Bifurcation from rotationally invariant states[a)]

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Bifurcation in the presence of the rotation group is investigated. The covariant bifurcation equations are derived using the familiar angular momentum operators of quantum mechanics. Variational methods are also discussed. It is shown that the quadratic terms either vanish for odd l or possess a gradient structure for even l. This result is generalized to the case of an arbitrary simply reducible group. Applications to problems in geophysics and elasticity theory are discussed.

1. SYMMETRY BREAKING INSTABILITIES

There are a number of situations in classical mechanics in which the onset of instability of a physical system is accompanied by a spontaneous symmetry-breaking bifurcation. For example, the onset of convection in a spherical mass or the buckling of a perfectly uniform spherical shell leads to a bifurcation which breaks complete rotational symmetry. In such cases one is led to an investigation of the branching of solutions of a nonlinear functional equation $G(\lambda, u) = 0$ in the neighborhood of a known solution (λ_0, u_0). If $G_u(\lambda_0, u_0)$ (G_u denotes the Frechet derivative of G) is a Fredholm operator of index 0, the problem is reduced, via the Lyapounov—Schmidt method, to a finite-dimensional problem

$$F_i(\lambda, z_1, \ldots, z_n) = 0, \quad i = 1, \ldots, n \tag{1.1}$$

where $n = \dim \ker G_u(\lambda_0, u_0)$.

If the original equations $G(\lambda, u)$ are covariant with respect to a representation T_g of a group $\mathcal{G}$—that is, if $T_g\, G(\lambda, u) = G(\lambda, T_g u)$—then the bifurcation equations (1.1) are covariant with respect to a finite-dimensional representation of $\mathcal{G}$. A direct computation of Eqs. (1.1) by numerical methods is often a major obstacle in their analysis, certainly if the original system of equations is very complicated. Using the covariance of the equations, however, the structure of the bifurcation equations can be computed up to unknown constants. In the case of the rotation group Busse,[1] using classical formulas of Gaunt for triple integrals of spherical harmonics, constructed the quadratic terms of (1.1) when $\ker G_u$ transforms according to an even irreducible representation of SO(3). In this paper we give an algorithm for obtaining the full structure of Eqs. (1.1) at all orders based on the Lie algebra of infinitesimal generators of the rotation group. The methods are familiar in the theory of angular momentum coupling in elementary quantum mechanics.

Group theoretic methods allow one to determine the bifurcation equations only up to unknown scalar constants, the dependence of these scalars on the original physical parameters of a particular problem could be determined by a direct computation of the bifurcation equations, say from the Lyapounov—Schmidt method.

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Rather than proceed in that direction, one can follow an approach similar in spirit to Thom's catastrophe theory[2,3]: The unknown parameters are regarded as free parameters, or control parameters, and one seeks a classification of the types of transitions (i.e., singularities) which may occur. In this way one can obtain a universal classification of the bifurcations which may occur in a physical system which is based on the geometry of the problem and is independent of the particular physical mechanism involved.

In resolving a bifurcation problem one is interested in determining the stability of the bifurcating solutions, and these questions are also discussed in the present paper. Since there is a three-parameter group present, the solutions appear in three- (or sometimes two-) dimensional orbits; hence they will at best be orbitally stable, with two or three neutral modes.

In Sec. 2 we review some of the basic ideas of bifurcation theory, adding some modest improvements to cover the present case. In Sec. 3 we discuss the Lie algebra of angular momentum operators J_+, J_-, and J_3 and show how these may be used to construct Eqs. (1.1) when $\ker G_u$ is irreducible; in Sec. 4 we discuss the modifications which must be made when the kernel is reducible. We also construct the generating function for the number of covariant terms in (1.1) of any given degree. The derivation is closely related to that of the Molien function (Jaric and Birman[4]). Given a finite-dimensional representation Γ of a compact group $\mathcal{G}$, the Molien function counts the number of times the identity representation is contained in the symmetric part of $\Gamma^{\otimes n}$. In the present case we are interested in counting the number of times Γ is contained in the symmetric part of $\Gamma^{\otimes n}$; the generating function in that case is

$$M_1(\Gamma; \mathcal{G}, z) = \int \det[I - z\Gamma(g)]^{-1} \overline{X}(g)\, d\mu(g), \tag{1.2}$$

where $d\mu(g)$ is the normalized invariant measure on $\mathcal{G}$ and X is the character of Γ. We calculate M_1 explicitly for the rotation group O(3).

The extremum principle discovered by Busse[1] is discussed in Sec. 5 and its relationship to the symmetry of the 3-j symbols for SO(3) explained. More generally we show that the result continues to hold whenever Eqs. (1.1) are covariant with respect to an irreducible representation of any simply reducible group. A theorem of Wigner[5,6] on representations of simply reducible groups then implies that the quadratic

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terms in (1.1) vanish for an odd representation or possess a gradient structure for an even representation. This is a striking result, since it implies the bifurcation equations may possess a gradient structure even though the original problem did not arise as a variational problem.

In Sec. 5 we also show that the variational problem associated with the bifurcation equations can, in the case of SO(3), be formulated as

$$\min \tfrac{1}{3}\mathrm{tr}A^3$$

subject to the constraints

$$\tfrac{1}{2}\mathrm{tr}A^2 = 1, \quad \mathrm{tr}A = 0, \quad \mathrm{tr}AB_j = 0,$$

where A is a symmetric matrix and the B_j are symmetric matrices which transform according to certain representations of SO(3). For $l = 2$ this leads to the Euler–Lagrange equations

$$A^2 = \lambda A + \gamma I, \tag{1.3}$$

where A is a 3×3 symmetric traceless matrix and I is the 3×3 identity matrix. This problem is easily resolved, giving an especially simple resolution of the bifurcation problem in the case $l = 2$. (The results described in this paragraph were obtained jointly with L. Green) The approach is compared with that discussed by Michel and Radicati[7,8] in their investigations of symmetry breaking in elementary particle physics.

Section 6 contains an analysis of the relationship of the stability properties of the bifurcating solutions to the extremal properties of the solutions of the variational problem. Results of this type have previously been obtained by Sather.[9]

In Sec. 7 we discuss the resolution of the bifurcation equations when $\ker G_u$ transforms according to an irreducible representation D^l of SO(3) for low values of l. Busse's solutions for even l are discussed, and their stability is analyzed.

Finally, in Sec. 7 we discuss situations in classical physics in which questions of bifurcation in the presence of O(3) arise. These are generally problems in geophysics[10] and elasticity theory[11,12,9] which are modeled by nonlinear systems of partial differential equations. We close with a brief discussion of some of the open mathematical problems.

2. LYAPOUNOV-SCHMIDT METHOD

The Lyapounov–Schnidt method, or alternative method, discussed at length by many authors, enables one to reduce an infinite-dimensional problem to a finite-dimensional one. We present here, very briefly, a slight modification of the argument in Ref. 13 which deals with the case in which the equations are covariant with respect to a transformation group.

Suppose the equilibrium states of a physical system are represented by solutions of the nonlinear system of equations

$$G(\lambda, u) = 0, \tag{2.1}$$

where $G: \Lambda \times X \to Y$ is a smooth (Frechet differentiable) mapping, Λ is a finite-dimensional vector space, and X and Y are Banach spaces. We assume here that all spaces are Banach spaces over the complex numbers. Let (λ_0, u_0) be a solution pair of (2.1) and let $L_0 = G_u(\lambda_0, u_0)$ (G_u denotes the Frechet derivative of G). Let $\mathcal{N} = \ker L_0 \subset X$ and $\mathcal{R} = \mathrm{Range} L_0 \subset Y$. We assume that G is regular in the sense that for any (λ_0, u_0) $\mathcal{R}$ is always a closed subspace of finite codimension, $\mathcal{N}$ is finite-dimensional, and $\dim \mathcal{N} = \mathrm{codim} \mathcal{R}$. If $\mathcal{N}$ is trivial and $\mathcal{R} = Y$, then by the implicit function theorem there is an analytic curve of solutions $u = u(\lambda)$, defined for sufficiently small $|\lambda - \lambda_0|$, with $u(\lambda_0) = u_0$. From now on, for simplicity, we shall always assume $\lambda_0 = 0$,

$u_0 = 0$.

If $\mathcal{N}$ is nontrivial, then (0,0) may be a bifurcation point of solutions of (2.1): That is, there may be several distinct solution branches which confluesce at (λ_0, u_0). Let $\dim \mathcal{N} = n$ and choose vectors $\varphi_1^*, \ldots, \varphi_n^*$ in Y^* such that

$$\mathcal{R} = \{f: \langle f, \varphi_j^* \rangle = 0, \ j = 1, \ldots, n\}.$$

Then the φ_j^* must be null vectors of the adjoint operator L_0^*. Choose vectors $\varphi_1, \ldots, \varphi_n \in Y$ such that $\langle \varphi_i, \varphi_j^* \rangle = \delta_{ij}$; then the linear operator

$$P_2 f = \sum_{j=1}^{n} \langle f, \varphi_j^* \rangle \varphi_j$$

is a projection, and $Q_2 = I - P_2$ is a projection of Y onto $\mathcal{R}$. Similarly, let P_1 be the projection onto the kernel $\mathcal{N}(L_0 P = 0)$. We can write

$$P_1 u = \sum_{j=1}^{n} \langle u, \psi_j^* \rangle \psi_j,$$

where the vectors ψ_j span $\mathcal{N}$. Let $Q_1 = I - P_1$.

