varphi \, dx = \int_D \left\{L^*\varphi \frac{1}{T} \int_0^T u(x, t) \, dt + \varphi \frac{1}{T} \int_0^T f(x, u) \, dt\right\} dx.$$

Now

$$\frac{u(x, T) - u(x, 0)}{T} \underset{T\to\infty}{\longrightarrow} 0,$$

$$\frac{1}{T} \int_0^T u(x, t) \, dt \underset{T\to\infty}{\longrightarrow} \tilde{u}(x),$$

$$\frac{1}{T} \int_0^T f(x, u(x, t)) \, dt \underset{T\to\infty}{\longrightarrow} f(x, \tilde{u}(x)).$$

Furthermore, the three quantities on the left above remain bounded uniformly as $T \to \infty$. Therefore, by the Lebesgue dominated convergence theorem we get in the limit as $T \to \infty$,

$$0 = \int_D \{\tilde{u}L^*\varphi + f(x, \tilde{u})\varphi\} \, dx;$$

or

$$(\tilde{u}, L^*\varphi) = (g, \varphi) \tag{3.11}$$

where $g(x) = -f(x, \tilde{u}(x))$.

We now want to show that if $(u, L^*\varphi) + (f(x, u), \varphi) = 0$ for all $\varphi \in \mathfrak{D}(L^*)$, then u is a classical solution of the boundary value problem. First we note that L and L^* are invertible. (This follows by the maximum principle.) Let $\mathcal{G}$ be the inverse of L. Putting $w = -\mathcal{G}f(x, u)$ we have

$$(w, L^*\varphi) = -(\mathcal{G}f, L^*\varphi) = -(f, \mathcal{G}^*L^*\varphi) = -(f, \varphi),$$

hence $(u - w, L^*\varphi) = 0$ for all $\varphi \in \mathfrak{D}(L^*)$. But the range of L^* is all of L_2 (since L^* is invertible), so this means $u = w = -\mathcal{G}f(x, u)$.

Since $u + \mathcal{G}f(x, u) = 0$, u is a weak solution to the nonlinear boundary problem $Lu + f(x, u) = 0$. To show that u is a strong solution we need to prove the regularity of u. Again, by the L_p estimates of Agmon, Douglis, Nirenberg ([20], Theorem 15.2) $u \in W_{2,p}(D)$ for any p, $1 < p < \infty$ since $\mathcal{G}$ takes L_p into $W_{2,p}$ and since $f(x, u)$ is bounded and measureable if u is. By the embedding lemma, for $p > n$ $u \in C_{1+\alpha}(\bar{D})$. Finally, by the classical Schauder estimates, $u \in C_{2+\alpha}(\bar{D})$. This concludes the proof of Theorem 3.6.

The results in this section will be of great importance in investigating the asymptotic behavior of solutions of the parabolic problem and the stability properties of stationary solutions.

4. Stability of solutions.

Definition. A solution U of the boundary value problem (1.1) is stable in the maximum norm if given any $\epsilon > 0$ there exists a $\delta > 0$ such that if $||u_0(x) - U||_\infty < \delta$ then $||u(x, t) - U||_\infty < \epsilon$ for all $t \geqq 0$, where u satisfies the initial value problem

$$u_t = Lu + f(x, u), \qquad Bu = g, \qquad u(x, 0) = u_0(x). \tag{4.1}$$

U is asymptotically stable if in addition $||u(x, t) - U||_\infty \to 0$ as $t \to \infty$. U is unstable if it is not stable; that is, if the above definition is complemented.

Stability analyses have been carried out by a number of authors (see *e.g.* [1], [4], [5], [14]) but generally with assumptions on the nonlinear term f or on L (for example self-adjointness). We are going to prove that any solution of (1.1) that is obtained by the monotone procedures of §2 is stable, without any assumptions about f or L. We shall also show how the upper and lower solutions can be used to estimate the *extent* of stability. (This procedure has also been carried out in [14] along similar lines.)

From the standpoint of applications, of course, it is important to know whether a given solution is stable. But from the standpoint of the application of the monotone methods it is also important to realize that *only* stable solutions can be obtained by these procedures. Other solutions which might exist but would be unstable must be obtained by other means. We shall come back to this point when we discuss examples in §5.

