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Group Representation Theory, Bifurcation Theory and Pattern Formation*

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1. SYMMETRY-BREAKING INSTABILITIES

In this paper the phenomenon of pattern formation is regarded as a "symmetry breaking instability". For a given range of a parameter λ we suppose that a physical system possesses a stable solution invariant under a symmetry group $\mathscr{G}$, but that as λ crosses a critical parameter λ_c new solutions appear which are invariant only under a subgroup H. The appearance of convection cells in the Bénard problem constitutes a classical example of such a bifurcation phenomenon. Prior to the onset of instability the fluid is motionless and the solution is invariant under the entire group of rigid motions in the plane; but after the onset of instability hexagonal or roll-like cellular motions appear which suggest the existence of stable doubly periodic motions in the plane. An excellent account of the Bénard problem is given in the review articles by L. A. Segel [32], and Palm, Ellingsen, and Gjevck [20]. See also [2], [5], and [9]. Plates showing the hexagonal convection cells discovered by Bénard may be found in the article by Koschmieder [12].

Other familiar examples of symmetry breaking instabilities are the bifurcation of time periodic solutions from an equilibrium solution (the Hopf bifurcation theorem); the buckling of spheres; the onset of convection in spherical

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fluid masses; and pattern formation in chemical reaction — diffusion processes. A survey of symmetry breaking transitions in particle physics was given by L. Michel in his 1975 address.

The phenomenon of crystallization, or phase transitions, in statistical mechanics is another important example of a symmetry breaking transition. In 1941 Kirkwood and Monroe [11] analyzed a nonlinear integral equation for the single particle density function ρ. Homogeneous solutions ($\rho \equiv$ constant) correspond to a fluid phase; but at certain critical values of the temperature the equation possesses bifurcation points where multiply periodic solutions appear. These solutions are interpreted as a crystalline phase of the material. These ideas have been pursued more recently by Raveché and Stuart [23]. See also [21].

Another example from statistical mechanics is Landau's theory of second order phase transitions. (See [1], [12], [15]).

One of the important features which these physical problems have in common is the covariance of the equations with respect to a transformation group, for example the group of rigid motions in $\mathbb{R}^2$ or $\mathbb{R}^3$. An immediate consequence of this invariance is the multiplicity of the bifurcation points. The difficulties then encountered are largely algebraic whenever the problem may be reduced via the classical Lyapounov-Schmidt method to solving a system of n algebraic equations in n unknowns. In this paper we address ourselves to the questions of pattern formation in physical systems and on the role which group representation theory plays in the analysis of such phenomena.

F. Busse [2] discussed extensively the bifurcation of cellular solutions of the Boussinesq equations. The use of group representation theory permits one to simplify and clarify Busse's analysis, as well as to extend it to a general class of problems. We derive the explicit form of the bifurcation equations and carry out an analysis of the bifurcation and stability of cellular solutions for the case of hexagonal, square, and rectangular cellular patterns. Our approach is not restricted to the Bénard problem but depends only on the covariance of the equations. In this way Busse's results may be seen in a much more general context. For example, motivated by Malkus' hypothesis of maximum heat transport, Busse showed that the low order terms in the bifurcation equations possess a gradient structure. This result, however, is not peculiar to the Boussinesq equations, but is an artifact of the group invariance of the problem, as we shall prove in § 10. Furthermore, Table 6.1, which shows the possible bifurcating patterns together with their stability properties, follows entirely from symmetry considerations and is independent of the particular structure of the equations. The parameters a and b, however, do depend on the physical parameters of the particular problem.

The introduction of group theory into bifurcation theory suggests a natural way to classify bifurcation problems. Rather than treat each physical situation separately we consider bifurcation problems according to classes, each class identified by the symmetry group natural to the problem. In this paper we consider a general equation in a Banach space of the form

$$G(\lambda, u) = 0 \tag{1.1}$$

which is covariant (in a sense to be prescribed later on) with respect to the group of rigid motions in the plane. We suppose that the trivial solution $u = 0$ loses stability as λ crosses some critical value λ_c, and then investigate the bifurcation of doubly periodic solutions in the plane. (Henceforth we shall take λ_c to be zero). Under fairly standard assumptions the branching problem is reduced, via the Lyapounov-Schmidt method, to a finite dimensional problem of the form

$$F(\lambda, v) = 0 \tag{1.2}$$

where $v \in \mathcal{N}$, the kernel of the linearized operator $G_u(0, 0)$. In (1.2) F is a mapping from $\mathbb{C} \times \mathcal{N}$ into $\mathcal{N}$. Upon introducing a basis for $\mathcal{N}$ we can write (1.2) as a system of n equations in n unknowns, where $n = \dim \mathcal{N}$. Equations (1.2), known as the bifurcation equations, are covariant under a subgroup of the group of rigid motions.