To reduce (2.1) to a finite-dimensional problem in a neighborhood of (0,0), we decompose the problem as follows:

$$u = P_1 u + Q_1 u = v + \psi,$$

$$G(\lambda, u) = P_2 G(\lambda, u) + Q_2 G(\lambda, u) = 0.$$

We first solve

$$H(\lambda, v, \psi) = Q_2 G(\lambda, v + \psi) = 0. \tag{2.2}$$

At the point $\lambda = 0$, $v = 0$, $\psi = 0$, the Frechet derivative of $H(\lambda, v, \psi)$ with respect to ψ is

$$H_\psi(0,0,0) = Q_2 G_u(0,0) = Q_2 L_0.$$

Now $Q_2 L_0$ is an isomorphism from the subspace $Q_1 X$ to $Q_2 Y$. In fact, $Q_2 L_0 u = 0$ implies $L_0 u = 0$ and therefore that $u \in \mathcal{N}$; but if $u \in Q_1 X \cap \mathcal{N}$ then $u = 0$. Therefore, L_0 is a bounded one-to-one mapping from $Q_1 X$ to $Q_2 Y$. By the closed graph theorem L_0 is invertible, hence an isomorphism. It follows from the implicit function theorem on a Banach space that there is a smooth solution $\psi = \psi(\lambda, v)$ of (2.2). Since X and Y are complex Banach spaces, ψ is analytic in λ and v. The solutions of the full equations (2.1) are obtained now as solutions of the *bifurcation equations*

$$F(\lambda, v) = P_2 G(\lambda, v + \psi(\lambda, v)) = 0. \tag{2.3}$$

Equations (2.3) comprise a system of n equations in n unknowns; by writing $v = z_1 \psi_1 + \ldots + z_n \psi_n$ we can rewrite (2.3) as

$$F_j\cdot(\lambda, z_1, \ldots, z_n) = \langle G(\lambda, z_1\psi_1 + \ldots + z_n\psi_n + \psi(\lambda, v), \varphi_j^*\rangle = 0.$$

Now suppose the nonlinear mapping G is covariant with respect to a representation T_g of a group $\mathcal{G}$:

$$T_g G(\lambda, u) = G(\lambda, T_g u). \tag{2.4}$$

We have

Theorem 2.1: Let $G(\lambda, u)$ be covariant with respect to a representation T_g of $\mathcal{G}$. Then $\mathcal{N}$ reduces T_g. Assume the projections P_i, Q_i commute with T_g. Then the bifurcation equations themselves are covariant with respect to the finite-dimensional representation $\Gamma = T_g|\mathcal{N}$: that is, $\Gamma F(\lambda, v) = F(\lambda, \Gamma v)$, where F is given by (2.3).

Theorem (2.1) was proved in Ref. 13 where it was shown that commuting projections P_i and Q_i can be constructed if $X\subset Y$. [In that case the resolvent operator $(\lambda - L)^{-1}$ is well defined and the commuting projections can be obtained by the standard residue formula

$$P = \frac{1}{2\pi i}\int_C (\lambda - L)^{-1} d\lambda,$$

where C encloses the isolated eigenvalue of L at the origin.] The assumption $X\subset Y$ is quite natural if G is an elliptic system of partial differential operators; then a natural choice for X and Y is typically $X = C_{m+k+\alpha}$, $Y = C_{m+\alpha}$, where $C_{j+\alpha}$ are the Banach spaces of functions with Hölder continuous derivatives. We note here, nevertheless, that if $\mathcal{G}$ is compact, we can drop the assumption $X\subset Y$ and construct commuting projections as follows.

Lemma 2.2: Let T_g be a representation of the compact group $\mathcal{G}$ on the Banach spaces X and Y. Let L be a bounded mapping from X to Y which intertwines with T_g; $T_g L = L T_g$. Let $\mathcal{N} = \ker L \subset X$ be a closed finite-dimensional subspace and let $R = \text{Range}L \subset Y$ be a closed subspace of finite codimension. Let Q be any projection onto R and $P = I - Q$. Then the projections $\hat{Q}$ and $\hat{P} = I - \hat{Q}$, where

$$\hat{Q} = \int_{\mathcal{G}} T_{g^{-1}} Q T_g d\mu(g) \tag{2.5}$$

commute with T_h for all $h\in G$. The same result holds for the projections P_1 and $Q_1 = I - P_1$, where P_1 is any projection onto $\ker L$ in X.

Proof: The fact that $\hat{Q}$ as given in (2.5) commutes with T_h follows from the invariance of the measure $d\mu(g)$. Since $\mathcal{R}$ is invariant under T_g and Q, it is clear that $\mathcal{R}$ is invariant under $\hat{Q}$ as well, and also its range is contained in $\mathcal{R}$. It remains to show that $\hat{Q}$ is a projection, and to that end it is enough to show that $\hat{Q}f = f$ if $f\in R$. We have, whenever $f = Lu$,

$$\begin{aligned}\hat{Q}f = \hat{Q}Lu &= \int T_{g^{-1}} Q T_g L u\, d\mu(g)\\ &= \int T_{g^{-1}} Q L T_g u\, d\mu(g)\\ &= \int T_{g^{-1}} L T_g u\, d\mu(g) = Lu = f.\end{aligned}$$

The proof for the case that P_1 is a projection onto $\ker L$ in X goes similarly.

We remark that in the case of representations of a noncompact group a reducing subspace need not possess a projection which commutes with the representation. For example, the action of $\mathbb{R}^1$ on $\mathbb{R}^2$ by $\left(\begin{smallmatrix}1 & t\\ 0 & 1\end{smallmatrix}\right)$ leaves the x axis invariant, but all projections onto the x axis take the form $\left(\begin{smallmatrix}1 & a\\ 0 & 0\end{smallmatrix}\right)$, and none of these commute with the action.

In order to analyze the bifurcation equations (2.3), it is often convenient to reduce them further by scaling them, as follows. A uniformizing parameter ϵ is introduced by setting

$$\lambda = \epsilon^m \sigma, \quad v = \epsilon^n w, \tag{2.6}$$

where $w = w_0 + \epsilon w_1 + \epsilon^2 w_2 + \ldots$. The appropriate powers of m and n are determined by a Newton diagram.[13] Now suppose

$$F(\epsilon^m\sigma, \epsilon^n w) = \epsilon^k Q(\sigma, w_0) + O(\epsilon^{k+1}),$$

where $k > \max\{m, n\}$. Dividing by ϵ^k and letting $\epsilon\to 0$, we arrive at the *reduced bifurcation equations*

$$Q(\sigma, w_0) = 0, \tag{2.7}$$

where σ may be chosen conveniently. If the Jacobian $Q_w(\sigma, w_0)$ is invertible at a solution (σ, w_0) of (2.7), then solutions of the full bifurcation equations may be obtained from the implicit function theorem. In the case, however, that Eq. (2.3), hence (2.7), are invariant under a Lie group the solutions of (2.7) may appear in k-parameter sheets; and in that case the Jacobian $Q_w(\sigma, w_0)$ will possess a k-dimensional kernel, spanned by the vectors $L_j w_0$, where the operators L_j are the infinitesimal generators of the Lie group $\mathcal{G}$.

Given a solution (σ, w_0) of (2.7) we examine the full solution curve (2.6). If m is even, the bifurcation is one-sided (that is, solutions appear for $\lambda > 0$ or $\lambda < 0$). When $\sigma > 0$, the bifurcation is supercritical, and it is subcritical when $\sigma < 0$. When m is odd, the branches appear on both sides of criticality (transcritical case).

Stability of the bifurcating solutions: Let a nontrivial one-parameter branch of solutions of (2.1) be given by $(\lambda(\epsilon), u(\epsilon))$ and put $L(\epsilon) = G_u(\lambda(\epsilon), u(\epsilon))$. According to the principle of linearized stability the local stability of the solution $u(\epsilon)$ is determined by the eigenvalues of $L(\epsilon)$. When $\epsilon = 0$, $L(0) = L_0$ has (by assumption) an eigenvalue of multiplicity n at the origin; and, if the trivial solution $u = 0$ is just losing stability as λ crosses zero, all other eigenvalues of L_0 must lie strictly in the left half-plane. The stability of the bifurcating branch is therefore determined by the behavior of the n-fold eigenvalue at the origin as ϵ varies from zero. The following theorem is proved in Ref. 14.

Theorem 2.3: Let $E(\epsilon)$ denote the analytic projection valued operator whose range is the n-dimensional invariant subspace of $L(\epsilon)$ corresponding to the n-fold eigenvalue at the origin. Then the eigenvalues of $L(\epsilon)$ in the vicinity of the origin are precisely those of the n-dimensional operator $B(\epsilon) = L(\epsilon)E(\epsilon)$. Furthermore, if the scaling of the solutions have the form (2.6), then

$$B(\epsilon) = \epsilon^{k-n} Q_w(\sigma, w_0) + O(\epsilon^{k-n+1}).$$

Accordingly, to lowest order in ϵ, the behavior of the multiple eigenvalue 0 under the perturbation along the bifurcating branch is determined by the eigenvalues of the Jacobian of the reduced bifurcation equations. Supercritical solutions are stable if all eigenvalues of $Q_w(\sigma, w_0)$ are negative and subcritical solutions are stable if all eigenvalues of $Q_w(\sigma, w_0)$ are positive.

When a continuous transformation group is present, one or more of the eigenvalues of $Q_w(\sigma, w_0)$ are zero (depending on the dimension of the manifold of solutions); in that case one can at best conclude orbital stability from an analysis of the reduced equations: There will always be a number of neutral modes present.

3. CONSTRUCTION OF THE COVARIANT BIFURCATION EQUATIONS IN THE CASE SO(3)

We denote by Γ the representation $T_g|\mathcal{N}$. Let us expand $F(\lambda, v)$ of (2.3) in a power series in v:

$$F(\lambda, v) = A(\lambda)v + B_2(\lambda, v, v) + B_3(\lambda, v, v, v) + \ldots .$$

Then we must have

$$\begin{aligned} &\Gamma A(\lambda)v = A(\lambda)\Gamma v, \\ &\Gamma B_2(\lambda, v, w) = B_2(\lambda, \Gamma v, \Gamma w), \\ &\quad\vdots \end{aligned} \tag{3.1}$$

Therefore, each multilinear operator B is covariant with respect to the representation Γ.

We first make the assumption that Γ is irreducible, that is, that $\Gamma = D^l$, where D^l is one of the irreducible representations of the irreducible representations of SO(3). The contrary case, when $\mathcal{N}$ is reducible, is sometimes called "accidental degeneracy" by physicists (Ref. 5 p. 161); Ruelle[15] suggests the situation is nongeneric. Indeed, that is clearly the case in a problem analyzed in detail by Chow, Hale, and Mallet-Paret.[16] They consider the buckling of a rectangular plate. Since the symmetry group of the rectangle is Abelian, the irreducible representations are all one dimensional; but when the ratio of length to width is $\sqrt{2}$, the principle eigenvalue has multiplicity 2. This situation is clearly nongeneric, for it depends on a specific choice of physical parameters.

