The Mathematical Problem of Hydrodynamic Stability[*]

D. H. SATTINGER

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1. Introduction. The Navier-Stokes equations governing viscous incompressible flow are

$$\partial_t u_i + u_j \partial_j u_i = -\partial_i p + \frac{1}{R} \Delta u_i, \qquad (1.1)$$
$$\partial_i u_i = 0,$$

where $\partial_j = \partial/\partial x_j$, $\partial_t = \partial/\partial_t$, and repeated indices denote summation. The functions u_i, $i = 1, 2, 3$, are the components of the velocity field in rectilinear coordinates, and p denotes the pressure. The quantities in (1.1) are non-dimensional. The parameter R is called the Reynolds number for the flow and is a dimensionless quantity.

Let D be a bounded region of $\mathbf{R}^3$ with boundary ∂D. Due to the viscosity, the fluid must adhere to the walls of D. If the boundaries are moving at a constant, tangential rate (such as a sphere or cylinder rotating about its axis of symmetry) then the fluid must move with the walls at the boundary. Let $\tilde{u}$ be a stationary (time independent) solution of (1.1) satisfying non-homogeneous boundary conditions. An interesting question, and one which has occupied an important position in the research of experimentalists and applied mathematicians, is that of the *stability* of the flow $\tilde{u}$. That is, if the flow is disturbed slightly, do the perturbations grow or decay? If the solution of (1.1) is written in the form $u = \tilde{u} + v$, where v denotes the perturbed flow, then we derive for v the system of equations

$$\partial_t v_i + v_j \, \partial_j v_i + \tilde{u}_j \, \partial_j v_i + v_j \, \partial_j \tilde{u}_i = -\partial_i p + \frac{1}{R} \Delta v_i, \qquad (1.2)$$
$$\partial_i v_i = 0, \qquad v_i = 0 \quad \text{on} \quad \partial D.$$

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The question of the stability of $\tilde{u}$ is now reduced to a study of the asymptotic behavior of v as $t \to \infty$.

Now $\tilde{u}$ depends on R and the basic problem in hydrodynamic stability is to determine the critical values of R at which the flow becomes unstable. The equations (1.2) being nonlinear, it is hard to get any meaningful quantitative information (*i.e.* numbers) from a direct analysis of them. Accordingly much of the literature in hydrodynamic stability is devoted to an analysis of the linearized equations

$$\begin{aligned} \partial_t v_i + \tilde{u}_j\, \partial_j v_i + v_j\, \partial_j \tilde{u}_i &= -\partial_i p + \frac{1}{R}\, \Delta v_i\,, \\ \partial_i v_i = 0, \qquad v_i &= 0 \quad \text{on} \quad \partial D. \end{aligned} \tag{1.2'}$$

(*Cf.* [4], [10].) It is argued that if the perturbations are small then the nonlinear term is of second order magnitude and can be neglected. Of course, this argument is open to question, especially since the nonlinear term involves derivatives of the flow, which may not be small. Nevertheless, a considerable body of experimental evidence tends to support this "linearization hypothesis," and the point of this paper is to give a rigorous, general mathematical proof of its validity in the case of a bounded domain.

In order to give a precise formulation of the results the following norms must be introduced:

$$\begin{aligned} |u|^2 &= \int_D u_i u_i \, d\mathbf{x} \\ \|u\|^2 &= \int_D u_{i,j} u_{i,j} \, d\mathbf{x}, \end{aligned} \tag{1.3}$$

where $u_{i,j} = \partial_j u_i$. We have

Definition. Let $v(t)$ denote a solution of (1.2). The flow $\tilde{u}$ is *unconditionally stable* if $\lim_{t\to\infty} |v(t)| = 0$ whenever $|v(0)|$ is finite. The flow is *conditionally stable* if given any $\epsilon > 0$ there is a $\delta > 0$ such that $|v(t)| < \epsilon$ for all $t \geqq 0$ and $\lim_{t\to\infty} |v(t)| = 0$ whenever $|v(0)| < \delta$. The flow is *unstable* if it is not conditionally stable.

In this paper it is shown that the flow is (conditionally) stable or unstable according as the solutions of the linearized system (1.2′) decay or grow. Thus the stability or instability of the flow can in principle be determined from an analysis of the linearized equations. (See, however, the comments in §6.)

The asymptotic behavior of the solutions of (1.2′) is determined by the spectrum of a certain linear, nonsymmetric operator L closely associated with (1.2′). In §3 it is shown that the spectrum of L consists of a discrete set of eigenvalues, and that the eigenfunctions are complete in $L_2(D)$ (the Hilbert space of all measurable v for which $|v|$ is finite). These facts are established by a method of

Carleman [3] concerning operators of Hilbert-Schmidt class which has been suitably generalized in [7]. The technique of proving the stability and instability theorems makes important use of this spectral decomposition of L.

Prodi [11] has shown that if the spectrum of L lies in the right half plane (in which case the solutions of (1.2′) decay exponentially) then $\tilde{u}$ is conditionally stable in the double norm. He assumes that $\tilde{u}$ has L_2 second derivatives. In this paper we assume that $\tilde{u}$ and the eigenfunctions of L (and its adjoint L^*) have bounded first derivatives in D. In that case our stability theorem shows that it is sufficient to assume only that a disturbance is small in $| \ |$ in order to prove that it decays.

Considerable work has been done recently on related mathematical problems in hydrodynamic stability. We mention especially the papers of Velte [17], [18], Rabinowitz [16], and Kirchgässner and Sorger [15] who treat the bifurcation of stationary solutions in specific problems of physical interest. In these papers it is shown that a second stationary solution of the equations of hydrodynamics bifurcates from the basic flow $\tilde{u}$ just as the critical value of the stability parameter is crossed. In [15] an instability theorem is proved under certain assumptions. Also, see [19], where some stability and instability theorems are given for solutions which possess some degree of regularity.

No proof of global, strong solutions of (1.1) or of (1.2) for arbitrary initial data in $L_2(D)$ exists; and in fact it may be that this is not the case. Therefore the stability and instability theorems are proved for weak solutions of (1.2). In §2 an account of the existence and important properties of weak solutions of (1.2) is given. Since the proofs, by and large, are similar to the corresponding proofs for system (1.1), they will only be sketched briefly.

Since the value of R is not explicitly important in our discussion we will take $R = 1$ from now on.

2. General facts about the equations for perturbed flow.

In what follows we shall be dealing with a nonsymmetric operator whose spectrum may be complex. Therefore it is necessary to extend $L_2(D)$ to include complex fields u, and the norms (1.3) and their corresponding inner products must be modified accordingly.

We say that u is *weakly divergence free* if

$$\int_D u_i \, \partial_i p \, dx = 0$$

for all $p \in C^1(\bar{D})$. (Given a set Ω, the class of functions n times continuously differentiable on Ω is denoted by $C^n(\Omega)$.) By Gauss' Integral Theorem it is not hard to see that if u is weakly divergence free and continuously differentiable then $\operatorname{div} u = 0$ in the strong sense in the interior of D and u has zero normal component at boundary points of D where ∂D is smooth. The class of all weakly divergence free fields forms a closed subspace of $L_2(D)$ which we denote by H_σ. The orthogonal complement of H_σ, denoted by H_π, is the class of all gradients

$u = \operatorname{grad} p$ (in the weak sense), where p is an element of the Sobolov space $W^{1,2}(D)$. (See [8].)