The class of doubly periodic functions can be characterized as the subclass of functions which are invariant under a lattice subgroup of the translation group. This characterization, which provides us with a precise formulation of the bifurcation of cellular patterns in a physical problem, was given by Kirchgassner and Kielhofer in their survey article [9]. The reader may well ask whether there is any justification for the restriction to cellular motions. While we know of no rigorous justification, the repeated appearance of regular cellular phenomena in nature strongly suggests their relevance.

By a scaling argument due to L. Graves (see Kirchgassner [7], Sather [25], [26], and Sattinger [28]) equations (1.2) may sometimes be formally reduced to a set of equations of the form

$$Av + B_k(v) = 0 \tag{1.3}$$

where B_k is a homogeneous operator of degree k and A is a linear operator. (It turns out that A is a scalar multiple of the identity if the kernel $\mathcal{N}$ of the linearized operator is irreducible under the group representation). If equations (1.3), which we shall call the ***reduced bifurcation equations***, have a solution v_0, and if the Jacobian of these equations is non singular at v_0, then a simple application of the implicit function theorem allows one to extend v_0 to a solution of the full bifurcation equations (1.2), hence of the original equation (1.1).

In the present case the situation is complicated by the fact that (1.2) and hence also (1.3) are invariant under a two-parameter Lie group. This invariance is a consequence of the original translational invariance of the equations (1.1).

Solutions of (1.2) and (1.3) therefore appear in two parameter sheets, and the Jacobian of (1.3) always has a null space of dimension (at least) two. (In the case of hexagonal patterns it is sometimes three dimensional). Nevertheless, the analysis may be modified in the present case to extend solutions of the reduced bifurcation equations to the full equations by an implicit function argument. This is done in § 6.

The reduced bifurcation equations (1.3) are of a simple enough structure when $k = 2$ or 3 that they may be completely analyzed by quadratures. The results of this analysis for the square, rectangular, and hexagonal solutions are summarized in Table 6.1.

The Jacobian of the reduced equations is evaluated and it is proved that the eigenvalues of the Jacobian of the reduced equations determine, to lowest non-vanishing order, the stability of the bifurcating solutions. This is done in § 7. Theorem 7.2 is a general result which is independently of interest in itself and has as an immediate corollary a number of previous results on the stability of bifurcating solutions due to Sattinger [27], Crandall and Rabinowitz [3], and McLeod and Sattinger [13]. (See also L. Nirenberg [17], pp. 102–110).

In § 4 we analyze the kernel $\mathcal{N}$ of the linearized operator $G_u(0, 0)$ in the case of the hexagonal lattice. By the use of projection methods of group representation theory we are able to decompose $\mathcal{N}$ into irreducible invariant subspaces of D_6 (the symmetry group of the hexagon), thereby obtaining, in a systematic fashion, a complete set of the so-called "planform functions" of hydrodynamics. (See [31]) These functions have previously been sought by *ad hoc* methods which did not yield a complete set. A knowledge of their structure will prove useful in our discussion of the stability of hexagonal patterns in § 9.

The multiplicity of the branch point is an intrinsic aspect of the problem of pattern formation, a fact which is underscored by our analysis of the stability of the various cellular patterns. Some authors (e.g. Judovic [6] and Rabinowitz [22], but not Busse [2]) have constructed solutions of the Bénard problem by restricting the problem to subspaces of doubly periodic solutions in which the branch point is simple. This technique artificially excludes some solutions; but, more importantly for the present case, the simplicity of the eigenvalue leads to incorrect conclusions about the stability of the bifurcating solutions. At a simple eigenvalue supercritical solutions are stable and subcritical solutions are unstable, but this conclusion breaks down at multiple eigenvalues, as we shall see in this paper. For a further discussion, see the final section of this paper.