The reducible case is discussed in the next section. When Γ is irreducible, the linear term in (3.1) is a scalar multiple of the identity by Schur's lemma. Thus $A(\lambda) = \sigma(\lambda)I$ for some scalar σ. The quadratic term $B(\lambda, v, w)$ must be symmetric in v and w and transform as D^l. The quadratic mapping B may be regarded as a subspace of symmetric second order tensors which transform as D^l under the action of SO(3). The Clebsch–Gordan series

$$D^l \otimes D^l = D^{2l} \oplus D^{2l-1} \oplus \cdots \oplus D^0 \tag{3.2}$$

tells us that the tensor product space $\mathcal{N} \otimes \mathcal{N}$ decomposes into a direct sum of subspaces, precisely one of which transforms according to D^l, as follows:

$$\mathcal{N} \otimes \mathcal{N} = V^{2l} \oplus \cdots \oplus V^l \oplus \cdots \oplus \cdots \oplus V^0.$$

In this decomposition V^{2l} consists of symmetric tensors, V^{2l-1} antisymmetric tensors, and so forth. Accordingly V^l consists of symmetric tensors *iff* l is even. Therefore, *for odd* l *the quadratic term vanishes*, and we must go to cubic terms to get the reduced bifurcation equations. (We shall show below that a similar result holds more generally when $\mathcal{G}$ is a simply reducible group.)

Since we are interested solely in symmetric tensors over $\mathcal{N}$, we can work with polynomials (due to the natural isomorphism between the ring of polynomials and symmetric tensors over $\mathcal{N}$). We therefore identify $\mathcal{N}$ with the vector space of linear polynomials in the variables $z_{-l}, \ldots, z_l$ [since $\dim D^l = (2l+1)$] and denote by $K[z_{-l}, \ldots, z_l]$ the ring of polynomials in the independent variables $z_{-l}, \ldots, z_l$. K is then isomorphic to the algebra of symmetric tensors over $\mathcal{N}$.

Let the bifurcation equations be

$$F_m(z_{-l}, \ldots, z_l) = 0, \quad m = -l, \ldots, l.$$

The linear terms of F_m are of the form

$$F_m(z_{-l}, \ldots, z_l) = a z_m$$

since, as we have said, the linear term must be a scalar multiple of the identity. The quadratic terms are given by

$$F_m = \sum_{m_1 + m_2 = m} C(l, m_1; m_2; l, m_2; l, m) z_{m_1} z_{m_2}, \tag{3.3}$$

where $C(l, m_1; l, m_2; l, m)$ are the *Clebsch–Gordan* coefficients for SO(3), or

$$F_m = (-1)^m \sum_{m_1 + m_2 = m} \begin{pmatrix} l & l & l \\ m_1 & m_2 & -m \end{pmatrix} z_{m_1} z_{m_2}, \tag{3.4}$$

where

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

are the Wigner 3-j coefficients for SO(3).

In the general case the terms F_m can be constructed by the following algorithm. Let the infinitesimal generators of SO(3) be J_1, J_2, J_3; these satisfy the commutation relations

$$[J_i, J_j] = \epsilon_{ijk} J_k,$$

where ϵ_{ijk} is the completely antisymmetric tensor.

Putting $J^{\pm} = \pm J_2 + iJ_1$ and $J^3 = -iJ_3$, we obtain instead the commutation relations for sl(2)

$$[J^+, J^-] = 2J^3, \quad [J^3, J^{\pm}] = \pm J^{\pm}. \tag{3.5}$$

By well-known algorithms (see Ref. 17, p. 234), using the commutation relations (3.5), we can construct a basis f_m for the complexification of the vector space $\mathcal{N}$ such that

$$J_3 f_m = m f_m,$$

$$J_{\pm} f_m = \beta_{\pm m} f_{m\pm 1},$$

where

$$-l \leq m \leq l \quad \text{and} \quad \beta_m = [(l-m)(l+m+1)]^{1/2}.$$

In addition, the f_m can be normalized so that the reality condition

$$\bar{f}_m = (-1)^m f_{-m} \tag{3.6}$$

is satisfied.

Since I have not found (3.6) in the standard references I will give a proof here. First note that the operators J_1, J_2, J_3 are real operators, and so $\overline{J^3} = -J^3$, $\overline{J^+} = -J^-$, $\overline{J^-} = -J^+$. It follows that $\overline{J^3 f_m} = \overline{m f_m} = -J^3 \overline{f_m}$ and therefore that $J_3 \overline{f_m} = -m \overline{f_m}$. The vector $\overline{f_m}$ has weight $-m$. Since $\mathcal{N}$ is irreducible there is only one vector with weight $-m$, and that is f_{-m}. So $\overline{f_m} = c_m f_{-m}$. On the other hand $J^+ f_m = \beta_m f_{m+1}$ so $\overline{J^+ f_m} = \beta_m \overline{f_{m+1}} = \beta_m c_{m+1} f_{-(m+1)}$ $= -J^- \overline{f_m} = -J^- c_m f_{-m} = -c_m J^- f_{-m} = -c_m \beta_m f_{-m-1}$. Consequently, $c_{m+1} = -c_m$ and we can take $c_m = (-1)^m c$. For $m=0$ we have $\overline{f_0} = c f_0$. Choosing $c=1$, we obtain that f_0 is real and $\overline{f_m} = (-1)^m f_{-m}$.

The reality condition (3.6) is important when we wish to restrict ourselves to real solutions of the bifurcation equations (1.1).

We now require the variables z_m to transform as the f_m under J_3 and $J_\pm$. We extend J_3 and $J_\pm$ to be derivations over K:

$$J(\alpha f + \beta g) = \alpha J f + \beta J g,$$

$$J(fg) = f Jg + (Jf) g,$$

where $f, g \in K$ and α and β are scalars. It is natural to extend the J's in this way since they are Lie derivatives.

If the functions $F_m(z_{-l}, \ldots, z_l)$ are to transform as D^l they also must transform as the z_m:

$$J_3 F_m = m F_m, \quad J_\pm F_m = \beta_{\pm m} F_{m\pm 1}. \tag{3.7}$$

For example, the quadratic polynomials F_m are obtained as follows. The action of J_3 on $z_j z_k$ is

$$J_3(z_j z_k) = (J_3 z_j) z_k + z_j (J_3 z_k) = (j+k) z_j z_k$$

so $J_3 z_j z_k = m z_j z_k$ if $j+k=m$. Therefore,

$$F_m(z_{-l}, \ldots, z_l) = \sum_{m_1+m_2=m} A_{m_1 m_2 m} z_{m_1} z_{m_2}.$$

In particular, when l is even,

$$F_l = a_0 z_l z_0 + a_1 z_{l-1} z_1 + \cdots + a_{l/2} (z_{l/2})^2.$$

Furthermore, $J_+ F_l = \beta_l F_l = 0$, and this condition gives us a set of linear equations for the coefficients $a_0 \cdots a_{l/2}$. In the case $l=2$ we have

$$F_2 = a z_2 z_0 + b z_1^2,$$

$$J_+ F_2 = a\beta_0 z_2 z_1 + 2b\beta_1 z_1 z_2$$

$$= (a\beta_0 + 2b\beta_1) z_1 z_2 = 0,$$

$$a\beta_0 + 2b\beta_1 = 0.$$

The last equation determines the coefficients a and b, hence F_2, up to a scalar multiple. Once F_l is known we get F_{l-1} from

$$J_- F_l = \beta_{-l} F_{l-1}$$

and so forth. In this way we can construct all the F_m's.

This procedure extends immediately to higher order terms. For example, to get third order terms we write

$$F_l = \sum_{i+j+k=l} a_{ijk} z_i z_j z_k$$

and apply

$$J_+ F_l = 0$$

to get a linear system of equations for the a_{ijk}. For $l=1$ there is only one solution but for $l=3$ there are two independent solutions. In fact, the condition $J_+ F_l = 0$ in that case leads to five equations in seven unknowns (see Sec. 7).

4. THE CASE Γ REDUCIBLE

We begin by deriving a generating function which gives the number of covariant n-linear symmetric mappings for arbitrary n. We first derive a general formula for arbitrary compact groups, and then apply the formula in the specific case of O(3). We denote the irreducible representations of $\mathcal{G}$ by Γ_ν and suppose $\Gamma = \Sigma a_\nu \Gamma^\nu$, where a_ν is the multiplicity of Γ^ν in Γ. The characters of Γ and Γ^ν are denoted by χ and χ^ν respectively.

Theorem 4.1: Let Γ be a representation on the vector space $\mathcal{N}$ of the compact group $\mathcal{G}$ and let $c_n = c_n(\Gamma, \mathcal{G})$ denote the number of completely symmetric n-linear operators B on $\mathcal{N} \otimes \mathcal{N}$ to $\mathcal{N}$ which are covariant with respect to Γ. Then a generating function for the coefficients c_n is

$$\sum_{n=0}^{\infty} c_n(\Gamma, \mathcal{G}) z^n = M_1(\mathcal{G}, \Gamma, z) = \int_{\mathcal{G}} \det[I - z\Gamma(g)]^{-1} \overline{\chi}(g)\, d\mu(g). \tag{4.1}$$

In the above expression $\mu(g)$ denotes the normalized invariant measure on $\mathcal{G}$; we set $c_0(\Gamma, \mathcal{G}) = 1$ by convention.

Proof: Let $\mathcal{N}^*$ be the dual of $\mathcal{N}$; let the n-linear map B be covariant with respect to Γ; and put

$$F(u_1, \ldots, u_n; u_{n+1}) = \langle B(u_1, \ldots, u_n), u_{n+1} \rangle, \tag{4.2}$$

where $u_{n+1} \in \mathcal{N}^*$ and $\langle\ ,\ \rangle$ denotes the bilinear pairing between $\mathcal{N}$ and $\mathcal{N}^*$. Let $\tilde{\Gamma}(g) = \Gamma^+(g^{-1})$ be the contragradient representation. (Here Γ^+ denotes the adjoint of Γ relative to the bilinear pairing $\langle\ ,\ \rangle$). F is a tensor in $\mathcal{N}^{\otimes n} \otimes \mathcal{N}^*$ which is invariant under the representation $\Gamma^{\otimes n} \otimes \tilde{\Gamma}$; in fact,

$$\Gamma^{\otimes n} \otimes \tilde{\Gamma} F(u_1, \ldots, u_n; u_{n+1})$$

$$= F(\Gamma u_1, \ldots, \Gamma u_n; \tilde{\Gamma} u_{n+1})$$

$$= \langle B(\Gamma u_1, \ldots, \Gamma u_n), \tilde{\Gamma} u_{n+1} \rangle$$

$$= \langle \Gamma B(u_1, \ldots, u_n), \tilde{\Gamma} u_{n+1} \rangle$$

$$= \langle B(u_1, \ldots, u_n), u_{n+1} \rangle = F(u_1, \ldots, u_n; u_{n+1}).$$

The correspondence (4.2) between covariant n-linear maps and $(n+1)$-linear invariants is one—one. On the other hand, the number of invariants is equal to the number of times the identity representation is contained in the tensor product representation $\Gamma^{\otimes n} \otimes \tilde{\Gamma}$. This number is given by the expression

$$\int_{\mathcal{G}} \chi^n(g) \overline{\chi}(g)\, d\mu(g).$$

Now, however, we must modify the argument to take into account the fact that B is completely symmetric. We do this by restricting the representation $\Gamma^{\otimes n}$ to the symmetric part of $\mathcal{N}^{\otimes n}$; the restriction of $\Gamma^{\otimes n}$ to this subspace is denoted by $(\Gamma^{\otimes n})s$. The character $\chi_{(n)}$ of $(\Gamma^{\otimes n})s$ is given by the generating function

$$\det[I - z\Gamma(g)]^{-1} = \sum_{n=0}^{\infty} \chi_n(g) z^n, \qquad (4.3)$$

where $\chi_{(0)}(g) \equiv 1$. The result (4.1) now follows immediately.