As a preliminary result we have

Theorem 4.1. *Let U be a solution of* (1.1) *and let φ and ψ be lower and upper solutions respectively with $\varphi < U < \psi$ on $\bar{D}$. Let v satisfy the initial value problem*

$$(4.1) \qquad Lv + f(x, v) - v_t = 0,$$
$$Bv = g, \qquad v(x, 0) = v_0(x).$$

If $\varphi < v_0 < \psi$, then $\varphi(x, t) < v(x, t) < \psi(x, t)$, where $\varphi(x, t)$ and $\psi(x, t)$ are solutions of (4.1) *with initial data φ and ψ. If $T^m\varphi \uparrow U$ and $T^m\psi \downarrow U$, then U is asymptotically stable and $v(x, t) \to U$ as $t \to +\infty$.*

Proof. This is an immediate consequence of the comparison theorem for parabolic equations [15]. By Theorem 3.6 we know that $\varphi(x, t) \uparrow \tilde{\varphi}(x)$ and $\psi(x, t) \downarrow \tilde{\psi}(x)$ as t tends to infinity. If $\tilde{\varphi}(x) = \tilde{\psi}(x)$, then necessarily $U(x) = \tilde{\varphi} = \tilde{\psi}$ and $v(x, t) \to U$. This will be the case if $\varphi(\psi)$ generates a monotone increasing (decreasing) sequence which converges to U. In particular, it follows from Corollary 2.2 that if $T^m\varphi \uparrow U$ and $T^m\psi \downarrow U$ then U is asymptotically stable, and any solution of (4.1) with initial data $\varphi < v_0 < \psi$ tends to U as $t \to \infty$. Q.E.D.

A converse to Theorem 4.1 also holds: if U is a stable solution of (1.1) then it can be obtained as a limit of upper and lower solutions. To prove this we consider first the derivative operator

$$(4.2) \qquad L\varphi + f'_u(x, U)\varphi,$$
$$B\varphi = 0$$

evaluated at the solution U. It is well known that the eigenvalue λ_1 of (4.2) with smallest real part has associated with it a positive eigenfunction $\varphi_1 \geqq 0$. This is a consequence of Krein's theorem on operators preserving a cone in a Banach space. We have

Theorem 4.2. *If $\lambda_1 < 0$, then U is stable and is the limit of a sequence of upper solutions from above and lower solutions from below. If $\lambda_1 > 0$, then U is unstable and is the limit of a sequence of lower solutions from above and upper solutions from below.*

Proof. Suppose $\lambda_1 < 0$ and consider the function $U + \epsilon\varphi_1$:

$$L(U + \epsilon\varphi_1) + f(x, U + \epsilon\varphi_1) = LU + f(x, U)$$
$$+ \epsilon[L\varphi_1 + f'_u(x, U)\varphi_1] + O(\epsilon^2\varphi_1^2) = \epsilon\lambda_1\varphi_1 + O(\epsilon^2\varphi_1^2),$$
$$B(U + \epsilon\varphi_1) = BU = g.$$

Since $\varphi_1 \geqq 0$ and $\lambda_1\varphi_1$ dominates the term $O(\epsilon^2\varphi_1^2)$ for small ϵ, $U + \epsilon\varphi_1$ is an upper solution for $\epsilon > 0$ and a lower solution for $\epsilon < 0$. This establishes the first statement above. (The stability of U follows from Theorem 4.1.) If $\lambda_1 > 0$, then $U + \epsilon\varphi_1$ is a lower solution for $\epsilon < 0$ and an upper solution for $\epsilon > 0$. To establish the instability of U, let φ_δ be a solution of the initial value problem with $\varphi_\delta(x, 0) = U + \delta\varphi_1$ (say $\delta > 0$). Then $\varphi_\delta(x, t)$ is increasing for $t > 0$ (assuming δ is sufficiently small so that $\varphi_\delta(x, 0)$ is a lower solution). Consequently we have solutions with small initial data which do not remain small, and this amounts to a statement of instability. Q.E.D.