Let $C_{0,\sigma}^{\infty}$ denote the class of infinitely differentiable divergence free fields with compact support in D. The closure of $C_{0,\sigma}^{\infty}$ in the norm $\| \; \|$ is denoted by $\dot{H}_{\sigma}^{1}$. It is the class of divergence free fields with L_2 first derivatives which vanish on the boundary in a weak sense. For $T > 0$ we define the norm $| \; |_T$ by

$$|v|_T^2 = \int_0^T \{|u|^2 + \|u\|^2\}\, dt.$$

Let V_{σ}^{∞} denote the class of infinitely differentiable divergence free fields $v(x, t)$ in $D \times [0, \infty)$ which vanish on ∂D. We say that a field is in the class V_T if $|v(t)|$ is bounded for $0 \leqq t \leqq T$ and v is in the closure of V_{σ}^{∞} under the norm $| \; |_T$.

A weak solution of (1.2) satisfying the initial condition $u(0) = u^{(0)}$ is a real valued measurable field which belongs to V_T for all $T > 0$ and which satisfies

$$(u^{(0)}, \zeta) + \int_0^{\infty} \int_D \{u_i\, \partial_t \bar{\zeta}_i - u_{i,j} \bar{\zeta}_{i,j} - \tilde{u}_j \bar{\zeta}_i\, \partial_j u_i - u_j \bar{\zeta}_i\, \partial_j \tilde{u}_i + u_i u_j\, \partial_j \bar{\zeta}_i\}\, d\mathbf{x}\, dt = 0 \tag{2.1}$$

for all ζ in V_{σ}^{∞} which vanish identically for large t. Equation (2.1) is obtained formally from (1.2) by multiplying (1.2) by $\bar{\zeta}$, integrating over $D \times [0, \infty)$, and integrating by parts. It can be extended by continuity to hold for all ζ such that $\zeta(t)$ belongs to the class $\dot{H}_{\sigma}^{1}$ for $t \geqq 0$. If u is a weak solution, is twice continuously differentiable in the space variables, and is continuously differentiable in t, then u can be shown to satisfy (1.2) for a suitable choice of the pressure p [9]. In that case u is called a strong solution.

It is convenient to denote by $L[u, v]$ the bilinear form

$$L[u, v] = \int_D \{u_{i,j} \bar{v}_{i,j} + \tilde{u}_j \bar{v}_i (\partial_j u_i) + (\partial_j \tilde{u}_i) u_j \bar{v}_i\}\, d\mathbf{x},$$

defined on pairs u, v in $\dot{H}_{\sigma}^{1}$. Equation (2.1) can then be written

$$(u^{(0)}, \zeta) + \int_0^{\infty} \{(u, \zeta_t) - L[u, \zeta] + (u, u \cdot \operatorname{grad} \zeta)\}\, dt = 0. \tag{2.2}$$

Strong solutions of (1.2) satisfy the energy identity

$$\frac{d}{dt} \frac{|u|^2}{2} + L[u, u] = 0. \tag{2.3}$$

This is obtained by multiplying (1.2) by u and integrating over D in the usual way. The terms $(\operatorname{grad} p, u)$ and $(u, u \cdot \operatorname{grad} u)$ vanish because of the divergence free character of u. Integrating (2.3) over any interval (t_1, t_2) we get

$$|u(t_2)|^2 + 2 \int_{t_1}^{t_2} L[u, u]\, dt = |u(t_1)|^2. \tag{2.4}$$

Now the second term in $L[u, u]$ has zero real part. In fact, by integration by parts, we get

$$\int_D \tilde{u}_j(\partial_j u_i)\bar{u}_i \, d\mathbf{x} = -\int_D \tilde{u}_j u_i(\partial_j \bar{u}_i) \, d\mathbf{x},$$

since div $\tilde{u} = 0$ and $\tilde{u}$ is real. Since u is real, the imaginary part also vanishes and (2.4) can also be written

$$|u(t_2)|^2 + 2\int_{t_1}^{t_2} \left\{ \|u\|^2 + \int_D (\partial_j \tilde{u}_i)u_i u_j \, d\mathbf{x} \right\} dt = |u(t_1)|^2. \tag{2.5}$$

The energy relations for strong solutions can be written either in differential or integral form. In the case of weak solutions, however, the energy relations must be written in integral form since, so far at any rate, the continuity of $|u(t)|$ has not been established. Furthermore, one can only assert [9], [13] the inequalities

$$|u(t_2)|^2 + 2\int_{t_1}^{t_2} L[u, u] \, dt \leqq |u(t_1)|^2, \tag{2.6}$$

$$|u(t_2)|^2 + 2\int_{t_1}^{t_2} \left\{ \|u\|^2 + \int_D (\partial_j \tilde{u}_i)u_i u_j \, d\mathbf{x} \right\} dt \leqq |u(t_1)|^2. \tag{2.7}$$

Let

$$c = \sup_{x \varepsilon D} \frac{\partial_j \tilde{u}_i \xi_i \xi_j}{\xi_i \xi_j}.$$

Assuming $\tilde{u}_i$ is continuously differentiable on $\bar{D}$, c is finite. From (2.3) we get

$$\frac{d}{dt}\frac{|u|^2}{2} + \|u\|^2 \leqq c\,|u|^2.$$

Integrating this differential inequality we obtain

$$|u(t)|^2 + 2\int_0^t e^{2c(t-s)} \|u\|^2 \, ds \leqq e^{2ct}\,|u(0)|^2.$$

Using this formal inequality weak solutions of (1.2) which satisfy the energy inequalities (2.6) or (2.7) can be constructed by the Galerkin approximation method described by E. Hopf [9].

Any weak solution of (1.2) can be modified on a set of t of measure zero in such a way that [13]

$$\int_0^t \{(u, \zeta_t) - L[u, \zeta] + (u, u \cdot \operatorname{grad} \zeta)\} \, dt = (u, \zeta) - (u^{(0)}, \zeta(0))$$

for any ζ in $\dot{H}_\sigma^1$. When u is modified in this way we have immediately that $(u(t), \zeta)$ is absolutely continuous in t for any $\zeta \varepsilon \dot{H}_\sigma^1$. For arbitrary ζ in H_σ we can approximate ζ by a sequence $\{\zeta_n\}$ in $\dot{H}_\sigma^1$. Since $|u(t)|$ is uniformly bounded,

$(u(t), \zeta_n)$ tends uniformly to $(u(t), \zeta)$. Hence $(u(t), \zeta)$ is continuous for any ζ in H_σ, and $u(t)$ is weakly continuous in t.

From the weak continuity of u we get $|u(\tau)| \leqq \liminf_{t\to\tau}|u(t)|$ for any $\tau > 0$. On the other hand, from (2.7) with $t_1 = \tau$, $t_2 = t$ we have $\limsup_{t\to\tau^+}|u(t)| \leqq |u(\tau)|$. Therefore the right hand limit of $|u(t)|$ exists, and $|u(t)|$ is right continuous. From the well known theorem on weak and strong continuity it follows that $\lim_{t\to\tau^+}|u(t) - u(\tau)| = 0$ for any $\tau \geqq 0$.

We shall need the following lemmas. They are quite standard but we include their proofs for completeness.

Lemma 2.1. *Let $\alpha(t)$ be bounded, right continuous, and satisfy*

$$\alpha(t_2) - \alpha(t_1) \leqq \int_{t_1}^{t_2} F(s, \alpha)\, ds \tag{2.8}$$

for all $0 \leqq t_1 < t_2$. Let F be continuous in t and Lipschitz continuous in α. Then $\alpha(t) \leqq \beta(t)$, where β satisfies $\dot\beta = F(t, \beta)$, $\beta(0) = \alpha(0)$.

Proof. First let β satisfy $\dot\beta = F(t, \beta) + \epsilon$, $\beta(0) = \alpha(0)$ for some $\epsilon > 0$. From (2.8) there is a constant K such that

$$\frac{\alpha(t_2) - \alpha(t_1)}{t_2 - t_1} \leqq K \tag{2.9}$$

for all $0 \leqq t_1 < t_2$. Suppose there is a point $t_* > 0$ at which $\alpha(t_*) > \beta(t_*)$. Then there is an interval $(t_* - \delta, t_*)$ on which $\alpha(t) > \beta(t)$. Otherwise there would exist a sequence of points $t_k \to t_*-$ at which $\alpha(t_k) \leqq \beta(t_k)$; since $\beta(t)$ is continuous, condition (2.9) would be contradicted.