A derivation of (4.3) may be found in Littlewood[18] in the chapter on Schur functions; but for completeness I will present a simpler derivation in the Appendix.

The expression (4.2) for the covariant mappings is very closely related to the Molien function (see Jarić and Birman[4]) which counts the number of completely symmetric invariant tensors of each order. The Molien function has been calculated by Jarić and Birman for various space groups. Let us calculate the function (4.1) in the case $\mathcal{G} = \mathrm{O}(3)$. First note that the determinant of the direct sum $A \oplus B$ of two matrices is $\det A \oplus B = (\det A)(\det B)$; for the determinant of an operator is the product of its eigenvalues, and the eigenvalues of $A \oplus B$ are the union of those of A and of B. Therefore

$$\det(I - z\sum a_\nu \Gamma_\nu)^{-1} = \prod_\nu [\det(I - z\Gamma_\nu)]^{-a_\nu},$$

where I on the left is the $N\times N$ identity matrix ($N = \dim\Gamma$) and the I's on the right are the $\mathcal{N}_\nu \times \mathcal{N}_\nu$ identity matrices ($\mathcal{N}_\nu = \dim\Gamma_\nu$).

It remains to calculate $\det(I - z\Gamma_\nu)$ for the irreducible representations O(3). We first carry out the calculation for SO(3) and then indicate the modifications which must be made to treat O(3). Let g be a rotation through an angle θ. The eigenvalues of $D^l(g)$ are then $\exp(\pm im\theta$, $m = -l, \ldots, l$, and therefore

$$\det(I - zD^l) = \prod_{m=-l}^{l} (1 - z\exp(im\theta))(1 - z\exp(-im\theta))$$

$$= \left(\prod_{m=0}^{l} (1 - 2z\cos m\theta + z^2)\right).$$

The invariant integral for SO(3) is

$$\frac{1}{\pi}\int_0^\pi (1 - \cos\theta) d\theta$$

and so our expression for (4.1) is

$$M(\mathrm{SO}(3);\Gamma;z) = \frac{1}{\pi}\int_0^\pi \prod_l \prod_{m=0}^{l} (1 - 2z\cos m\theta + z^2)^{-a_l} \times \sum_l a_l \chi_l(\theta)(1 - \cos\theta) d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \prod_l \prod_{m=0}^{l} (1 - 2z\cos m\theta + z^2)^{-a_l} \times \sum_l a_l \chi_l(\theta)(1 - \cos\theta) d\theta, \qquad (4.4)$$

which can be evaluated by residues.

As to O(3) there are two types of representations, positive and negative, which are closely related to those of SO(3). (See Miller,[17] p. 249). When g is a pure rotation, $D^l_+(g) = D^l_-(g) = D^l(g)$; but when g is a rotation reflection, $D^l_+(g) = -D^l_-(g) = D^l(g)$. The negative representations thus contain the inversion $v \to -v$. Since the spherical harmonics satisfy $Y^l_m(\theta, l) = (-1)^l Y^l_m(\pi - \theta, \pi + \varphi)$ the subspaces V^l transform according to positive representations for even l, and negative for odd l. In order to correct (4.4) for the case O(3), we cut (4.4) in half and add another term corresponding to the integral over the rotation-reflections of the group. For this portion of the integral the eigenvalues of the representations are multiplied by a factor of $(-1)^l$. Therefore, the correction term is

$$\frac{1}{2\pi}\int_0^\pi \prod_l \prod_{m=0}^{l} (1 - 2z(-1)^l \cos m\theta + z^2)^{-a_l} \times \sum_l a_l(-1)^l \chi_l(\theta)(1 - \cos\theta) d\theta.$$

Now let us turn to algorithms for constructing the covariant bifurcation equations in the reducible case. Such an algorithm was given for the case of a finite group in Ref. 13. Here we present a method using again the Lie algebra of infinitesimal operators. We look first at two simple examples, from which the general algorithm will be clear.

Suppose first that the kernel $\mathcal{N}$ transforms according to the representation $D^1 \oplus D^2$, and represent the vector space $\mathcal{N}$ as linear polynomials in the variables $x_0, x_{\pm 1}$ and $y_0, y_{\pm 1}, y_{\pm 2}$. We then seek polynomials $F_0, F_{\pm 1}$ and $G_0, G_{\pm 1}, G_{\pm 2}$ in these variables which transform as D^1 and D^2 respectively. Let $J_\pm$ and J_3 be the operators discussed in Sec. 3. We have

$$J_3 x_j = j x_j, \quad J_3 y_j = j y_j, \quad J_3 F_j = jF_j, \quad J_3 G_j = jG_j,$$

$$J_+ x_j = \beta_{l,j} x_{j+1}, \quad J_- x_j = \beta_{l,-j} x_{j-1},$$

$$J_\pm y_k = \beta_{2,\pm j} y_{j\pm 1},$$

$$J_\pm F_j = \beta_{1,k} F_{j\pm 1}, \quad J_\pm G_j = \beta_{2,j} G_{j\pm 1}$$

where $\beta_{l,j} = [(l - j)(l + j + 1)]^{1/2}$.

Since F_1 must have weight 1, we write it as a sum of all possible terms of weight 1. For the quadratic case we have

$$F_1 = ay_1x_0 + bx_0y_1 + cy_2x_{-1},$$

and we require $J_+F_1 = 0$. We have omitted such terms as x_1x_0 and y_1y_0 because D^1 is not contained in $(D^1 \otimes D^1)s$ or $(D^2 \otimes D^2)s$. (In fact, $D^1 \otimes D^1 = D^2 \oplus D^1 \oplus D^0$, where the first and third invariant subspaces are symmetric tensors, and the subspace which transforms according to D^1 is antisymmetric. A similar situation is true in the case $D^2 \otimes D^2$.) Terms such as $x_i x_j$ show up in the representation $D^1 \otimes D^1$; terms $y_i y_j$ come from $D^2 \otimes D^2$; and terms $x_i y_j$ come from $D^1 \otimes D^2$.

The condition $J_+F_1 = 0$ leads to the equations

$$b\beta_{2,1} + c\beta_{1,-1} = 0, \quad a\beta_{2,0} + b\beta_{1,0} = 0,$$

of which there is one solution. Similarly, for G_2 we take

$$G_2 = ax_1^2 + by_2y_0 + cy_1^2 + dx_1y_1 + ex_0y_2$$

and apply the condition $J_+G_2 = 0$. This leads to the equations $d\beta_{2,1} + e\beta_{1,0} = 0$, $b\beta_{2,0} + 2c\beta_{2,1} = 0$, and no restriction on a. We get three linearly independent solutions in all, so there are four covariant polynomials of degree 2: one of weight 7 and three of weight 2. There are therefore four parameters in the reduced bifurcation equations.

For the second example consider the case that $\mathcal{N}$

transforms as $2D^1$, or $D^1 \oplus D^1$. In this case choose variables $x_0, x_{\pm 1}$ and $y_0, y_{\pm 1}$, and look for polynomials F_1 of weight one such that $J_+F_1 = 0$. The reader will easily determine that there is only one such polynomial, namely $(x_1y_0 - x_0y_1)$. This may be repeated *twice* in the bifurcation equations, so there are two parameters. The reduced bifurcation equations are therefore

$$x_1 = A(x_1y_0 - x_0y_1), \quad x_0 = A\sqrt{2}(x_1y_{-1} - x_{-1}y_1),$$

$$x_{-1} = A(x_0y_1 - x_{-1}y_0), \quad y_1 = B(x_1y_0 - x_0y_1),$$

$$y_0 = B\sqrt{2}(x_1y_{-1} - x_{-1}y_1), \quad y_{-1} = B(x_0y_{-1} - x_{-1}y_0).$$

The general algorithm now follows. When $\Gamma = \sum a_l D^l$ put $N = \sum a_l$ and choose N different sets of variables $x, y, \cdots$ and N different sets of polynomials $F, G, \cdots$. Each set of variables and polynomials is to transform irreducibly under the Lie algebra so(3)—with a one—one correspondance among variables, polynomials, and the irreducible representations D^l occurring in Γ. Each chain of polynomials of a given weight can occur in any part of the bifurcation equations of the same weight. Thus, to a_ν occurrences of D^ν and b_ν covariant polynomial chains of weight ν there correspond $a_\nu b_\nu$ independent parameter in the bifurcation equations—that is, $a_\nu b_\nu$ independent occurrences of the polynomials of weight ν.

5. GRADIENT STRUCTURE OF THE BIFURCATION EQUATIONS; SIMPLY REDUCIBLE GROUPS

Suppose that the kernel $\mathcal{N}$ is irreducible and that the reduced bifurcation equations take the simple form

$$\sigma w + B(w, w) = 0. \qquad (5.1)$$

As we have already seen, $\mathcal{N}$ must transform according to an even representation of O(3) [$D^l(g)$, where l is even], for otherwise the quadratic term is antisymmetric. In that case Eq. (5.1) possess a gradient structure, as Busse[1] has observed. This gradient structure is a consequence of a symmetry property of the 3-j symbols which holds in the more general context of a "simply reducible group."