Remark. We actually proved only that $\partial\varphi_\delta/\partial t \geqq 0$ in Theorem 3.3. However, either $\varphi_\delta(x, t)$ is bounded above for all $t > 0$, in which case it tends to a stationary solution $\tilde{\varphi}(x)$, or $\varphi_\delta(x, t) \uparrow +\infty$. In either case, it must grow as t increases.

One final question: Suppose we know only that U is (say) the limit from above of upper solutions. What can we conclude about the stability of U in this case? First, it is clear from the previous arguments that U is stable to sufficiently small perturbations from above. Moreover we must have $\lambda_1 \leqq 0$. Let u_n be a sequence of upper solutions converging downward to U. If $\lambda_1 > 0$ we can construct a lower solution $U + \delta\varphi_1$ for small $\delta > 0$, with $U + \epsilon\varphi_1 \leqq u_k$ for some fixed integer k. Let $v(x, t)$ and $u_k(x, t)$ be solutions of the initial value problem with $v(x, 0) = U(x) + \delta\varphi_1(x)$, $u_k(x, 0) = u_k(x)$. Then as $t \to +\infty$, $u_k(x, t) \downarrow U(x)$ while $v(x, t)$ increases. This is in contradiction to the comparison theorem for parabolic equations, which implies we must have $v(x, t) \leqq u_k(x, t)$ for all $t > 0$.

5. Examples and applications.

(a) Consider first the boundary value problem

$$-(\Delta + \mu)u + u^3 = 0, \qquad u|_{\partial D} = 0 \tag{5.1}$$

in some domain D. Let φ_1 be the first eigenfunction of the Laplacian:

$$\Delta\varphi_1 + \lambda_1\varphi_1 = 0, \qquad \varphi_1|_{\partial D} = 0.$$

Then $\lambda_1 > 0$ and $\varphi_1 > 0$ on the interior of D. To construct upper and lower solutions we try $u = \sigma\varphi_1$, where σ is a constant. We get

$$\begin{aligned}(\Delta + \mu)u - u^3 &= (\mu - \lambda_1)\sigma\varphi_1 - (\sigma\varphi_1)^3 \\ &= \sigma\varphi_1[(\mu - \lambda_1) - \sigma^2\varphi_1^2].\end{aligned} \tag{5.2}$$

If $\mu < \lambda_1$, then the quantity in brackets is always negative, so $u = \sigma\varphi_1$ is an upper solution when $\sigma > 0$ and a lower solution when $\sigma < 0$. This shows that the null solution is stable when $\mu < \lambda_1$.

Now suppose $\mu > \lambda_1$. Then for small $|\sigma|$ the quantity in brackets is positive and $\sigma\varphi_1$ is a lower solution when $\sigma > 0$ and an upper solution when $\sigma < 0$. The trivial solution is now unstable. Let us now show there exist stable positive and negative (non-trivial) solutions of (5.1) when $\mu > \lambda_1$.

Let $\tilde{\varphi}_1$ be the principal eigenfunction of the Laplacian with boundary conditions

$$\epsilon\frac{\partial\tilde{\varphi}_1}{\partial\nu} + \tilde{\varphi}_1 = 0.$$

For small enough ϵ, $\tilde{\lambda}_1$ is close to λ_1, and $\tilde{\lambda}_1 > 0$ for any ϵ; furthermore, $\tilde{\varphi}_1 > 0$ on $\bar{D}$ (a consequence of the maximum principle at the boundary). For sufficiently large $|\sigma|$ the quantity $[(\mu - \lambda_1) - \sigma^2\tilde{\varphi}_1^2]$ is negative, so $\sigma\tilde{\varphi}_1$ is an upper solution for large positive σ and a lower solution for large negative σ. These functions taken together with $\sigma\varphi_1$ imply the desired result.