Let $(\tau, t_*]$ be the largest interval on which $\alpha(t) > \beta(t)$. By right continuity, $\alpha(\tau + 0) = \alpha(\tau) = \beta(\tau)$; otherwise the interval could be extended to the left by the argument above. From (2.8) we have

$$\overline{\lim_{t\to\tau^+}} \frac{\alpha(t) - \alpha(\tau)}{t - \tau} \leqq \overline{\lim_{t\to\tau^+}} \frac{1}{t - \tau} \int_\tau^t F(s, \alpha)\, ds$$
$$= F(\tau, \alpha(\tau)) = F(\tau, \beta(\tau)) = \dot\beta(\tau) - \epsilon.$$

The limit on the right hand side above exists by the right continuity of α. Thus we have

$$\dot\beta(\tau) \geqq \overline{\lim_{t\to\tau^+}} \frac{\alpha(t) - \alpha(\tau)}{t - \tau} + \epsilon,$$

which implies that $\beta > \alpha$ in some interval to the right of τ. Thus $\alpha(t) > \beta(t)$ on $[0, t_*]$ and the initial condition $\beta(0) = \alpha(0)$ cannot be satisfied.

We have shown, therefore, that $\alpha(t) \leqq \beta_\epsilon(t)$, where $\dot\beta_\epsilon = F(t, \beta_\epsilon) + \epsilon$, $\beta_\epsilon(0) = \alpha(0)$. As $\epsilon \to 0$, β_ϵ tends uniformly to β, the unique solution of $\dot\beta = F(t, \beta)$, $\beta(0) = \alpha(0)$. Thus $\alpha(t) \leqq \lim_{\epsilon\to 0} \beta_\epsilon(t) = \beta(t)$. This proves Lemma 2.1.

Remark. Letting $t_1 = 0$, $t_2 = t$ in (2.8) we get

$$\alpha(t) - \alpha(0) \leqq \int_0^t F(s, \alpha)\, ds. \tag{2.10}$$

The conclusion of Lemma 2.1 does not in general follow from (2.10) (however, see Lemma 2.2 below), and it will be necessary in the proofs in §§4, 5 to make use of the stronger statement (2.8).

Remark. The condition of right continuity on α can be dropped. In fact, (2.9) shows that α cannot oscillate, so that its left and right hand limits exist at each point. Letting $\tilde{\alpha}$ be α modified so as to be right continuous, the argument above shows that $\tilde{\alpha}(t) \leqq \beta(t)$. On the other hand, (2.9) implies that $\alpha(t + 0) \geqq \alpha(t) \geqq \alpha(t - 0)$, so that $\alpha(t) \leqq \beta(t)$ as well. The right continuity of the weak solutions of (1.2) is not necessary to what we prove in this paper. We have simply included the fact as an interesting observation.

Lemma 2.2. *Let $\alpha(t)$ be measurable and satisfy*

$$\alpha(t) \leqq A + \int_0^t F(s, \alpha)\, ds, \tag{2.11}$$

where F is continuous in s, Lipschitz continuous in α, and monotone increasing in α. Then $\alpha(t) \leqq \beta(t)$, where $\beta(0) = A$ and $\dot{\beta} = F(t, \beta)$.

Proof. Let $\alpha_1(t)$ denote the right hand side of (2.11). Then α_1 is absolutely continuous and $\dot{\alpha}_1 = F(t, \alpha)$ a.e. By the monotonicity of F we have $\dot{\alpha}_1 = F(t, \alpha) \leqq F(t, \alpha_1)$ a.e.; hence

$$\alpha_1(t) \leqq A + \int_0^t F(s, \alpha_1)\, ds. \tag{2.12}$$

Letting α_2 denote the right side of (2.12) we get $\dot{\alpha}_2 \leqq F(t, \alpha_2)$ for all t; since α_2 is continuously differentiable we have immediately that $\alpha_2 \leqq \beta$.

Serrin's criterion for unconditional stability. In order to give some insight into the methods of proof of the stability theorem we consider Serrin's criterion for unconditional stability. Serrin's energy criterion [12] for unconditional stability can be formulated as a variational problem as follows. Let

$$d_{ij} = (\tilde{u}_{i,j} + \tilde{u}_{j,i})/2$$

and let

$$\lambda = \min \int_D u_{i,j}u_{i,j} + d_{ij}u_iu_j\, dx,$$

subject to the constraints $|u|^2 = 1$, $\operatorname{div} u = 0$, and $u = 0$ on ∂D. *If $\lambda > 0$ then all weak solutions of* (1.2) *are stable and decay exponentially in the mean to zero.* In fact, going back to (2.7) we get, since $u(t) \in \dot{H}^1_\sigma$ for almost all t by (2.7),

$$|u(t_2)|^2 + 2\lambda \int_{t_1}^{t_2} |u(s)|^2\, ds \leqq |u(t_1)|^2$$

for all $0 \leqq t_1 < t_2$. By Lemma 2.1 this implies that $|u(t)| \leqq e^{-\lambda t}\, |u(0)|$.

Associated with the bilinear form $L[u, v]$ is a linear operator L such that $L[u, v] = (Lu, v)$ for all u in the domain of L and all v in $\dot{H}^1_\sigma$. (See §3.) The energy criterion then becomes $\operatorname{Re}(Lu, u) \geqq \lambda(u, u)$. If $\lambda > 0$ then all eigenvalues of L have positive real parts, but the converse need not be true, since L in general is not symmetric. In Lemma 3.7, however, it is proved that $\operatorname{Re}(Lu, u) \geqq \|u\|^2/2$ for all u in an invariant subspace of L with finite codimension. This fact allows us to prove the stability theorem even though $\operatorname{Re} L[u, u]$ may not be positive on the entire space $\dot{H}^1_\sigma$.

The operator A. In the Galerkin argument of Hopf a system of orthogonal functions complete in H_σ is required. Such a system can be constructed in the following way. Consider the eigenvalue problem

$$\begin{gathered}\Delta\varphi_i + \mu\varphi_i = -\partial_i p, \\ \partial_i\varphi_i = 0, \qquad \varphi_i = 0 \quad \text{on} \quad \partial D.\end{gathered} \tag{2.12}$$

We say that μ is an eigenvalue if there exists a non-trivial field φ and pressure p satisfying (2.12). A complete set of eigenfunctions $\{\varphi^{(k)}\}$ with eigenvalues $0 < \mu_1 \leqq \mu_2 \leqq \cdots$ tending to infinity can be constructed by the usual arguments of the calculus of variations. The n^{th} eigenvalue and eigenfunction are obtained as solutions of the variational problem

$$\mu = \min \|\varphi\|^2,$$

subject to the conditions $\operatorname{div} \varphi = 0$, $\varphi = 0$ on ∂D, $|\varphi| = 1$, and $(\varphi, \varphi^{(k)}) = 0$, $k = 1, \cdots, n - 1$. In fact, writing the divergence free condition in weak form $(u, \operatorname{grad} p) = 0$ for all $p \in C^1(D))$, we append it to the functional $\|\varphi\|^2$ and treat p as a Lagrange multiplier. We thus arrive at equations (2.12) as the Euler equations for this variational problem.