Recall that the quadratic terms of the bifurcation equations are given (for even l) by

$$F_m(z_{-l}, \ldots, z_l) = \sum (-1)^m \begin{pmatrix} l & l & l \\ m_1 & m_2 & -m \end{pmatrix} z_{m_1} z_{m_2}$$

(The 3-j symbols vanish whenever $|m| > l$ or $m_1 + m_2 + m_3 \neq 0$, hence we may drop the limits of summation.) Consider the homogeneous polynomial of degree 3

$$p(z_{-l}, \ldots z_l) = \tfrac{1}{3} \sum_{-l}^{l} F_m \overline{z_m}$$

restricted to the real subspace of $\mathcal{N}$ for which $\overline{z_m} = (-1)^m z_{-m}$. There we have

$$p(z_{-l}, \ldots z_l) = \tfrac{1}{3} \sum_{-l}^{l} (-1)^m F_m z_{-m} = \tfrac{1}{3} \sum_{m_1, m_2, m_3 = -l}^{l} \begin{pmatrix} l & l & l \\ m_1 & m_2 & m_3 \end{pmatrix} z_{m_1} z_{m_2} z_{m_3}.$$

For l even the 3-j symbols are completely symmetric in the integers m_1, m_2, m_3, (Ref. 5, p. 159), and therefore

$$\frac{\partial p}{\partial z_m} = F_m(z_{-l}, \ldots, z_l).$$

In as much as (5.1) can be written in the component form as

$$\sigma z_m + F_m(z_{-l}, \ldots, z_l) = 0, \qquad (5.1')$$

we see that these equations possess a gradient structure and in fact are the Euler—Lagrange equations for the variational problem

$$\min_{|z|=1} \tfrac{1}{3} p,$$

where

$$|z|^2 = \sum_{m=-l}^{l} z_m \overline{z_m} = \sum_{-l}^{l} (-1)^m z_m z_{-m}.$$

The function p is the third order invariant for the representation $D^l(g)$ of O(3); that is, $p(D^l(g)z) = p(z)$ for all $g \in$ O(3). The norm $|z|^2$ is the second order invariant. In vector form these invariants take the form $\langle u, u \rangle$ and $\langle B(u,u), u \rangle$, where $\langle\ ,\ \rangle$ denotes the scalar product on the vector space $\mathcal{N}$. The variational problem is accordingly

$$\min_{\langle u,u \rangle = 1} \tfrac{1}{3} \langle B(u,u), u \rangle. \qquad (5.2)$$

In general, let $\mathcal{N}$ be a complex inner product space, T_g a unitary representation, and B a covariant symmetric bilinear mapping from $\mathcal{N} \times \mathcal{N}$ to $\mathcal{N}$. The trilinear form $F(u,v,w) = \langle B(u,v), w \rangle$ is always an invariant of $T^{\otimes 3}$, as we saw in Sec. 4. Under what conditions, however, is $B(u,u)$ also the gradient of the function $\mathcal{J}(u) = \tfrac{1}{3} F(u,u,u)$? The answer is given in the following lemma.

Lemma 5.1: Let B, F and $\mathcal{N}$ be as above. Then B is the gradient of the functional $\mathcal{J}$ iff the trilinear form $\langle B(u, v), w \rangle$ is completely symmetric.

Proof: The operator $B(u)$ is the gradient of the functional $\mathcal{J}(u)$ if $\mathcal{J}(u + \epsilon h) = \mathcal{J}(u) + \epsilon \langle B(u), h \rangle + O(\epsilon^2)$. First suppose the trilinear form $\langle B(u,v), w \rangle$ is completely symmetric. Then

$$\begin{aligned} \mathcal{J}(x + \epsilon h) &= \mathcal{J}(x) + (\epsilon/3)[\langle B(h,x), x \rangle + \langle B(x,h), x \rangle + \langle B(x,x), h \rangle] + O(\epsilon^2) \\ &= \mathcal{J}(x) + \epsilon \langle B(x,x), h \rangle + O(\epsilon^2). \end{aligned}$$

By definition, then, the gradient of $\mathcal{J}$ is the bilinear operator B.

The converse is a little harder to prove. By the symmetry of B,

$$\mathcal{J}(x + \epsilon h) = \mathcal{J}(x) + \epsilon[\langle B(x,x), h \rangle + 2\langle B(h,x), x \rangle] + O(\epsilon^2),$$

whereas, if B is the gradient of $\mathcal{J}$,

$$\mathcal{J}(x + \epsilon h) = \mathcal{J}(x) + \epsilon \langle B(x,x), h \rangle + O(\epsilon^2).$$

Comparing terms of order ϵ, we have

$$\langle B(x,x), h \rangle = \langle B(x,h), x \rangle \qquad (5.3)$$

for all vectors x and h. Replacing x by $x + y$ in (5.3) and using the symmetry of B, we obtain the identity

$$\langle B(x,y), h \rangle = \tfrac{1}{2}[\langle B(x,h), y \rangle + \langle B(y,h), x \rangle]. \qquad (5.4)$$

Interchanging x and h in (5.4), we get

$$\langle B(h,y),x\rangle = \tfrac{1}{2}[\langle B(h,x),y\rangle + \langle B(y,x),h\rangle]$$
$$= \tfrac{1}{2}[\langle B(h,x),y\rangle + \tfrac{1}{2}[\langle B(x,h),y\rangle + \langle B(y,h),x\rangle]]$$
$$= \tfrac{3}{4}\langle B(x,h),y\rangle + \tfrac{1}{4}\langle B(y,h),x\rangle;$$

hence $\langle B(y,h),x\rangle = \langle B(h,y),x\rangle = \langle B(x,h),y\rangle$. Consequently, the trilinear form F is invariant under the permutations (12) and (13) and so is completely symmetric.

Summarizing the above discussion, we have

Theorem 5.2: When the kernel $\mathcal{N}$ transforms according to the irreducible representation $D^l(g)$ of SO(3) the quadratic term in the bifurcation equations vanishes altogether for l odd and is the gradient of the third order invariant if l is even.

The above theorem continues to hold, with appropriate modifications, in the more general context of simply reducible groups. Simply reducible groups (S.R. groups) were introduced by Wigner in 1940. (References for the following remarks may be found in Wigner[6] and Hammermesh,[5] pp. 151–59). A group is simply reducible if

(a) Every element is equivalent (conjugate) to its inverse (i.e., for every p there is an h such that $p = hp^{-1}h^{-1}$).

(b) The tensor product of any two irreducible representations contains no irreducible representation more than once.

Many of the groups occurring in applications are S.R. groups: the symmetric groups S_3 and S_4, the quaternion group, the three-dimensional rotation group, the two-dimensional unimodular group, and most of the crystal point groups. An immediate consequence of property a is that all the group characters are real [since $\chi(g^{-1}) = \overline{\chi(g)}$ and the character is constant on conjugacy classes] and so every representation is equivalent to its complex conjugate representation.

The irreducible representations of a compact group can be classified into three groups: Those which possess a real matrix representation; those which do not possess a real representation but which are nevertheless unitarily equivalent to their complex conjugate representation; and those which are not equivalent to their complex conjugate representations. Representations of the first kind are called *integer representations*; those of the second are called *half-integer* representations.

Lemma 5.3: The tensor product of two integer representations or of two half-integer representations of an S.R. group contains only integer representations, while the tensor product of an integer and a half-integer representation contains only half-integer representations.

The proof of this lemma is given in Wigner's article.[6]

The tensor product of a representation with itself can be decomposed into a symmetric and an antisymmetric part. For a representation T denote the symmetric part of $T\otimes T$ by $(T\otimes T)$s and the antisymmetric part by $(T\otimes T)$a. If T is an integer representation, the irreducible parts of $(T\otimes T)$s are called *even* representations and and those in $(T\otimes T)$a are called *odd* representations; on the other hand, this nomenclature is reversed if T is a half-integer representation. Wigner has proved that no representation can be simultaneously odd and even, but there are integer representations which are neither even nor odd.

Let the irreducible representations be denoted by D^j and introduce the convention that $(-1)^j = 1(-1)$ if j is an even (odd) representation and $(-1)^{2j} = 1(-1)$ if j is an integer (half-integer) representation.

Theorem 5.4: (Wigner[6] p. 92) For an S.R. group it is possible to normalize the 3-j symbols in such a way that

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ \kappa_1 & \kappa_2 & \kappa_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_2 & j_1 & j_3 \\ \kappa_2 & \kappa_1 & \kappa_3 \end{pmatrix}.$$

Hence the 3-j symbols remain unchanged under an even permutation of the columns but are multiplied by $(-1)^{j_1}(-1)^{j_2}(-1)^{j_3}$ for an odd permutation.

The following is an immediate consequence of Wigner's theorem.

Theorem 5.5: Let $\mathcal{G}$ be a simply reducible group and let D^j be an irreducible (unitary) integer representation of $\mathcal{G}$ acting on the vector space $\mathcal{N}$. Let the bilinear mapping B be covariant with respect to D^j. Then the third order trilinear invariant $\langle B\langle u,v),w\rangle$ is completely symmetric if D^j is an even representation and completely antisymmetric if D^j is an odd representation. Consequently, the quadratic terms of the bifurcation equations vanish for an odd representation and possess a gradient structure for even representations.

More generally,

Theorem 5.6: Let Γ be a unitary representation on a Hilbert space $\mathcal{H}$ such that $\Gamma^{\otimes(n+1)}$ contains the identity representation precisely once. Then there is a convariant n-linear map $B:\mathcal{H}\to\mathcal{H}$ which is either completely antisymmetric or is completely symmetric and the gradient of a completely symmetric invariant of order $(n+1)$.

Proof: Since $\Gamma^{(n+1)}$ contains the identity representation once there exists a unique invariant $F(x_1, \ldots, x_{n+1})$. Define a representation of S_{n+1} by $T_\sigma F(x_1, \ldots, x_{n+1}) = F(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n+1)})$ for $\sigma\epsilon S_{n+1}$. Since the subspace of invariants is one-dimensional, $T_\sigma F = \chi_{(\sigma)}F$ where χ is a character of S_{n+1}. Since the only one dimensional representations of the symmetric group are the identity and alternating actions, F is either completely symmetric or completely antisymmetric. The associated covariant operator B is therefore the same. Remark: Given the $(n+1)$ linear form F the mapping B is obtained as follows: Fix $x_1, \ldots, x_n$ and consider the linear functional $u \to F(x_1, \ldots, x_n, u)$. This may be represented as $u \to (B(x_1, \ldots, x_n), u)$ where $B(x_1, \ldots, x_n) \in \mathcal{H}$. The linearity and transformation properties of B are readily derived.) Finally, if F is completely symmetric, then its gradient is easily seen to be the mapping $x \to (n+1)B(x, \ldots, x)$ by the argument of the first part of Lemma 5.1. Theorem 5.5 may be obtained directly from Theorem 5.6 without recourse to Wigner's theorem. For if Γ is an irreducible integer representation of a simply reducible group, then it is equivalent to its contragradient. If Γ is contained in $\Gamma \otimes \Gamma$ precisely once, then the identity representation is contained precisely once in $\Gamma^{\otimes 3}$ ($R \otimes S$ contains the identity representation precisely once iff R and S are unitarily equivalent irreducible representations; see Ref. 13, Theorem 8.1). If Γ is an even representation, it is contained in $(\Gamma \otimes \Gamma)s$, hence the third order invariant must be symmetric, while if Γ is odd it is contained in $(\Gamma \otimes \Gamma)a$ and the third order invariant is antisymmetric.