If we were to reverse the sign of u^3 in the previous example and consider instead

$$(\Delta + \mu)u + u^3 = 0,$$

then the situation regarding upper and lower solutions is altered as follows. For $\mu < \lambda_1$, $\sigma\varphi_1$ would still be a positive upper solution, but $\sigma\tilde{\varphi}_1$ for large σ is a lower solution. We now have a lower solution lying *above* an upper solution, the reverse of the situation in the theorems in §§1–4. Nothing can rigorously be deduced from the monotone arguments, although it seems plausible to conjecture an unstable positive solution exists when $\mu < \lambda_1$ (also an unstable negative solution). In this case the existence and stability of such solutions can be established by birfurcation arguments for $\mu < \lambda_1$ [16] when μ is near λ_1.

Finally, note the following for the problem (5.1). Let $\mu > \lambda_1$ and normalize the eigenfunction φ_1 so that $\sup \varphi_1 = 1$: $0 \leqq \varphi_1 \leqq 1$. Then in constructing the positive lower solution we may take σ so large that

$$(\mu - \lambda_1) - \sigma^2\varphi_1^2 \geqq \mu - \lambda_1 - \sigma^2 \geqq 0.$$

Thus we can choose $\sigma \leqq (\mu - \lambda_1)^{1/2}$. This shows that the positive solution has a maximum value exceeding $(\mu - \lambda_1)^{1/2}$ and tends to $+\infty$ as $\mu \to +\infty$.

(b) Now consider the example

$$\Delta u + u^2 = 0, \tag{5.3}$$

$$u|_{\partial D} = 0.$$

One solution to this boundary value problem is $u \equiv 0$; we might also expect a positive one. In fact, positive solutions to (5.3) can be constructed in special cases (for example $n = 1$ space dimension).

Let us show here that any such solution is *a priori* unstable—we don't even need to be able to solve the equation. Let w be such a solution. First note that $\Delta w = -w^2 \leqq 0$; hence w cannot have an interior negative minimum by the maximum principle. Since $w|_{\partial D} = 0$, $w \geqq 0$ in the interior of D. We now consider the comparison functions $\varphi = \sigma w$ for $\sigma > 0$. We have

$$\begin{aligned}\Delta\varphi + \varphi^2 &= \sigma\Delta w + \sigma^2 w^2\\ &= \sigma[\Delta w + \sigma w^2]\\ &= \sigma[\Delta w + w^2 + (\sigma - 1)w^2]\\ &= \sigma(\sigma - 1)w^2\end{aligned}$$

so φ is an upper solution if $0 < \sigma < 1$ and a lower solution if $\sigma > 1$. This shows that w would be unstable. In fact, taking $z(\sigma, x, t)$ to be a solution of the initial value problem with initial data $z(\sigma, x, 0) = \sigma w(x)$ we see that z is monotone increasing in t for $\sigma > 1$ and monotone decreasing in t for $\sigma < 1$.

Note also in the above example that if $\sigma < 0$, then φ is again a lower solution. This shows that the trivial solution is stable. Moreover, we see that $u = 0$ is stable to any perturbations which vanish on ∂D and lie strictly below the nontrivial solution w. These arguments, however, presuppose the existence of the non-trivial solution w. Thus we see that unstable solutions might also be of physical interest in as much as they may determine the extent of stability.

(c) We now consider the following singular perturbation problem discussed previously by D. S. Cohen [4]:

$$\begin{aligned} \beta u'' - u' + f(u) &= 0, \qquad 0 \leqq x \leqq 1, \\ -u'(0) + au(0) &= 0, \\ u'(1) &= 0. \end{aligned} \tag{5.4}$$

This problem arises in simple models of tubular chemical reactors.

The function f has the qualitative appearance shown in Fig. 1. The singular perturbation arguments for (5.4) are outlined in [4]. The gist of the discussion there is that solutions of (5.4) are given to order $O(\beta)$ by solutions of the first order equation

$$u' = f(u) \tag{5.5}$$

where $u(0)$ is chosen so as to satisfy (5.5) and the left boundary condition. Taking $u(0) = \alpha$ we get $u'(0) = f(\alpha)$ from (5.5) and $u'(0) = a\alpha$ from (5.4), or

$$a\alpha = f(\alpha). \tag{5.6}$$

The solutions of this equation are indicated graphically in Fig. 1. For appropriate choices of a we get one or three solutions (in some cases only two, with one