From this variational formulation for the eigenvalues we can show that $\sum_k \mu_k^{-2}$ is finite. In fact, if the divergence free condition is dropped we get the eigenvalue problem for the Laplacian on D repeated three times. Denote the eigenvalues of the Laplacian, each repeated three times, by μ_k'. By the minimax property of the eigenvalues [6] $\mu_k' \leqq \mu_k$, since the μ_k' are obtained from a variational problem with fewer constraints. Therefore

$$\sum_k \mu_k^{-2} \leqq \sum_k (\mu_k')^2 < +\infty.$$

This fact establishes that the inverse of a certain operator A associated with (2.12) is of Hilbert-Schmidt class. That operator can be described as follows. Let P be the orthogonal projection onto H_σ. Given the operator $(-\Delta)$ defined in H_σ, A is given by $P(-\Delta)$ (or, more precisely, the Friedrichs extension of this operator). The eigenvalues of A, considered as a linear transformation from H_σ to itself, are μ_i; and the inverse of A is a symmetric Hilbert–Schmidt operator on H_σ. These facts will be of use in §3.

3. The spectral decomposition of L. The starting point of this section is the bilinear form $L[u, v]$ introduced in §2. We assume that $\tilde{u}$ is continuously differentiable in D and has bounded derivatives there. L is then defined on pairs u, v in $\dot{H}^1_\sigma$ and satisfies the inequalities

$$|L[u, v]| \leqq c_1 \, ||u|| \cdot ||v||, \tag{3.1}$$

$$\operatorname{Re} L[u, u] \geqq ||u||^2 - c_2 \, |u|^2. \tag{3.2}$$

Inequality (3.2) follows from the fact that the second term in $L[u, u]$ has zero real part.

The theory of such bilinear forms is well known [1], [2]. Associated with $L[u, v]$ is an operator L on H_σ to itself defined in the following way. If there is an element f in H_σ such that $L[u, \varphi] = (f, \varphi)$ for all φ in $\dot{H}^1_\sigma$, then u belongs to the domain of L, denoted by $\mathfrak{D}(L)$, and $Lu = f$. If $u \in C^\infty_{0,\sigma}$ then $u \in \mathfrak{D}(L)$ and

$$Lu = P[-\Delta u_i + \tilde{u}_j \partial_j u_i + u_j \partial_j \tilde{u}_i],$$

where P is the projection on H_σ. The quantity in brackets above need not be divergence free, even though u is; so if the projection is dropped the equation $Lu = f$ must be written

$$-\Delta u_i + \tilde{u}_j \, \partial_j u_i + u_j \, \partial_j \tilde{u}_i = \partial_i p + f_i \, ,$$
$$\partial_i u_i = 0,$$

for some appropriate choice of the "pressure" p.

We have the following lemma.

Lemma 3.1. L is closed and its spectrum consists of a discrete set of eigenvalues in the plane, each of finite multiplicity, which can cluster only at infinity. The resolvent $(L - \lambda)^{-1}$ is compact when it exists. The eigenvalues of L lie inside the parabolic region

$$c_3 \tau^2 < \sigma + c_4 \qquad (\lambda = \sigma + i\tau), \tag{3.3}$$

where c_3 and c_4 are some fixed positive constants.

Proof. The fact that the eigenvalues lie in the parabolic region (3.3) has been proved by Prodi [11]. The proof of the first two statements of Lemma 3.1 is standard [1], [2], and we will only sketch it here. Let r be a positive number, $r \geqq c_2$, and let $L_r[u, v] = L[u, v] + r(u, v)$. The bilinear form L_r corresponds to the operator $L + r$, and from (3.2) we get, for $u \in \mathfrak{D}(L)$,

$$\begin{aligned} ||u||^2 &\leqq \operatorname{Re} L_r[u, u] = \operatorname{Re} ((L + r)u, u) \\ &\leqq |(L + r)u| \cdot |u| \leqq \mu_1^{-1/2} \, |(L + r)u| \cdot ||u|| \, , \end{aligned}$$

where μ_1 is the first eigenvalue of the operator A (see §2). Thus $|(L + r)u| \geqq \mu_1^{1/2} \, ||u||$. The closure of $(L + r)$, hence that of L, follows from this inequality and the continuity of $L_r[u, v]$ in the double norm (inequality (3.1)). From the inequality $\|u\|^2 \leqq \operatorname{Re} L_r[u, u]$ and the Lax-Milgram Theorem [1], [2], the "generalized Dirichlet problem"

$$L_r[u, \varphi] = (f, \varphi) \qquad \text{for all} \quad \varphi \in \dot{H}^1_\sigma$$

has a unique solution $u \in \dot{H}^1_\sigma$ for any $f \in H_\sigma$. Thus $(L + r)$ has a bounded inverse on H_σ; furthermore, $\|u\| \leqq \mu_1^{-1/2} |(L + r)u|$. Therefore $(L + r)^{-1}$ maps H_σ continuously into $\dot{H}^1_\sigma$; since bounded sets in $\dot{H}^1_\sigma$ are compact in H_σ by Rellich's Selection Theorem, $(L + r)^{-1}$ is compact. The rest of the results in the first two statements now follow.

Let λ be a complex number in the resolvent set of L. Then given any $f \in H_\sigma$ there is a u in $\dot{H}^1_\sigma$ such that $L[u, \varphi] - \lambda(u, \varphi) = (f, \varphi)$ for all $\varphi \in \dot{H}^1_\sigma$; and in particular,

$$L[u, u] - \lambda(u, u) = (f, u).$$

Taking the real part of this equation we get

$$\operatorname{Re} L[u, u] - \sigma|u|^2 = \operatorname{Re} (f, u),$$

$$\|u\|^2 - \sigma |u|^2 \leqq |f| \cdot |u| + c_2 |u|^2 \leqq \frac{|f|^2}{2} + c_6 |u|^2. \tag{3.4}$$

Taking imaginary parts we obtain

$$\operatorname{Im} L[u, u] - \tau |u|^2 = \operatorname{Im} (f, u),$$

$$\tau |u|^2 \leqq \operatorname{Im} L[u, u] + |f| \cdot |u|$$

$$\leqq c_7 |u| \cdot \|u\| + |f| \cdot |u|,$$

$$\tau |u| \leqq c_7 \|u\| + |f|.$$

Hence

$$\frac{\tau^2}{2c_7^2} |u|^2 \leqq \frac{|f|^2}{c_7^2} + \|u\|^2. \tag{3.5}$$

Adding (3.4) and (3.5) we get

$$(c_3\tau^2 - \sigma - c_4) |u|^2 \leqq c_8 |f|^2, \tag{3.6}$$

for some constants $c_3 > 0$ and $c_4 > 0$. If $c_3\tau^2 - \sigma - c_4 > 0$ then from (3.6) it follows that

$$|(L - \lambda)^{-1}| \leqq \frac{c_8^{1/2}}{(c_3\tau^2 - \sigma - c_4)^{1/2}}. \tag{3.7}$$

Now $(L - \lambda)^{-1}$ is holomorphic in λ and can be continued analytically through any domain as long as $|(L - \lambda)^{-1}|$ remains bounded. Thus (3.7) shows that $(L - \lambda)^{-1}$ exists everywhere outside the parabolic region (3.3), since it exists at one point there.

Definition 3.2. A *ray of minimal growth* of the operator L is a ray $\{\rho e^{i\theta}; \rho > 0\}$ in the complex plane such that $(L - \lambda)^{-1}$ exists ($\lambda = \rho e^{i\theta}$) for sufficiently large $|\lambda|$, and $|(L - \lambda)^{-1}| = O(|\lambda|^{-1})$ as λ tends to infinity along the ray.

From (3.7) it follows that every ray except the positive real axis is a ray of minimal growth for L.

Definition 3.3. A bounded operator T on a Hilbert space $\mathcal{H}$ is said to be of *Hilbert–Schmidt class* (HS) if

$$\sum_k |T\varphi_k|^2 < +\infty$$

for every orthonormal sequence $\{\varphi_k\}$ in $\mathcal{H}$.

An excellent reference for the theory of HS operators is [7].

Lemma 3.4. For complex λ, $(A - \lambda)^{-1}$ maps H_σ into $\dot{H}_\sigma^1$, and

$$||(A - \lambda)^{-1}u|| \leqq [2|\lambda|(1 - \cos\theta)]^{1/2}|u|,$$

where $\lambda = |\lambda|e^{i\theta}$.