In the case of the rotation group Professor L. Green (School of Mathematics) and I have succeeded in casting the variational problem in slightly different way. For even l we have the Clebsch–Gordan series

$$D^{l/2} \otimes D^{l/2} = D^l \oplus D^{l-1} \oplus \cdots \oplus D^0 \tag{5.5}$$

and the associated representation

$$U_g A = D^{l/2}(g) A D^{l/2}(g^{-1}) \tag{5.6}$$

on $(l+1)\times(l+1)$ matrices A. This representation is unitary relative to the inner product

$$\langle A, B\rangle = \tfrac{1}{2}\mathrm{tr} AB^* \tag{5.7}$$

(B^* = Hermitian conjugate of B). The third order invariant (there is only one, since $D^l \otimes D^l \otimes D^l$ contains D^0 only once) is

$$p(A) = \tfrac{1}{3}\,\mathrm{tr} A^2 A^*.$$

The highest weight space, the one that transforms like D^l in (5.5), consists of symmetric tensors, so we may restrict ourselves to Hermitian symmetric matrices and rephrase our variational problem as

$$\min \tfrac{1}{3}\,\mathrm{tr} A^3$$

subject to

$$\tfrac{1}{2}\mathrm{tr} A^2 = 1 \text{ and } \mathrm{tr} AB_j = 0,$$

where the B_j are symmetric matrices which lie in the lower weight invariant subspaces. In particular $\mathrm{tr} AI = \mathrm{tr} A = 0$. For $l=2$, (5.5) reads $D^1 \otimes D^1 = D^2 \oplus D^1 \oplus D^0$; but the tensors transforming according to D^1 are antisymmetric, so we have only the constraint $\mathrm{tr} A = 0$, $\mathrm{tr} A^2 = 2$. The Euler–Langrange equations are therefore

$$A^2 = \lambda A + \gamma I, \tag{5.8}$$

where A and I are 3×3 matrices. (The gradient of the functional $\frac{1}{3}\mathrm{tr} A^3$ is the mapping $A \to A^2$). Equations (5.8) can be completely solved as follows.

Taking the trace of (5.8), we get $\gamma = 2/3$. In the case $l=2$, A is a 3×3 matrix and we can choose a rotation g as that $D^1(g) A D^1(g^{-1})$ is diagonal, since $D^1(g)$ ranges over over all orthogonal matrices as g ranges over O(3). So, assuming A is diagonal, we can write (5.8) as

$$\mu_i^2 = \lambda \mu_i + \tfrac{2}{3}, \quad i = 1, 2, 3, \tag{5.9}$$

where μ_1, μ_2, μ_3 are the eigenvalues of A. The constraints are

$$\mu_1^2 + \mu_2^2 + \mu_3^2 = 2, \tag{5.9a}$$

$$\mu_1 + \mu_2 + \mu_3 = 0, \tag{5.9b}$$

There are two sets of solutions to (5.9), (5.10); viz.,

$$\lambda = -1/\sqrt{3}, \quad A = \begin{pmatrix} 1/\sqrt{3} & 0 & 0 \\ 0 & 1/\sqrt{3} & 0 \\ 0 & 0 & -2/\sqrt{3} \end{pmatrix}$$

and

$$\lambda = 1/\sqrt{3}, \quad A = \begin{pmatrix} -1/\sqrt{3} & 0 & \\ 0 & -1/\sqrt{3} & \\ 0 & 0 & 2/\sqrt{3} \end{pmatrix}.$$

The order of the eigenvalues on the diagonal is immaterial, for any permutation of the diagonal entries of A produces a point on the same orbit. Indeed, any such permutation is accomplished by the operation PAP^{-1}, where P is a permutation matrix, and such a P is an element of O(3). One of the orbits above gives the maximum of the functional $\frac{1}{3}\mathrm{tr} A^3$ on the sphere $\frac{1}{2}\mathrm{tr} A^2 = 1$, the other orbit is the minimum. The isotropy subgroup in each case is O(2) (rotations which leave $\binom{0}{1}$ invariant), so each extremal is axisymmetric.

The results above were obtained jointly with Professor L. Green. The method, while quite straightforward in the case $l=2$, becomes extremely complicated already in the case $l=4$ and so does not seem to be a practical approach to the resolution of the bifurcation equations (5.1) in the general case. It is interesting, nevertheless, to compare this approach with that of Michel and Radicati[7] in their work on symmetry breaking problems in physics.

They study the action of $\mathrm{SU}(n)$ on the vector space Q of Hermitian traceless matrices A with the inner product (5.7). It can be proved that there are two linearly independent trilinear invariants of this action, viz.,

$$\{A, B, C\} = \tfrac{1}{2}\sqrt{n}\ \mathrm{tr}(AB + BA)C, \quad [A, B, C] = -\tfrac{1}{2} i\, \mathrm{tr}[A, B]C$$

with $\{\ ,\ ,\ \}$ completely symmetric and $[\ ,\ ,\ ]$ completely antisymmetric. The bilinear form (5.7) is the only second order invariant. From this it can be concluded that there are only two linearly independent algebras on $\mathcal{A}$ with $\mathrm{SU}(n)$ as automorphism group. One is the Lie algebra whose multiplication law is

$$x \wedge y = -\tfrac{1}{2} i [x, y]$$

and the other is that with multiplication law

$$x \vee y = \tfrac{1}{2}\sqrt{n}\,(xy + yx) - (1/\sqrt{n})\,\mathrm{tr} xy.$$

Michel and Radicati are led to study the equation [(III.17), p. 194 of Ref. 7]

$$q \vee q + N(q) q = 0, \tag{5.10}$$

where $N(q)$ is a real number. Equation (5.10) is precisely equivalent to (5.8).

6. EXTREMAL METHODS AND STABILITY OF BIFURCATING SOLUTIONS

Having shown in the previous section that the reduced bifurcation equations sometimes possess a gradient structure as a consequence of their symmetry, we investigate in this section the relationship between the extremal properties of solutions and stability of the bifurcating solutions. Let us again assume v is a solution of the reduced bifurcation equations (5.1). The Jacobian of these equations at v is the linear operator

$$J_\sigma(v)x = \sigma x + 2B(x,x).$$

We know from Theorem 2.3 that the stability of the bifurcating solutions is determined by the eigenvalues of the linear operator $J_\sigma(v)$. Now suppose B is the gradient of the functional

$$\mathcal{J}(v) = \tfrac{1}{3}\langle B(v,v), v\rangle$$

and that v is the extremal of the variational problem

$$\max_{\langle v,v\rangle=1} \mathcal{J}(v). \tag{6.1}$$

We calculate the second variation of the variational problem at v. Let $x(t)$ be a curve on the unit sphere such that $x(0)=v$. Then, if $\mathcal{J}(v)$ attains a maximum at v,

$$\frac{d^2}{dt^2}\mathcal{J}(x(t)) = \langle B(v,v), \ddot{x}\rangle + 2\langle B(v,\dot{x}),\dot{x}\rangle \leqslant 0$$

and

$$\frac{d^2}{dt^2}\frac{1}{2}\langle x,x\rangle = \langle v,\ddot{x}\rangle + \langle \dot{x},\dot{x}\rangle = 0.$$

From (5.1), $\sigma v + B(v,v) = 0$, so

$$-\sigma\langle v,\ddot{x}\rangle + 2\langle B(v,\dot{x}),\dot{x}\rangle \leqslant 0, \quad \langle \sigma\dot{x} + 2B(v,\dot{x}),\dot{x}\rangle \leqslant 0$$

for all tangent vectors $\dot{x}$ and v. Consequently,

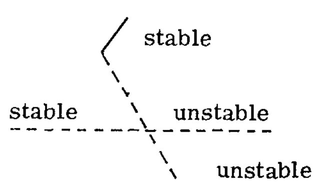
$$\langle J_\sigma(v)\dot{x},\dot{x}\rangle \leqslant 0 \tag{6.2}$$

for all tangent vectors $\dot{x}$. Furthermore, $J_\sigma(v)$ leaves the tangent plane at v invariant. In fact, the equation $\langle v,\dot{x}\rangle = 0$ describes the tangent plane at v and also $\langle B(v,\dot{x}),v\rangle = \langle B(v,v),\dot{x}\rangle = -\sigma\langle v,\dot{x}\rangle = 0$. Therefore, if $\langle v,\dot{x}\rangle = 0$, then $\langle J_\sigma(v)\dot{x},v\rangle = 0$ as well, and the tangent plane at v is preserved. So $J_\sigma(v)$ maps the tangent plane to itself, and (6.2) tells us $J_\sigma(v)$ is negative semidefinite at a local maximum v. The normal vector to the tangent plane is v itself, and $J_\sigma(v)v = \sigma v + 2B(v,v) = B(v,v) = -\sigma v$. Therefore, the remaining eigenvalue of $J_\sigma(v)$ is $-\sigma$. Since at an extremal v

$$-\sigma = \langle B(v,v),v\rangle / \langle v,v\rangle,$$

the eigenvalue $-\sigma$ is positive at a positive maximum of $\langle B(v,v),v\rangle$. We have proved

Theorem 6.1: Suppose the reduced bifurcation equations (5.1) have a gradient structure and that a solution v is obtained as a maximum of the variational problem (5.2). Then one eigenvalue of the Jacobian $J_\sigma(v) = \sigma I + 2B(v)$ is positive and the rest are nonpositive. Accordingly, from Theorem 2.3 it follows that the corresponding branch of solutions, which in this case is transcritical, has one unstable subcritical mode:



This situation occurs often in bifurcation problems and is depicted schematically in the above figure. When the effect of higher order terms is included the subcritical branch may "bend back" and regain stability. Such a situation is called "hard excitation" in nonlinear oscillation theory or "snap through" instability in the case of buckling theory. The situation depicted in the figure of Theorem 6.1 can lead to sudden jump discontinuities and hysteresis effects as the parameter λ is varied in the vicinity of its critical value.