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2. The Onset of Instatility

We consider the formal system of equations (1.1) where G is defined on continuous functions u mapping $\mathbb{R}^k(k = 2, 3)$ into a complex Banach space B_1. We denote this class by $\mathscr{F}_1(\mathbb{R}^k, B_1)$ and assume that the image under G lies in a class $\mathscr{F}_2(\mathbb{R}^k, B_2)$, where B_2 is another complex Banach space, with B_1 a dense subspace in B_2. Let us further suppose that the classes $\mathscr{F}_j(\mathbb{R}^k, B_j)$ are themselves Banach spaces of continuous functions under some appropriate norm. The operation G is real in the sense that it commutes with complex conjugation. The Banach spaces B_1 and B_2 may be finite or infinite dimensional, and the bilinear pairing with their dual spaces is denoted by the symbol $\langle\ ,\ \rangle$. The principle assumptions on (1.1) are that G is invariant under the group $\mathscr{E}(k)$ of rigid motions; that G is Frechet differentiable as a mapping from $\mathscr{F}_1$ to $\mathscr{F}_2$; and that $G_u(0, 0)$ is a Fredholm operator of index zero when restricted to subclasses of k-periodic functions in $\mathscr{F}_1$ and $\mathscr{F}_2$ (this assumption is clarified below).

Let us denote points of $\mathbb{R}^k$ by $\mathbf{x}$ and elements $\sigma \in \mathscr{E}(k)$ by $\sigma = \{r, \mathbf{a}\}$ where r is a rotation-reflection and $\mathbf{a}$ is a vector in $\mathbb{R}^k$. Then $\sigma\mathbf{x} = r\mathbf{x} + \mathbf{a}$. Let S_σ be a representation of $\mathscr{E}(k)$ onto B_2 (hence on B_1 also) with the property that S_σ is the identity whenever σ is a pure translation. We define the representation T_σ of $\mathscr{E}(k)$ onto $\mathscr{F}_j$ by

$$(T_\sigma u)(\mathbf{x}) = (S_\sigma)\, u(\sigma^{-1}\mathbf{x}). \tag{2.1}$$

Then $T_{\sigma_1} T_{\sigma_2} = T\sigma_1\sigma_2$ and $T_e = I$, where e is the identity in $\mathscr{E}(k)$.

Lemma 2.1. *Let L be a bounded linear operator from $\mathscr{F}_1$ to $\mathscr{F}_2$ which is real and covariant under the representation $T\sigma$. Then there is an operator $\hat{L}(\boldsymbol{\omega})(\boldsymbol{\omega} \in \mathbb{R}^k)$ such that $\hat{L}(\boldsymbol{\omega}) : B_1 \to B_2$ and for $v \in B_1$*

(i) $L(ve^{i\langle\boldsymbol{\omega},\mathbf{x}\rangle}) = (\hat{L}(\boldsymbol{\omega})v)\, e^{i\langle\boldsymbol{\omega},\mathbf{x}\rangle}$

(ii) $S_r\hat{L}(\boldsymbol{\omega}) = \hat{L}(r\boldsymbol{\omega})\, S_r$ *(r a rotation)*

(iii) $\overline{\hat{L}(\boldsymbol{\omega})\, v} = \hat{L}(-\boldsymbol{\omega})\, \bar{v}$.

(The expression $\langle\boldsymbol{\omega}, \mathbf{x}\rangle$ is the Euclidean inner product on $\mathbb{R}^k$.)

Proof. By the translational invariance of L we have

$$\begin{aligned} T_a L v e^{i\langle\boldsymbol{\omega},\mathbf{x}\rangle} &= L T_a(ve^{i\langle\boldsymbol{\omega},\mathbf{x}\rangle}) \\ &= Lve^{i\langle\boldsymbol{\omega},\mathbf{x}+a\rangle} \\ &= e^{i\langle\boldsymbol{\omega},a\rangle} L v e^{i\langle\boldsymbol{\omega},\mathbf{x}\rangle} \end{aligned} \tag{2.2}$$

Recall that if $f(t)$ is a continuous function of t for which $T_a f = e^{i\omega a} f$ then $f(t) = e^{i\omega t} f(0)$. (This follows immediately from the relation $f(t + a) = e^{i\omega a} f(t)$

by putting $t = 0$). By the same reasoning (i) follows from (2.2) by setting

$$\hat{L}(\omega)\, v = \lim_{\mathbf{x}\to 0} Lve^{i\langle \omega, \mathbf{x}\rangle}.$$