Proof. Given a vector u in $\dot{H}_\sigma^1$ we have

$$||u||^2 = \sum_{k=1}^{\infty} \mu_k \, |(u, \varphi^{(k)})|^2.$$

Now

$$(A - \lambda)^{-1}u = \sum_{k=1}^{\infty} \frac{(u, \varphi^{(k)})\varphi^{(k)}}{\mu_k - \lambda},$$

so

$$||(A - \lambda)^{-1}||^2 = \sum_{k=1}^{\infty} \frac{\mu_k \, |(u, \varphi^{(k)})|^2}{|\lambda - \mu_k|^2}.$$

The result now follows by maximizing $\mu|\lambda - \mu|^{-2}$ over $\mu \geqq 0$.

Lemma 3.5. For large positive r the operator $(L + r)$ has an inverse of Hilbert–Schmidt class.

Proof. The operator L can be written as the sum $A + B$, where B is the "first order" part of L. For u in $\dot{H}_\sigma^1$, Bu is given by

$$Bu = P[\bar{u}_j \, \partial_j u_i + u_j \, \partial_j \bar{u}_i].$$

It is easy to show that $|Bu| \leqq c_9 \, ||u||$ for some positive constant c_9.

Let $A_r = (A + r)^{-1}$; this operator exists and is HS for large positive r. Since A_r maps H_σ into $\dot{H}_\sigma^1$ and B maps $\dot{H}_\sigma^1$ into H_σ, BA_r can be considered as a map from H_σ to itself. From Lemma 3.4,

$$|BA_r u| \leqq c_9 \, ||A_r u|| \leqq \frac{c_9}{2r^{1/2}} \, |u|.$$

For sufficiently large r, $|BA_r| < 1$ and $(I + BA_r)^{-1}$ exists and is bounded. We have

$$\begin{aligned}(L + r)^{-1} &= (A + B + r)^{-1} = ((I + BA_r)(A + r))^{-1} \\ &= A_r(I + BA_r)^{-1}.\end{aligned}$$

Since A_r is HS, and since the product of a bounded operator with a Hilbert–Schmidt operator is obviously also HS, the conclusion of Lemma 3.5 now follows.

We have now verified all the conditions of Corollary 31, p. 1042 in [7]. This corollary is an extension to general HS operators of a theorem due to Carleman [3] concerning the existence and completeness of the eigenfunctions of second order elliptic operators. We can state, then, the following lemma.

Lemma 3.6. *L has an infinite number of eigenvalues $\lambda_1, \lambda_2, \cdots$ lying in the parabolic region given in (3.3). The corresponding eigenfunctions (and higher order eigenfunctions) are complete in H_σ. That is, the class of all finite linear combinations of them is dense in H_σ.*

Since L is nonsymmetric the higher order eigenfunctions of L must be included in the completeness statement above. For an invariant subspace $\mathfrak{M}_\lambda$ corresponding to an eigenvalue λ, we have a Jordan basis

$$L\psi^{(1)} = \lambda\psi^{(1)}, \qquad L\psi^{(2)} = \lambda\psi^{(2)} + \psi^{(1)}, \cdots .$$

However, all these eigenfunctions of L will simply be denoted by $\psi_1, \psi_2, \cdots$.

The operator L^* is defined as follows. An element $u \varepsilon \dot{H}_\sigma^1$ belongs to $\mathfrak{D}(L^*)$ if there is an $f \varepsilon H_\sigma$ such that $L[\varphi, u] = (\varphi, f)$ for all $\varphi \varepsilon \dot{H}_\sigma^1$. In this case we write $f = L^*u$. L^* is the adjoint of L [1]; and L^* has eigenvalues λ_j^*, $j = 1, 2, \cdots$ and eigenfunctions $\psi_1^*, \psi_2^*, \cdots$. The usual Fredholm theory applies to L and L^*, and a subspace $\mathfrak{M}$ is invariant under L if and only if $\mathfrak{M}^\perp$ is invariant under L^*. These facts concerning L and L^* are all standard and will not be proved here.

The following lemma is central to the method used to prove the stability and instability theorems in the next two sections.

Lemma 3.7. *Let $\mathfrak{M}_n$ be the invariant subspace of L corresponding to $\{\lambda_1, \cdots, \lambda_n\}$ and let $\mathfrak{N}_n$ be the invariant subspace corresponding to $\{\lambda_{n+1}, \cdots\}$. Then for sufficiently large n,* $\operatorname{Re} L[u, u] \geqq ||u||^2/2$ *for all $u \varepsilon \dot{H}_\sigma^1 \cap \mathfrak{N}_n$.*

Proof. From (3.2)

$$\begin{aligned}\operatorname{Re} L[u, u] &\geqq ||u||^2 - c_2 |u|^2 \\ &= ||u||^2 \left(1 - c_2 \frac{|u|^2}{||u||^2}\right),\end{aligned}$$

so it suffices to show that $||u||^2 \geqq 2c_2|u|^2$ for $u \varepsilon \mathfrak{N}_n \cap \dot{H}_\sigma^1$ and n sufficiently large.

For any subspace $\mathfrak{M}$ of H_σ let $d(\mathfrak{M})$ be given by the variational condition

$$d(\mathfrak{M}) = \inf \frac{||u||^2}{|u|^2},$$

subject to $u \in \mathfrak{M}^{\perp}$. Furthermore, given functions $\{v_1, \cdots, v_k\}$ we denote by $d(v_1, \cdots, v_k)$ the number $d([v_1, \cdots, v_k])$, where $[v_1, \cdots, v_k]$ is the span of the given functions. Let μ_k be the k^{th} eigenvalue of A and assume $\mu_k < \mu_{k+1}$. We shall show that $d(\mathfrak{N}_n^{\perp}) \geqq \mu_k$ for sufficiently large n. Since $\mu_k \rightarrow \infty$ this will establish the lemma.

Now let $\mathfrak{M}_n^* = \mathfrak{N}_n^{\perp}$. Then $\mathfrak{M}_n^*$ is invariant under L^* and

$$\dim \mathfrak{M}_n^* = \dim \mathfrak{N}_n \geqq n.$$

Let $\{v_1, \cdots, v_{k+1}\}$ be any $(k+1)$ linearly independent functions in $\mathfrak{M}_n^*$, $(k+1) \leqq n$. Then

$$d(v_1, \cdots, v_{k+1}) \leqq d(\mathfrak{M}_n^*).$$

In fact the quantity on the right is obtained by taking the infimum over a *smaller* class of functions (more constraints).

Now the eigenfunctions of L^* are complete. Therefore, given any $\epsilon > 0$, for n sufficiently large, we can find vectors $\{v_1, \cdots, v_{k+1}\}$ in $\mathfrak{M}_n^*$ such that $|v_i - \varphi^{(i)}| \leqq \epsilon^{1/2}/(k+1)$, where $\varphi^{(1)}, \cdots, \varphi^{(k+1)}$ are the first $(k+1)$ eigenfunctions of A. Let $u \in \mathfrak{N}_n \cap \dot{H}_\sigma^1$ and $|u| = 1$. Let P_k be the projection on $[\varphi^{(1)}, \cdots, \varphi^{(k+1)}]$ and let $Q_k = I - P_k$. Then

$$P_k u = \sum_{i=1}^{k+1} (u, \varphi^{(i)})\varphi^{(i)}.$$

Since $|(u, \varphi^{(i)})| = |(u, \varphi^{(i)} - v_i)| \leqq \epsilon^{1/2}/(k+1)$ we have

$$|P_k u| \leqq \epsilon^{1/2}, \qquad |Q_k u|^2 \geqq 1 - \epsilon.$$

Furthermore,

$$||u||^2 = ||P_k u||^2 + ||Q_k u||^2,$$

since P_k and Q_k commute with A. Therefore,

$$||u||^2 \geqq ||Q_k u||^2 \geqq \mu_{k+1} |Qu|^2 \geqq \mu_{k+1}(1 - \epsilon).$$

Since this holds for any u in $\mathfrak{N}_n \cap \dot{H}_\sigma^1$ with $|u| = 1$ we have $d(\mathfrak{M}_n^*) = d(\mathfrak{N}_n^{\perp}) \geqq d(v_1, \cdots, v_{k+1}) \geqq \mu_{k+1}(1 - \epsilon)$. By choosing n large enough ϵ can be made so small that $\mu_{k+1}(1 - \epsilon) \geqq \mu_k$. Hence the proof of Lemma 3.7 is complete.