A similar analysis can be carried through the cubic case, when the reduced bifurcation equations take the form

$$\sigma x - B(x,x,x) = 0. \tag{6.3}$$

Again, if B is the gradient of the functional $\frac{1}{4}\langle B(x),x\rangle$ then equations (6.3) are the Euler-Lagrange equations for the variational problem

$$\min_{\langle x,x\rangle=1} \tfrac{1}{4}\langle B(x,x,x),x\rangle.$$

It can easily be shown, by the same analysis as before, that the eigenvalues of the Jacobian are always nonpositive at a positive minimum of the quartic $\langle B(x,x,x), x\rangle$ on the sphere $\langle x,x\rangle = 1$. Hence at a positive minimum (which does not necessarily exist) we get stable supercritically bifurcating solutions. The bifurcations are one-sided—supercritical positive extrema and subcritical at negative extrema. Subcritically bifurcating solutions are always unstable. (For further results see Sather.[9])

7. SPECIAL RESULTS

In this section we discuss the special results which can be obtained by direct calculations for low values of l: $l=1, 2, 3, 4$.

For $l=1$ the reduced bifurcation equations are

$$\begin{aligned} \sigma z_1 &= a z_1(z_0^2 - 2z_1 z_{-1}), \\ \sigma z_2 &= a z_0(z_0^2 - 2z_1 z_{-1}), \\ \sigma z_{-1} &= a z_{-1}(z_0^2 - 2z_1 z_{-1}), \end{aligned} \tag{7.1}$$

where the parameter a is to be considered a fixed real constant. For real solutions we require $\bar{z}_m = (-1)^m z_{-m}$, hence $z_0^2 - 2z_1 z_{-1} = z_0^2 + 2|z_i|^2$. A nontrivial solution of (6.1) must satisfy

$$z_0^2 + 2|z_1|^2 = \sigma/a, \tag{7.2}$$

from which we see that σ/a must be positive. Therefore, the bifurcation is supercritical ($\sigma > 0$) if $a > 0$ and subcritical if $a < 0$. The full set of solutions of (6.1) is

$$z_0 = \sqrt{\sigma}/a\cos\theta, \quad z_{\pm1} = \pm\sqrt{\sigma}/2a\sin\theta\exp(\pm i\varphi). \tag{7.3}$$

The eigenvalues of the Jacobian are constant on orbits and are most easily evaluated at $z_0 = \sqrt{\sigma/a}$, $z_{\pm1} = 0$; they are $0, 0, -2\lambda$, reflecting the fact that the orbit of solutions is two-dimensional. Since the Jacobian is not invertible, the implicit function theorem cannot be used to continue solutions of the reduced bifurcation equations to the full equations. One can *a fortiori* restrict oneself to a subspace of axisymmetric solutions where the problem reduces to bifurcation at a simple eigenvalue; but the question remains as to whether all solutions of the full bifurcation equations are obtained in this way. The case $l=1$ arises in spherical convection problems when the inner and outer surfaces are free surfaces (Chossat,[19] p. 19).

Now let us turn to the case $l=2$, which was first treated in full by Busse[1] and in Sec. 5 of the present paper. The reduced bifurcation equations are (setting $\sigma/a=1$ without loss of generality)

$$
\begin{aligned}
z_2 &= (2\sqrt{2}/7 z_2 z_0 - \sqrt{3}/7 z_1^2),\\
z_1 &= (-2\sqrt{1}/14 z_1 z_0 + 2\sqrt{3}/7 z_2 z_{-1}),\\
z_0 &= (-\sqrt{2}/7 z_0^2 + 2\sqrt{1}/14 z_1 z_{-1} + 2\sqrt{2}/7 z_2 z_{-2}), \qquad (7.4)\\
z_{-1} &= (-2\sqrt{1}/14 z_{-1} z_0 + 2\sqrt{3}/7 z_{-2} z_1),\\
z_{-2} &= (2\sqrt{2}/7 z_{-2} z_0 - \sqrt{3}/7 z_{-1}^2).
\end{aligned}
$$

This system of equations is already quite complicated (and is destined to get worse). However, one can obtain a special class of solutions by setting $z_{\pm 1}=0$ and taking z_2 to be real. The equations then reduce to two equations in two unknowns (since $z_{-2}=z_2$), viz.,

$$
\begin{aligned}
z_2 &= 2\sqrt{2}/7 z_2 z_0,\\
z_0 &= -\sqrt{2}/7 z_0^2 + 2\sqrt{2}/7 z_2^2.
\end{aligned}
$$

Two solution sets are

$$z_{\pm 2} = \sqrt{21}/16, \quad z_{\pm 1} = 0, \quad z_0 = \sqrt{7}/8 \qquad (7.5)$$

and

$$z_{\pm 2} = z_{\pm 1} = 0, \quad z_0 = -\sqrt{7}/2. \qquad (7.6)$$

One can evaluate the third order invariant $p_3(z) = \frac{1}{3} \sum F_m \overline{z_m}$ at the two solutions (properly normalized) and one finds $p_3 = \sqrt{2}/7$ for the first solution and $-\sqrt{2}/7$ for the second. Thus these two special solutions lie on the maximum and minimum orbits of the functional p. The Jacobian of the reduced bifurcation equations can be calculated and its eigenvalues determined. We omit the details, but the eigenvalues in both cases are $[3, 3, -1, 0, 0]$. Again, both orbits are axisymmetric. Due to the presence of the zero eigenvalues the implicit function theorem cannot be used directly here, but there is an alternative method. Let us restrict ourselves to solutions with the reflection symmetry property $z_{-m} = (-1)^m z_m$. Then all solutions have real values z_m. Under those conditions the last two equations of (6.4) are identical with the first two. More generally,

Theorem 7.1: Let the general bifurcation equations $F_m(z_{-l}, \ldots z_l) = 0$ be restricted to the subclass of solutions with the symmetry property

$$z_m = (-1)^m z_{-m}. \qquad (7.7)$$

Then $F_{-m}(z_{-l}, \ldots z_l) = (-1)^m F_m(z_{-l}, \ldots, z_l)$ and the bifurcation equations to $(l+1)$ real equations in $(l+1)$ real unknowns.

Proof: The reality condition $\overline{z_m} = (-1)^m z_{-m}$ and (7.7) imply that z_m is real. Furthermore, the bifurcation equations satisfy

$$\overline{F_m(z_{-l}, \ldots, z_l)} = (-1)^m F_{-m}(z_{-l}, \ldots, z_l), \qquad (7.8)$$

$$\overline{F_m(z_{-l}, \ldots, z_l)} = F_m(\overline{z_{-l}}, \ldots, \overline{z_l}). \qquad (7.9)$$

The property (7.8) following from the reality condition (3.6) and (7.9) being a consequence of the fact that all coupling coefficients are real. (They are obtainable by the Lie algebra methods outlined in Sec. 3.) Combining (7.7), (7.8), and (7.9), we obtain

$$
\begin{aligned}
F_{-m}(\cdots z_m \cdots) &= (-1)^m F_m(\cdots z_m \cdots)\\
&= (-1)^m F_m(\cdots z_m \cdots).
\end{aligned}
$$

Therefore, the last l equations coincide with the first l when z_{-m} is replaced by $(-1)^m z_{-m}$.

We omit the details, but if one computes the Jacobian of the first three equations of (7.4) at the special solutions (7.5) and (7.6) he obtains an invertible operator. Therefore, all solutions of the bifurcations can be obtained by solving the reduced bifurcation equations and applying the implicit function theorem when the restriction (7.7) is in force. A similar situation prevails in the case $l=4$. I conjecture that it is valid for all even l.

For $l=3$ there are two distinct covariant terms. They are obtained as follows. We must take F_3 to be

$$
\begin{aligned}
F_3 = a z_3^2 z_{-3} + b z_3 z_2 z_{-2} + c z_3 z_1 z_{-1}\\
+ d z_3 z_0^2 + e z_2 z_1 z_0 + f z_2^2 z_{-1} + g z_1^3.
\end{aligned}
$$

Applying the condition $J_+ F_3 = 0$, we are led to the system of five equations in seven unknowns:

$$
\begin{bmatrix}
\beta_{-3} & \beta_2 & 0 & 0 & 0 & 0 & 0\\
0 & \beta_{-2} & \beta_1 & 0 & 0 & 2\beta_2 & 0\\
0 & 0 & \beta_{-1} & 2\beta_1 & \beta_2 & 0 & 0\\
0 & 0 & 0 & 0 & \beta_1 & \beta_{-1} & 0\\
0 & 0 & 0 & 0 & \beta_0 & 0 & 3\beta_1
\end{bmatrix}
\begin{bmatrix} a\\ b\\ c\\ d\\ e\\ f\\ g \end{bmatrix} = 0,
$$

where $\beta_m = \sqrt{(3-m)(3+m+1)}$.

One solution is obtained by taking $g=0$ and $d=1$. Then $e=f=0$ and we get

$$F_3 = z_3(z_0^2 - 2z_1 z_{-1} + 2z_2 z_{-2} - 2z_3 z_{-3}).$$

The quantity in parentheses is the second order invariant, and so is annihilated by the application of any of the J operators. Therefore, one mapping is

$$F_m = z_m(z_0^2 - 2z_1 z_{-1} + 2z_2 z_{-2} - 2z_3 z_{-3}).$$

A second choice is $g \neq 0$, $d=0$. The choice $g=\sqrt{7}$ leads to

$$
\begin{aligned}
G_3 = 9\sqrt{60}/7 z_3^2 z_{-3} - 9\sqrt{60}/7 z_3 z_2 z_{-2}\\
+ 3\sqrt{60}\, z_3 z_1 z_{-1} - 3\sqrt{10}\, z_2 z_1 z_0\\
+ (30/\sqrt{7}) z_2^2 z_{-1} + \sqrt{7}\, z_1^3.
\end{aligned}
$$

The lower weight polynomials are obtained by successively applying the lowering operator J_-.