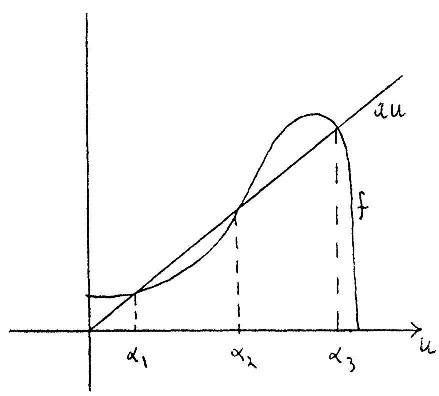

FIGURE 1

solution degenerate). Let us show that, for small β, the roots α_1 and α_3 in Fig. 1 lead to stable solutions of (5.4).

We solve the first order problem

$$-v' + f(v) = -\epsilon, \qquad v(0) = \alpha + \delta \tag{5.7}$$

where ϵ and δ are parameters and $\alpha = \alpha_1$ or α_3. Denoting by L the operator $Lu = \beta u'' - u'$ we have

$$\begin{aligned} Lv + f(v) &= \beta v'' - v' + f(v) \\ &= \beta v'' - \epsilon, \\ -v'(0) + av(0) &= -\epsilon - f(v(0)) + av(0) \\ &= -\epsilon + [a(\alpha + \delta) - f(\alpha + \delta)] \\ &= -\epsilon + a\alpha - f(\alpha) + [a\delta - f'(\alpha)\delta] + O(\delta^2) \\ &= -\epsilon + \delta[a - f'(\alpha)] + O(\delta^2). \end{aligned}$$

Now for $\alpha = \alpha_1$ or α_3 we have $a - f'(\alpha) > 0$ (see Fig. 1), so taking ϵ sufficiently small, say

$$\epsilon = \frac{\delta}{2}(a - f'(\alpha)),$$

we have (denoting the solution of 5.7 by v_δ)

$$Lv_\delta + f(v_\delta) < 0, \qquad -v_\delta'(0) + av_\delta(0) > 0 \tag{5.8}$$

for $\delta > 0$ and β sufficiently small. If $\delta < 0$ these inequalities are reversed. (Note that v_δ'' remains bounded for small $|\delta|$, so $\beta v_\delta'' = O(\beta)$. We first pick $\delta_0 > 0$ and then restrict β to be so small that (5.8) holds for $|\delta| < \delta_0$).

For small $\delta > 0$, v_δ is to be an upper solution and $v_{-\delta}$ a lower solution. We claim that $v_{-\delta} < v_\delta$ on $0 \leqq x \leqq 1$. In fact, $v_\delta(0) - v_{-\delta}(0) = 2\delta > 0$. Let x_0 be the first value of x at which $v_\delta(x_0) = v_{-\delta}(x_0)$. We have $v_{-\delta}'(x_0) \leqq v_\delta'(x_0)$; hence, from (5.7) $f(v_\delta) + \epsilon \leqq f(v_{-\delta}) - \epsilon$, which implies that $\epsilon \leqq -\epsilon$, a contradiction. So $v_\delta(x) > v_{-\delta}(x)$ on $[0, 1]$.

We have one final matter to deal with, and that is the boundary condition at $x = 1$. For $\delta > 0$ we are all right, since $v_\delta'(1) = f(v_\delta(1)) + \epsilon > 0$; so for $\delta > 0$, v_δ really is an upper solution. To actually get the lower solution we take

$$u_{-\delta}(x) = v_{-\delta}(x) - \beta v_{-\delta}'(1)e^{-(1-x)/\beta}.$$

Then $u_{-\delta}'(1) = 0$ and $u_{-\delta}(x) \leqq v_{-\delta}(x) \leqq v_\delta(x)$. This gives us the requisite lower solution.

For $\alpha = \alpha_1$ or α_3 we see that it is possible, for small β, to construct a lower solution below and an upper solution above the singular perturbation solution.