4. The Stability Theorem.

Everything said so far holds assuming only that the stationary flow has bounded first derivatives. In this and the next section we assume also that the eigenfunctions of L and L^* have bounded first derivatives in D. This will certainly be the case if the boundary of D and the stationary flow are sufficiently regular.

In this section we prove the following.

Theorem 4.1. *Let u be any weak solution of* (1.2) *satisfying the energy inequality* (2.6). *Let all solutions of the linearized equations* (1.2′) *decay, so that the* *eigenvalues of L lie in the right half plane. Then given any $\epsilon > 0$ there is a $\delta > 0$ such that $|u(0)| < \delta$ implies $|u(t)| < \epsilon$ for all $t > 0$ and $|u(t)|$ tends to zero exponentially as t tends to infinity.*

Proof. Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be invariant subspaces of L chosen as in Lemma 3.7. We take $\mathfrak{M}_1$ to be finite dimensional, and on $\mathfrak{M}_2 \cap \dot{H}^1_\sigma$ we have $L[u, u] \geqq \|u\|^2/2$. Let P_1 and P_2 be the projections onto $\mathfrak{M}_1$ and $\mathfrak{M}_2$ which commute with L. The adjoint P^*_1 is the projection onto $\mathfrak{M}^*_1 = \mathfrak{M}^{\perp}_2$. We write $u = P_1 u + P_2 u = u_1 + u_2$ for any $u \,\varepsilon\, H_\sigma$.

Let $\{\zeta_1, \cdots, \zeta\}$ be any orthonormal basis for $\mathfrak{M}_1$. In the weak form (2.2) of the Navier–Stokes equations put $\zeta = (P^*_1\zeta_i)c(t)$ for any smooth $c(t)$. Then

$$(u(0), P^*_1\zeta_i)\overline{c(0)} + \int_0^\infty \{(u, P^*_1\zeta_i) - L[u, \dot{\bar{c}}\, P^*_1\zeta_i]\bar{c} + (u, u\cdot\operatorname{grad} P^*_1\zeta_i)\bar{c}\}\, dt = 0;$$

hence

$$(4.1)\qquad (P_1u(0), \zeta_i)\overline{c(0)} + \int_0^\infty \{(u_1, \zeta_i)\dot{\bar{c}} - L[u_1, \zeta_i]\bar{c} + (u, u\cdot\operatorname{grad} P^*_1\zeta_i)\bar{c}\}\, dt = 0.$$

We may write

$$u_1(t) = P_1 u = \sum_{i=1}^{k} q_i(t)\zeta_i,$$

where $q_i(t) = (u_1, \zeta_i) = (P_1u, \zeta_i) = (u, P^*_1\zeta_i)$. Since $P^*_1\zeta_i$ lies in $\mathfrak{M}^*_1$, which is finite dimensional, $P^*_1\zeta_i$ belongs to $\dot{H}^1_\sigma$. By the remark in §2, $q_i(t)$ is absolutely continuous, and the term with the time derivative in (4.1) may be integrated by parts to obtain

$$0 = \int_0^\infty \left\{q_j + \sum_{i=1}^{k} L_{ij}q_i - (u, u\cdot\operatorname{grad} P^*_1\zeta_j)\right\}\bar{c}\, dt, \qquad j = 1, \cdots, k,$$

where $L_{ij} = [\zeta_i, \zeta_j]$. Since $c(t)$ is arbitrary we get the system of ordinary differential equations

$$(4.2)\qquad \dot{q}_j + \sum_{i=1}^{k} L_{ij}q_i = (u, u\cdot\operatorname{grad} P^*_1\zeta_j), \qquad j = 1, \cdots, k.$$

The subspace $\mathfrak{M}^*_1$ is spanned by the adjoint eigenfunctions $\{\psi^*_1, \cdots, \psi^*_k\}$. Since these have bounded first derivatives, so does $P^*_1\zeta_i$; and

$$|(u, u\cdot\operatorname{grad} P^*_1\zeta_j)| \leqq c_1 |u|^2$$

for some constant c_1. Let the eigenvalues of L all lie to the right of $\operatorname{Re}\lambda = \sigma$ in the complex plane. The same then holds when L is restricted to $\mathfrak{M}_1$; so for any γ, $0 < \gamma < \sigma$, there is a constant $c_2 = c_2(\gamma)$ [5], p. 315, such that the solutions of (4.2) satisfy

$$R(t) \leqq c_2 e^{-\gamma t} R(0) + c_1 c_2 \int_0^t e^{-\gamma(t-s)} \, |u(s)|^2 \, ds,$$

where

$$R(t) = \left(\sum_{i=1}^{k} |q_i|^2 \right)^{1/2} = |u_1|.$$

Squaring both sides of the above inequality we get

$$|u_1(t)|^2 \leqq 2c_2^2 e^{-2\gamma t} \, |u_1(0)|^2 + c_3 \left[\int_0^t e^{-2(\gamma-\gamma')(t-s)} \, ds \right] \left[\int_0^t e^{-2\gamma'(t-s)} \, |u(s)|^4 \, ds \right]$$

for any γ', $0 < \gamma' < \gamma < \sigma$. Thus for any γ, $0 < \gamma < \sigma$, there exist constants c_4 and c_5 such that

$$|u_1(t)|^2 \leqq c_4 e^{-2\gamma t} \, |u_1(0)|^2 + c_5 \int_0^t e^{-2\gamma(t-s)} \, |u|^4 \, ds. \tag{4.3}$$

On the other hand, we get from (2.6)

$$|u(t_2)|^2 + 2 \int_{t_1}^{t_2} \{ L[u_1 , u_1] + L[u_1 , u_2] + L[u_2 , u_1] + L[u_2 , u_2] \} \, ds \leqq |u(t_1)|^2$$

for all $0 \leq t_1 < t_2$. Since u belongs to the class V_T for all $T > 0$, u lies in $\dot{H}_\sigma^1$ for almost all $t \geq 0$; so from Lemma 3.7, $L[u_2 , u_2] \geqq ||u_2||^2/2$ a.e. Furthermore, since u_1 lies in a finite dimensional subspace there is a constant c_6 such that

$$\begin{aligned} |L[u_1 , u_1]| &= |(Lu_1 , u_1)| \leqq c_6 \, |u_1|^2, \\ |L[u_1 , u_2]| &= |(Lu_1 , u_2)| \leqq c_6 \, |u_1| \cdot |u_2|, \\ |L[u_2 , u_1]| &= |(u_2 , L^* u_1)| \\ &\leqq |(u_2 , Lu_1)| + |(u_2 , (L^* - L)u_1)| \\ &\leqq c_6 \, |u_2| \, |u_1|. \end{aligned}$$

The last inequality follows from the fact that $(L^* - L)$ is a first order "differential" operator, hence $|(L^* - L)u_1| \leqq c \, ||u_1|| \leqq c' \, |u_1|$ for some constants c, c'.