The general reduced bifurcation equations in this case take the form

$$\lambda z_m = A F_m + B G_m,$$

where the parameters A and B depend on the external physical parameters of the problem. Such a situation occurs in the Bénard problem and gives rise to mechanisms for pattern selection.[14]

$l=4$: The quadratic terms in this case are

$$F_4=(1/\sqrt{5})z_4z_0-(1/\sqrt{2})z_3z_1+(3/2\sqrt{14})z_2^2,$$
$$F_3=(1/\sqrt{2})z_4z_{-1}-(3/2\sqrt{5})z_3z_0+(1/\sqrt{14})z_2z_1,$$
$$F_2=(3/\sqrt{14})z_4z_{-2}-(1/\sqrt{14})z_3z_{-1}-(11/14\sqrt{5})z_2z_0+(3/7\sqrt{2})z_1^2,$$
$$F_1=(1/\sqrt{2})z_4z_{-3}+(1/\sqrt{14})z_3z_{-2}-(6/7\sqrt{2})z_2z_{-1}+(9/7\sqrt{20})z_1z_0,$$
$$F_0=(1/\sqrt{5})z_4z_{-4}+(3/2\sqrt{5})z_3z_{-3}-(11/14\sqrt{5})z_2z_{-2}-(9/14\sqrt{5})z_1z_{-1}+(9/14\sqrt{5})z_0^2.$$

The remaining polynomials are found from those above by the relationship $F_{-m}(z_{-4},\ldots,z_{+4})=(-1)^mF_m(z_{-4},\ldots,z_4)=(-1)^mF_m(\cdots(-1)^mz_{-m}\cdots)$. There are many possible solutions to the bifurcation equations in this case. Busse has found two special solutions:

(1) Axisymmetric solutions: $z_{\pm1}=\cdots=z_{\pm4}=0$, $z_0\neq0$; and (2) octahedral solutions: $z_4=z_{-4}=5/\sqrt{14}$, $z_0=\sqrt{5}$, $z_{\pm1}=z_{\pm2}=z_{\pm3}=0$. Busse conjectures, on the basis of numerical work, that the second solution is the one which maximizes the third order invariant. An analysis of the Jacobian shows that the axisymmetric solution is a saddle point and that the octahedral solution is a candidate for the maximum.

The Jacobian of the reduced bifurcation equations (we set $\sigma=1$ and drop the normalization condition $|z|=1$; the results are affected only by a possible change of scale) at the special solution $z_0=\sqrt{5}$, $z_4=z_{-4}=5/\sqrt{14}$, $z_{\pm1}=z_{\pm2}=z_{\pm3}=0$ is

$$J-I=\begin{bmatrix} 0 & 0 & 0 & 0 & \sqrt{5}/14 & 0 & 0 & 0 & 0\\ 0 & -5/2 & 0 & 0 & 0 & 5/\sqrt{7} & 0 & 0 & 0\\ 0 & 0 & -25/14 & 0 & 0 & 0 & 15/14 & 0 & 0\\ 0 & 0 & 0 & -5/14 & 0 & 0 & 0 & 5/2\sqrt{7} & 0\\ \sqrt{5}/14 & 0 & 0 & 0 & 2/7 & 0 & 0 & 0 & \sqrt{5}/14\\ 0 & 5/2\sqrt{7} & 0 & 0 & 0 & -5/14 & 0 & 0 & 0\\ 0 & 0 & 15/14 & 0 & 0 & 0 & -25/14 & 0 & 0\\ 0 & 0 & 0 & 5/2\sqrt{7} & 0 & 0 & 0 & -5/2 & 0\\ 0 & 0 & 0 & 0 & \sqrt{5}/14 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of this matrix can be determined by restricting the matrix to certain invariant subspaces (determined by inspection), as follows. Let $\mathbf{e}_i$ denote the column vector with a 1 in the ith row and zeros everywhere else. The subspaces $\{a\mathbf{e}_3+b\mathbf{e}_7\}$, $\{\mathbf{e}_1+\mathbf{e}_9\}$, $\{a\mathbf{e}_1+b\mathbf{e}_5+c\mathbf{e}_9\}$, $\{a\mathbf{e}_2+b\mathbf{e}_6\}$, $\{a\mathbf{e}_4+b\mathbf{e}_8\}$ are all invariant, and one has to calculate the eigenvalues at most of a 3×3 matrix. The complete set of eigenvalues is

$$\{0,0,0,-20/7,\ -20/7,\ -20/7,\ -5/7,\ -5/7,1\}.$$

Since only one eigenvalue is positive, this octahedral solution is a possible candidate for the maximum of the extremal problem. The axisymmetric solution above, however, is definitely a saddle point of the variational problem; the eigenvalues of the Jacobian are

$$\{0,0,20/9,\ 20/9,\ 10/3,\ 10/3,\ -5/9,\ -5/9,\ -1\}.$$

Busse's article also contains a discussion of the situation for higher values of l, and special solutions are given for $l=6,8$. His special solutions belong to one of two classes (besides the axisymmetric solutions):

$$z_0\neq0,\ \ z_n,\ z_{2n}\neq0,\ \ \tfrac{1}{3}l<n\leq\tfrac{1}{2}l, \tag{7.10a}$$

$z_m=0$ otherwise;

$$z_0\neq0,\ z_n\neq0 \text{ for a single } n>l/2, \tag{7.10b}$$

$z_m=0$ otherwise.

The axisymmetric solutions never give a maximum except in the case $l=2$.

8. APPLICATIONS

Convection problems in spherical geometries arise naturally in geophysical problems and have been discussed by many authors (Chandrasekar,[10] Busse,[1] Chossat[19]). Convective phenomena in fluid media are generally modeled by the Boussinesq equations, which, in dimensionless variables, take the form

$$\frac{\partial\mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}=\Delta\mathbf{u}-\nabla p+\lambda g_1(r)\theta\mathbf{r}+\epsilon\omega\mathbf{u}\times\hat{k}, \tag{8.1}$$
$$\frac{\partial\theta}{\partial t}=\frac{1}{\mathrm{Pr}}(\Delta\theta+\lambda\beta_1(r)\mathbf{u}\cdot\mathbf{r})-\mathbf{u}\cdot\nabla\theta,$$
$$\mathrm{div}\mathbf{u}=0,$$

where $\mathbf{u}$ is the fluid velocity field, θ is the temperature perturbation, p is the hydrodynamic pressure, and $\mathbf{r}=x\hat{i}+y\hat{j}+z\hat{k}$ is the position vector. Pr is the Prandtl number and λ is the Rayleigh number; $g_1(r)$ is the gravitational field and $\beta_1(r)$ is the steady state temperature gradient. The term $\omega\mathbf{u}\times\hat{k}$ is the coriolis term due to rotation of the fluid. The operator $\mathbf{u}\times\hat{k}$ breaks O(3) symmetry, as it is only invariant under rotations about the $\hat{k}$ axis.

In geophysical applications these equations are considered on a spherical shell $\mathcal{N}<r<1$ with appropriate boundary conditions. When both surfaces are free, the kernel of the linearized equations contains the space V^1 (which transforms as D^1) (Chossat,[19] p. 19). When both surfaces are rigid and $\mathcal{N}$ is in the vicinity of 0.3 the kernel of L_λ for the critical value of λ_c transforms as D^2; but, as $\mathcal{N}\to1$, the kernel of L_{λ_c} transforms as D^l for higher and higher values of l (Chossat, personal communication). Chossat's thesis contains an extensive discussion of the linearized eigenvalue problem for the

Boussinesq equations (7.1) in a spherical shell, and also discusses the effect of the symmetry breaking term $\omega \mathbf{u}\times k$ on the bifurcation point. Depending on the sign of ω, one gets either a bifurcation of stationary solutions or time periodic solutions.

The buckling of perfectly spherical shells has also been the subject of much investigation. (See esp. Sather.[9,12]) Many of the investigations have been limited to Axisymmetric buckling, as in Bauer, Keller and Reiss.[11] This restriction is certainly justified if $\ker G_u(0,0)$ transforms as D^l for $l=1,2$; but already in the case $l=4$ Busse's result shows that the axisymmetric solutions are generally not the relevant bifurcating solutions. If the equations of elasticity exhibit the same behavior as is supposed for the convection equations—that is, if $\ker G_u(0,0)$ transforms D^l for higher and higher l as $N\to 1$—then the bifurcation problem for higher values of l is also of interest in buckling problems. Actually, experiments on the buckling of very thin spherical shells indicates that this is precisely the case.

Finally, the symmetry breaking problems studied by Michel and Radicati[7,8] also lead directly to the analog of the reduced bifurcation equations (5.1) but with SO(3) [or SU(2)] replaced by SU(3).

APPENDIX

Let us derive the expression (4.3) for the generating function for the characters $\chi_n(g)$ of $(\Gamma^{\otimes n})s$. Fix the group element g and let the eigenvectors of Γ on V be $\mathbf{e}_1,\ldots,\mathbf{e}_r$ with eigenvalues $\lambda_1,\ldots,\lambda_s$. The vector space $(V^{\otimes n})s$ is spanned by the vectors $(\mathbf{e}_{i_1}\otimes\cdots\otimes\mathbf{e}_{i_r})s$, which we may represent as

$$\mathbf{w}=\sum_{\sigma\in S_r}\mathbf{e}_{i_{\sigma(1)}}\otimes\cdots\otimes\mathbf{e}_{i_{\sigma(r)}}.$$

The action of $\Gamma^{\otimes n}$ on $\mathbf{w}$ is simply $\Gamma^{\otimes n}\mathbf{w}=\lambda_1^{m_1}\cdots\lambda_r^{m_r}w$, where $m_1+\cdots+m_r=n$. Thus a vector in $(V^{\otimes n})s$ may be represented by its occupation numbers $m_1,\ldots,m_r$ (where m_1 = times $\mathbf{e}_1$ occurs, and so forth). The trace of $\Gamma^{\otimes n}$ is therefore

$$\chi_{(n)}(g)=\mathrm{tr}\,\Gamma^{\otimes n}(g)=\sum_{m_1+\cdots+m_r=n}\lambda_1^{m_1}\cdots\lambda_r^{m_r}.$$

Multiplying by z^n and summing, we get

$$\sum_{n=0}^{\infty}z^n\chi_{(n)}(g)=\sum_{n=0}^{\infty}\sum_{m_1+\cdots+m_r=n}(z\lambda_1)^{m_1}\cdots(z\lambda_r)^{m_r}$$

$$=\sum_{m_1,\ldots,m_r=0}^{\infty}(z\lambda_1)^{m_1}\cdots(z\lambda_r)^{m_r}$$

$$=\prod_{i=1}^{r}\frac{1}{(1-z\lambda_i)}=\det(I-z\Gamma_{(\mathbf{r})})^{-1}.$$

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