Therefore, for small β, there is a stable solution in the vicinity of the singular perturbation solution

$$v(x) = v_0(x) - \beta v_0'(1)e^{-(1-x)/\beta}.$$

For $\alpha = \alpha_2$ the equation and boundary conditions work against each other; this time we get a lower solution above and an upper solution below the singular perturbation solution—thus indicating, formally at least, that the middle solution might be unstable. With regard to this matter, see the discussion in [17].

(d) In the previous example there was no boundary layer at $x = 1$; however, if we require $u(1) = 0$ in (5.4) then a boundary layer occurs and we might expect the solution to be approximated by

$$w(x) = v(x) - v(1)e^{-(1-x)/\beta},$$

where v is the "outer solution" given by (5.7). This time we construct upper and lower solutions of the form

$$w_{\sigma,\delta}(x) = v_\delta(x) - v_\delta(1)\varphi_\sigma(x)$$

where

$$\varphi_\sigma(x) = e^{-\sigma(1-x)/\beta};$$

here $|\delta|$ is small and σ is close to unity. v_δ is the same function constructed previously. We have

$$\begin{aligned} Lw_{\sigma,\delta} + f(w_{\sigma,\delta}) &= \beta v_\delta'' - v_\delta' + f(v_\delta - v_\delta(1)\varphi_\sigma) - v_\delta(1)L\varphi_\sigma \\ &= \beta v_\delta'' - \left(\frac{a - f'(\alpha)}{2}\right)\delta + [f(v_\delta - v_\delta(1)\varphi_\sigma) - f(v_\delta)] \\ &\quad - \frac{\sigma v_\delta(1)}{\beta}(\sigma - 1)\varphi_\sigma \\ &= \beta v_\delta'' - v_\delta(1)\varphi_\sigma(x)\left[\frac{\sigma(\sigma-1)}{\beta} + h(x)\right] - \left(\frac{a - f'(\alpha)}{2}\right)\delta \end{aligned}$$

where

$$h(x) = \int_0^1 f'(v_\delta(x) - \tau v_\delta(1)\varphi_\sigma(x))\, d\tau.$$

The function h, which arises from Taylor's expansion of $f(v_\delta - v_\delta(1)\varphi_\sigma)$, is uniformly bounded for small enough $|\delta|$ and $|\sigma - 1|$.

To get an upper solution take $\delta > 0$ and $\sigma > 1$. Then $v_\delta(x) \geqq v_0(x)$ and, since the term in brackets above will be positive for small β, $w_{\sigma,\delta}$ is an upper solution for small β. Note that $w_{\sigma,\delta}(1) = 0$ and

$$-w_{\sigma,\delta}'(0) + aw_{\sigma,\delta}(0) = \delta\left(\frac{a - f'(\alpha)}{2}\right) + O(\delta^2) + O\left(\frac{e^{-1/\beta}}{\beta}\right)$$

is of the same sign as δ. To get a lower solution take $\delta < 0$ and $\sigma < 1$. Finally, let us show that if $\delta > 0$ and $\sigma > 1$, then $w_{\sigma,\delta} \geqq w_{1,0}(x)$. In fact, for $\sigma > 1$,

$$-e^{-\sigma(1-x)/\beta} > -e^{-(1-x)/\beta};$$

hence, since $v_\delta(x) > v_0(x)$,

$$-v_\delta(1)e^{-\sigma(1-x)/\beta} > -v_0(1)e^{-\sigma(1-x)/\beta},$$

$$v_\delta(x) - v_\delta(1)e^{-\sigma(1-x)/\beta} > v_0(x) - v_0(1)e^{-\sigma(1-x)/\beta}.$$

Similarly, if $\sigma' < 1$ and $\delta' < 0$, then $w_{1,0}(x) > w_{\sigma',\delta'}(x)$, so for small β the singular perturbation solution lies between upper and lower solutions.

6. Further results. The idea of upper and lower solutions also affords a method of comparing different boundary value problems—say on different domains or with different non-linear terms. We have

Theorem 6.1. *Consider two problems P_i $(i = 1, 2)$*

$$Lu + f_i(x, u) = 0, \qquad x \in D_i,$$

$$u|_{\partial D_i} = g_i$$

where $D_1 \subset D_2$, $f_1(x, u) \leqq f_2(x, u)$ for all $x \in D_1$, $\sup g_1 \leqq \inf g_2$. Let $\tilde{u}$ be a solution of P_2 and suppose that $f_2(x, \tilde{u}) \geqq 0$ everywhere on D_2. Then $\tilde{u}$ is an upper solution for P_1.