Remark. The norms $|| \; ||$ and $| \; |$ are equivalent on finite dimensional subspaces spanned by functions in $\dot{H}_\sigma^1$; that is, there are constants d and d' such that $d \, |u| \leqq ||u|| \leqq d' \, |u|$ for all u in some fixed subspace $\mathfrak{M}$. The first inequality holds on the entire space $\dot{H}_\sigma^1$ with $d = \mu_1^{1/2}$, where μ_1 is the first eigenvalue of A. (See §2.)

The second inequality is obtained as follows. Let $\{\psi_1 , \cdots , \psi_n\}$ be a basis for the subspace $\mathfrak{M}$ with each ψ_k in $\dot{H}_\sigma^1$, and write

$$u = \sum_{k=1}^{n} a_k \psi_k .$$

Then by Schwarz's inequality

$$||u|| \leqq \langle a, a\rangle^{1/2} \left(\sum_{k=1}^{n} ||\psi_k||^2\right)^{1/2},$$

where $\langle a, a\rangle^{1/2}$ denotes the Euclidean norm of the n-tuple $a = (a_1, \cdots, a_n)$. Now let $\{\psi_1', \cdots, \psi_n'\}$ be an orthonormal basis for $\mathfrak{M}$. The functions $\{\psi_k'\}$ can be obtained by a Gram-Schmidt process from the $\{\psi_k\}$. The expansion of u in terms of the functions $\{\psi_k'\}$ has coefficients $a_k' = (u, \psi_k')$, where (,) denotes the usual inner product on $L_2(D)$. Furthermore we have

$$\begin{aligned} |u|^2 &= \left(\sum_{k=1}^{n} a_k'\psi_k', \sum_{l=1}^{n} a_l'\psi_l'\right) \\ &= \sum_{k=1}^{n}\sum_{l=1}^{n} a_k'\overline{a_l'}(\psi_k', \psi_l') \\ &= \langle a', a'\rangle. \end{aligned}$$

Now the n-tuples $a = (a_1, \cdots, a_n)$ and $a' = (a_1', \cdots, a_n')$ are related by a non-singular transformation P: $a = Pa'$. Therefore $\langle a, a\rangle \leqq$ const. $\langle a', a'\rangle$. The desired inequality now follows.

Now returning to the proof we obtain

$$\begin{aligned} |u(t_2)|^2 + \int_{t_1}^{t_2} ||u_2||^2\, ds &\leqq c_6 \int_{t_1}^{t_2} |u_1|\,(|u_1| + |u_2|)\, ds + |u(t_1)|^2 \\ &\leqq \int_{t_1}^{t_2} \{c_7\, |u_1|^2 + \kappa\, |u_2|^2\}\, ds + |u(t_1)|^2 \end{aligned}$$

for any $\kappa > 0$ and some constant $c_7 = c_7(\kappa)$. From the proof of Lemma 3.7 we get that $||u_2||^2 \geqq \mu_l |u_2|^2$ for some eigenvalue μ_l of A. By choosing n sufficiently large, μ_l can be taken as large as we please. Therefore we can write

$$|u(t_2)|^2 + \frac{1}{2}\int_{t_1}^{t_2} ||u_2||^2\, ds \leqq \int_{t_1}^{t_2} \{c_8\, |u_1|^2 - c_9\, |u_2|^2\}\, ds + |u(t_1)|^2,$$

for some constants c_8 and c_9, where c_9 can be made large (at the cost, of course, of making c_8 large as well). Since $|u| \leqq |u_1| + |u_2| \leqq 2(|u_1|^2 + |u_2|^2)$ we have that $-|u_2|^2 \leqq |u_1|^2 - |u|^2/2$; so

$$|u(t_2)|^2 + \frac{1}{2}\int_{t_1}^{t_2} \{||u||^2 + c_9\, |u|^2\}\, ds \leqq |u(t_1)|^2 + c_{10}\int_{t_1}^{t_2} |u_1|^2\, ds. \tag{4.4}$$

Therefore

$$|u(t_2)|^2 + c_9\int_{t_1}^{t_2} |u|^2\, ds \leqq |u(t_1)|^2 + c_{10}\int_{t_1}^{t_2} |u_1|^2\, ds. \tag{4.5}$$

From (4.5) and Lemma 2.1 we get

$$|u(t)|^2 \leqq e^{-c_9 t}\, |u(0)|^2 + c_{10}\int_0^t e^{-c_9(t-s)}\, |u_1|^2\, ds. \tag{4.6}$$

Substituting (4.3) into (4.6) we get

$$|u(t)|^2 \leqq e^{-c_9 t}\,|u(0)|^2 + \int_0^t c_{10} e^{-c_9(t-s)} \left\{ c_3 e^{-2\gamma s}\,|u_1(0)|^2 + c_5 \int_0^s e^{-2\gamma(s-\tau)}\,|u|^4\,d\tau \right\} ds.$$

Since $u_1 = P_1 u$, $|u_1| \leqq$ const. $|u|$, so

$$|u(t)|^2 \leqq |u(0)|^2 \left\{ e^{-c_9 t} + c_{11} \int_0^t e^{-c_9(t-s)-2\gamma s}\,ds \right\} + c_{12} \int_0^t \int_0^s e^{-c_9(t-s)-2\gamma(s-\tau)}\,|u(\tau)|^4\,d\tau\,ds.$$

Now recall that c_9 can be made large by taking n large; in particular we can arrange that $c_9 - 2\gamma > 0$. In that case the first term is dominated by $c_{13}|u(0)|^2 e^{-2\gamma t}$. Inverting the order of integration in the iterated integral we get

$$|u(t)|^2 \leqq c_{13}\,|u(0)|^2\,e^{-2\gamma t} + c_{14} \int_0^t e^{-2\gamma(t-s)}\,|u(s)|^4\,ds,$$

hence

$$e^{2\gamma t}\,|u(t)|^2 \leqq c_{13}\,|u(0)|^2 + c_{14} \int_0^t e^{-2\gamma s}(e^{2\gamma s}\,|u(s)|^2)^2\,ds.$$

It now follows (Lemma 2.2) that

$$|u(t)|^2 \leqq \frac{2\gamma c_{13}\,|u(0)|^2\,e^{-2\gamma t}}{2\gamma - c_{13}c_{14}\,|u(0)|^2\,(1 - e^{-2\gamma t})}. \tag{4.7}$$

Hence the solution decays exponentially if $|u(0)|^2 < 2\gamma/c_{13}c_{14}$. This completes the proof of Theorem 4.1.

Remark. In (4.4) take $t_1 = 0$, $t_2 = t$. From (4.7) it follows that $\int_0^\infty \|u(s)\|^2 ds$ is finite. Arguing as in [14] we can prove that eventually the flow smooths out, the solution becomes a strong solution, and tends to zero uniformly.

5. The Instability Theorem.

Theorem 5.1. Let L have at least one eigenvalue with negative real part so that some solutions of (1.2′) *grow in time. Then $\bar{u}$ is unstable.*

Proof. In order to prove that $\bar{u}$ is unstable we must show that there is an $\epsilon_0 > 0$ such that for any $\delta > 0$ there are weak solutions $v(t)$ of (1.2) for which $|v(0)| < \delta$ but such that $|v(t)|$ does not remain less than ϵ_0 for all $t > 0$. We decompose H_σ into two linearly independent subspaces $\mathfrak{M}_1$ and $\mathfrak{M}_2$ as before, and derive the system of ordinary differential equations

$$\dot{q}_j + \sum_{i=1}^{k} L_{ij} q_i = (u, u\cdot\operatorname{grad} P_1^* \zeta_j), \qquad j = 1, \cdots, k. \tag{5.1}$$