Regarding the last hypothesis, we remark that $f_2 \geqq 0$ is an important case. Theorem 6.1 makes it possible to compare a problem on a domain of arbitrary shape with a similar problem on a larger or smaller domain of a more convenient geometry (such as a sphere). The equation $\Delta u + f(u)$ becomes an ordinary differential equation if one looks for solutions with spherical symmetry, and may therefore yield to a complete analysis. (See, for example, the methods in [2], [11].)

Theorem 6.1 may also be used to gain information on the variation of a solution with a parameter, *e.g.*, in the problem $\Delta u + \lambda f(u) = 0$. Results of this type have been obtained in [5].

Proof of Theorem 6.1. If $\tilde{u}$ is a solution of problem P_2, then

$$L\tilde{u} + f_1(x, \tilde{u}) \leqq L\tilde{u} + f_2(x, \tilde{u}) = 0 \quad \text{on } D_1.$$

Furthermore,

$$L\tilde{u} = -f_2(x, \tilde{u}) \leqq 0,$$

so for $w(x) = \tilde{u}(x) - g_{\inf}$, where $g_{\inf} = \inf_{\partial D_2} g_2(x)$, we have $Lw \leqq 0$ in D_2 and $w \geqq 0$ on ∂D_2. By the strong maximum principle we have $w > 0$ on the interior of D_2; hence $\tilde{u} > g_{\inf} \geqq g_2(x)$ on ∂D_1. Q.E.D.

Corollary 6.2. *Consider the two problems of Theorem* 6.1 *under the additional hypotheses that* $0 < f_1 \leqq f_2$ *and* $0 \leqq g_1 \leqq g_2$. *If problem P_1 does not have a positive* *solution, then neither does P_2. Equivalently, if P_2 has a positive solution, then so does P_1.*

Proof. If P_2 has a solution $\tilde{u}(x)$, then $\tilde{u}$ is an upper solution for P_1; on the other hand, P_1 always has the lower solution $\varphi \equiv 0$. So the existence of a positive solution to P_2 implies that of a positive solution of P_1.

Finally, let us indicate the extension of monotone methods to systems of equations satisfying certain restrictions. The method applies, for example, to systems of the form

$$\Delta u + f(u, v) = 0,$$
$$\Delta v + g(u, v) = 0,$$

with suitable boundary conditions $B_1 u = g$, $B_2 v = h$, where $f_v \geqq 0$ and $g_u \geqq 0$ or where $f_v \leqq 0$, $g_u \leqq 0$. If the reader will check through the proof of Theorem 2.1 he will see that the success of the monotone method hinges on the induction argument in the proof of the monotonicity of the successive iterations. Let us take first the case $f_v \geqq 0$ and $g_u \geqq 0$. We get for the difference of successive iterations

$$(6.1) \qquad \begin{aligned} (\Delta - \Omega)(u_{n+1} - u_n) &= -[F(u_n, v_n) - F(u_{n-1}, v_{n-1})], \\ (\Delta - \Omega)(v_{n+1} - v_n) &= -[G(u_n, v_n) - G(u_{n-1}, v_{n-1})] \end{aligned}$$

where

$$F(u, v) = f(u, v) + \Omega u,$$
$$G(u, v) = g(u, v) + \Omega v.$$

If $u_{n-1} < u_n$ and $v_{n-1} < v_n$, then the right hand sides of (6.1) are non-positive. Thus the maximum principle would yield $u_{n+1} > u_n$ and $v_{n+1} > v_n$, thus providing the induction step. If $u_{n-1} > u_n$ and $v_{n-1} > v_n$, then we obtain $u_{n+1} < u_n$ and $v_{n+1} < v_n$. So for this case it suffices to construct upper-upper (u_0, v_0) and lower-lower (φ_0, ψ_0) solutions:

$$\Delta u_0 + f(u_0, v_0) \leqq 0,$$
$$\Delta v_0 + g(u_0, v_0) \leqq 0,$$
$$\Delta \varphi_0 + f(\varphi_0, \psi_0) \geqq 0,$$
$$\Delta \psi_0 + g(\varphi_0, \psi_0) \geqq 0$$

with $\varphi_0 < u_0$ and $\psi_0 < v_0$. Then φ_0 and ψ_0 generate increasing and u_0, v_0 generate decreasing sequences. The remaining details are similar to those in the proof of Theorem 2.1 and are left to the reader.