We now regard $\mathfrak{M}_1$ as a finite dimensional vector space. Since L leaves $\mathfrak{M}_1$ invariant it can be regarded as a linear transformation on $\mathfrak{M}_1$ with the matrix $L_{ij} = L[\zeta_i, \zeta_j]$ relative to the basis $\{\zeta_1, \cdots, \zeta_k\}$. Let $\{\psi_1, \cdots, \psi_k\}$ be a Jordan basis for L on $\mathfrak{M}_1$. The system (5.1) can be transformed to the system

$$\dot{p}_j + \sum_{i=1}^{k} L'_{ij} p_i = f_j, \qquad j = 1, \cdots, k, \tag{5.2}$$

where the matrix L'_{ij} is in Jordan form and the p_j and f_j are related to q_j and $(u, u \cdot \operatorname{grad} P_1^* \zeta_j)$ by a nonsingular linear transformation on $\mathfrak{M}_1$. Thus the f_j's satisfy an estimate of the form $|f_j(t)| \leqq c_1 |u(t)|^2$ for some $c_1 > 0$, and there are positive constants c and c' such that

$$c\,|u_1(t)|^2 \leqq \sum_{j=1}^{k} |p_j|^2 \leqq c'\,|u_1(t)|^2.$$

For a suitable normalization of $\{\psi_1, \cdots, \psi_k\}$ the matrix L'_{ij} has zeros or γ's on the superdiagonal, where γ is an arbitrary positive number [5], p. 317. We may further assume that the equations (5.2) are ordered so that the entries $j = 1, 2, \cdots, l$, where $l \leqq k$, correspond to the eigenvalues of L'_{ij} with negative real parts. If these eigenvalues lie to the left of $\operatorname{Re} \lambda = -\sigma (\sigma > 0)$, then the following estimate holds:

$$\operatorname{Re} \sum_{i,j=1}^{l} L'_{ij} p_i \bar{p}_j \leqq (-\sigma + \gamma) \sum_{i=1}^{l} |p_i|^2.$$

Similarly, on the subspace of $\mathfrak{M}_1$ corresponding to the eigenvalues with nonnegative real parts we have

$$\operatorname{Re} \sum_{i,j=l+1}^{k} L'_{ij} p_i \bar{p}_j \geqq -\gamma \sum_{i=l+1}^{k} |p_i|^2.$$

Let

$$R(t) = \sum_{i=1}^{l} |p_i|^2, \qquad \rho(t) = \sum_{i=l+1}^{k} |p_i|^2.$$

Then $R(t)$ and $\rho(t)$ are absolutely continuous, and from (5.2) we get

$$\begin{aligned} \dot{R} &= 2 \operatorname{Re} \sum_{j=1}^{l} \dot{p}_j \bar{p}_j \\ &= 2 \operatorname{Re} \sum_{j=1}^{l} \bar{p}_j \left\{ -\sum_{i=1}^{l} L'_{ij} p_i + f_j \right\} \\ &\geqq 2(\sigma - \gamma) R - c_2 |u|^2 R^{1/2}. \end{aligned}$$

Now if $|u(t)| < \epsilon$ for all $t \geqq 0$ for some given ϵ, then

$$\begin{aligned} \dot{R} &\geqq 2(\sigma - \gamma) R - \frac{c_2 \epsilon}{2} R - \frac{c_2 \epsilon}{2} |u|^2 \\ &\geqq k_1 R - c_3 \epsilon |u|^2, \end{aligned}$$

where $k_1 = 2(\sigma - \gamma) - c_1\epsilon/2$ can be made positive for sufficiently small γ and ϵ. Integrating this inequality we obtain

$$R(t) \geqq R(0) + k_1 \int_0^t R(s)\,ds - c_3\epsilon \int_0^t |u|^2\,ds. \tag{5.3}$$

Similarly we get

$$\begin{aligned} \dot{\rho} &\leqq 2\gamma\rho + c_4\,|u|^2\,\rho^{1/2} \\ &\leqq \left(2\gamma + \frac{c_4\epsilon}{2}\right)\rho + \frac{c_4\epsilon}{2}\,|u|^2. \end{aligned}$$

Letting $k_2 = 2\gamma + c_4\epsilon/2$ and integrating this inequality we find

$$\rho(t) \leqq \rho(0) + k_2 \int_0^t \rho(s)\,ds \frac{c_4\epsilon}{2} \int_0^t |u|^2\,ds. \tag{5.4}$$

Subtracting (5.4) from (5.3) we get

$$\begin{aligned} R(t) - \rho(t) \geqq [R(0) - \rho(0)] + k_1 \int_0^t R(s)\,ds \\ - k_2 \int_0^t \rho(s)\,ds - \epsilon c_5 \int_0^t |u(s)|^2\,ds. \end{aligned} \tag{5.5}$$

Now in the same way (4.8) was derived it can be shown that

$$|u(t)|^2 \leqq e^{-c_6 t}\,|u(0)|^2 + c_7 \int_0^t e^{-c_6(t-s)}\,|u_1(s)|^2\,ds. \tag{5.6}$$

But

$$u_1(t) = \sum_{i=1}^{k} p_i(t)\psi_i\,,$$

so

$$|u_1(t)|^2 = \sum_{i,j=1}^{k} p_i\bar{p}_j(\psi_i\,,\,\psi_j) \leqq c_8(\rho + R),$$

for some $c_8 > 0$. Putting this back in (5.6) and integrating (5.6) with respect to t, we find that

$$\int_0^t |u(s)|^2\,ds \leqq c_9\,|u(0)|^2 + c_{10} \int_0^t (\rho + R)\,ds \tag{5.7}$$

for some constants c_9, $c_{10} > 0$ and all $t \geqq 0$.

Substituting (5.7) in (5.5) we obtain

$$\begin{aligned} (R - \rho) \geqq [R(0) - \rho(0) - \epsilon c_{11}\,|u(0)|^2] + k_1 \int_0^t R\,ds \\ - k_2 \int_0^t \rho\,ds - \epsilon c_{12} \int_0^t (\rho + R)\,ds. \end{aligned}$$

Now $|u(0)|^2 \leqq c_{13}(\rho + R) + 2|u_2(0)|^2$. Suppose ϵ is so small that $\epsilon c_{11}c_{13}$ and $2\epsilon c_{11}$ are less than $\frac{1}{2}$; and suppose γ and ϵ are so small that

$$k_1R - k_2\rho - \epsilon c_{12}(\rho + R) \geqq k_3(R - \rho)$$

for some $k_3 > 0$ and all positive R and ρ. Then

$$(R - \rho) \geqq \left[\frac{R(0)}{2} - \frac{3\rho(0)}{2} - \frac{|u_2(0)|^2}{2}\right] + k_3 \int_0^t (R - \rho)\, ds. \tag{5.8}$$

Now suppose that ϵ has been chosen as above so that (5.8) is satisfied for any weak solution. Let u be a weak solution of (1.2) with initial data such that

$$\left[\frac{R(0)}{2} - \frac{3\rho(0)}{2} - \frac{|u_2(0)|^2}{2}\right] \geqq \frac{R(0)}{4} > 0.$$

Then (5.8) implies that

$$(R(t) - \rho(t)) \geqq \frac{R(0)}{4} e^{k_3 t},$$

hence $R(t) \geqq R(0)e^{k_3 t}/4$. Since $R(t) \leqq c|u_1(t)|^2 \leqq c'|u(t)|^2$ for some constants c and c', u cannot then remain less than ϵ in norm for all time. This contradiction completes the proof of the instability theorem.

6. Concluding remarks. It must be pointed out that, in spite of the theorems in §§4, 5, it may not be sufficient in an actual physical case, to examine the linearized stability problem alone. In any theory of stability, attention must also be given to the *extent* of stability. We have shown that if the solutions of the linearized equations decay, then the stationary solution is stable relative to sufficiently small disturbances. Disturbances are inevitable in any physical situation, and the question then arises as to whether the magnitude of these disturbances is greater than the extent of stability of the given stationary flow. If so then instability (or turbulence) might be observed in a situation where the linearized theory predicts stability.

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University of California, Los Angeles

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