The second case is almost the same. We again get equations (6.1) for the difference between successive iterations. This time $F_u > 0$ and $F_v < 0$ for suitable choice of Ω, while $G_u < 0$ and $G_v > 0$. If $u_n > u_{n-1}$ and $v_n < v_{n-1}$, then

$$F(u_n, v_n) - F(u_{n-1}, v_{n-1}) > 0$$

so that $(\triangle - \Omega)(u_{n+1} - u_n) < 0$ and $u_{n+1} > u_n$ by the maximum principle. Similarly, $v_{n+1} < v_n$. So if we begin with a lower-upper solution (u_0, v_0):

$$\triangle u_0 + f(u_0, v_0) > 0,$$

$$\triangle v_0 + g(u_0, v_0) < 0,$$

we get an increasing sequence $\{u_n\}$ and a decreasing sequence $\{v_n\}$. Similarly an upper-lower solution (φ_0, ψ_0) leads to a decreasing sequence $\{\varphi_n\}$ and an increasing sequence $\{\psi_n\}$. The procedure for constructing solutions in this case should now be clear.

We have attempted to get these monotone methods to work in other cases, but without success. The second case above is applicable to systems governing simple endothermic chemical reactions, *viz.* (see, *e.g.*, [9])

$$\triangle c = \lambda_1 e^{-\gamma/T} c,$$

$$\triangle T = \lambda_2(\triangle H) e^{-\gamma/T} c$$

where $(\triangle H)$, the heat of reaction, is positive. In that case $f(c, T) = -\lambda_1 e^{-\gamma/T} c$ and $g(c, T) = -\lambda_2(\triangle H) e^{-\gamma/T} c$ are each decreasing in both variables (λ_1 and λ_2 are positive constants or functions).

7. Concluding remarks. We have indicated in this paper a number of types of problems to which monotone methods are applicable. We have seen that between a lower solution u_0 and an upper solution v_0, with $u_0 \leqq v_0$, there lies a stable solution φ. The question remains open if the situation is reversed: *i.e.*, $v_0 \leqq u_0$ where v_0 is an upper solution. The iteration procedure in this case does not work, since iterations beginning at v_0 tend downward, while those beginning at u_0 increase. We have tried, but without success, to prove the existence of a solution (presumably unstable) in this case as well.

There is a second area in which further questions remain, namely the case where the equation is non-linear in the derivatives as well: for example

$$\triangle u + H(x, u, \operatorname{grad} u) = 0, \tag{7.1}$$

or even the full quasi-linear or non-linear cases. Here again we have not been able to get the iteration schemes to work, but one possible alternative is the following: Suppose u_0 is a lower solution of (7.1) and solve the initial value problem

$$\begin{aligned} u_t &= \triangle u + H(x, u, \operatorname{grad} u), \\ u(x, 0) &= u_0(x). \end{aligned} \tag{7.2}$$

Then formally $u_t(x, 0) \geqq 0$ and again by the maximum principle $u(x, t) \leqq v(x, t)$ if v is a solution of (7.2) with initial data $v_0(x)$ which is an upper solution of (7.1). One might now hope to prove the existence of global solutions to the initial value problem with these initial data. Then the limits $\bar{u}(x) = \lim_{t\to\infty} u(x, t)$ and $\bar{v}(x) = \lim_{t\to\infty} v(x, t)$ exist, and one should try to prove that these functions are in fact stationary solutions of the associated elliptic boundary value problem. These arguments, which seem plausible, apparently depend on being able to get uniform estimates in time on the derivatives of the solution.

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