If r is a rotation we have

$$\begin{aligned} T_r(Lve^{i\langle\omega,\mathbf{x}\rangle}) &= L((S_r v)\, e^{i\langle \omega, r^{-1}\mathbf{x}\rangle}) \\ &= L((S_r v)\, e^{i\langle r\omega, \mathbf{x}\rangle}) \\ &= (\hat{L}(r\omega)\, S_r v)\, e^{i\langle r\omega,\mathbf{x}\rangle} \end{aligned}$$

on the one hand, and

$$T_r(Lve^{i\langle\omega,\mathbf{x}\rangle}) = (S_r\hat{L}(\omega)v)\, e^{i\langle r\omega,\mathbf{x}\rangle}$$

on the other. Property (ii) follows immediately. (Note that $\langle \omega, r^{-1}\mathbf{x}\rangle = \langle r\omega, \mathbf{x}\rangle$ since r is a unitary operator on $\mathbb{R}^k$). Property (iii) follows easily from the fact that $\overline{Lu} = L\bar{u}$.

An important example of such a covariant system of equations is provided by the Boussinesq equations in the theory of convection (see [2], [4], [22], and the Appendix. These comprise a system of nonlinear partial differential equations for five functions $u = (u_1, \dots, u_5)$ where $u_i = u_i(x, y, z)$. The components (u_1, u_2, u_3) form the Cartesian components of the fluid velocity, u_4 is the hydrodynamic pressure, and u_5 is the temperature. These equations are generally taken to hold on an infinite layer of fluid $-\infty < x, y, < \infty$, $0 \leqslant z \leqslant 1$. In that case the Banach spaces B_1 and B_2 may be taken to be the class of vector — valued functions $(f_1(z), \dots f_5(z))$, $0 \leqslant z \leqslant 1$. A suitable norm for the class $\mathscr{F}_1$ may be obtained by choosing appropriate Holder norms $\|\ \|_{j+a}$ on the various components. Our assumption that $G_u(0, 0)$ is a Fredholm operator of index zero when restricted to doubly periodic functions (satisfying appropriate boundary conditions at $z = 0, 1$) reduces to the standard theory of elliptic operators. (In the case of the Boussinesq equations the operator $G_u(0, 0)$ is an elliptic operator in the sense of Agmon, Douglis, and Nirenberg, See Fife [4].) The representation S_r is given by

$$S_r f = \left[\begin{array}{cc|ccc} \multicolumn{2}{c|}{\multirow{2}{*}{r}} & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} \tag{2.3}$$

where r denotes the standard 2×2 matrix representation of the two dimensional rotation r.

The onset of instability of the zero solution of (1.1) is formally characterized by the *dispersion relation*, as follows. From Lemma 2.1 we may write (with $L(\lambda) = G_u(\lambda, 0)$

$$L(\lambda)\, ve^{i\langle \boldsymbol{\omega}, \mathbf{x}\rangle} = (A(\lambda, \boldsymbol{\omega})v)\, e^{i\langle \boldsymbol{\omega}, \mathbf{x}\rangle}$$

where A possesses the properties (i)-(iii). Since A is covariant with respect to rotations nothing is lost in putting $\boldsymbol{\omega} = \omega(1, 0)$ and writing $\mathscr{D}(\lambda, \omega) = A(\lambda, \omega(1, 0))$. We then consider the eigenvalue problem

$$\sigma v + \mathscr{D}(\lambda, \omega)\, v = 0. \tag{2.4}$$

Let us suppose that the eigenvalues of (2.4) are discrete, and denote these by $\sigma_k(\lambda, \omega)$, with Re $\sigma_k(\lambda, \omega) \geqslant$ Re $\sigma_{k+1}(\lambda, \omega)$. The maximal branch, $\sigma_0 = \sigma_0(\lambda_c, \omega)$ will be called the dispersion relation, where λ_c is the critical value of the parameter λ.

The eigenvalue problem (2.4) arises from a formal consideration of infinitesimal disturbances of the equation $u_t + Lu = 0$ when one puts $u(\mathbf{x}, t) = ve^{i\langle \boldsymbol{\omega}, \mathbf{x}\rangle + \sigma t}$. Separation of variables then leads to equation (2.4). Instability is formally characterized by the existence of values of λ and ω for which Re $\sigma_0(\lambda, \omega) > 0$.

If B_1 is finite dimensional the eigenvalue problem (2.4) can be formulated in the form

$$\det[\sigma I + \mathscr{D}(\lambda, \omega)] = 0 \tag{2.5}$$

The *neutral stability curve* is obtained by setting Re $\sigma_0(\omega, \lambda) = 0$. In the finite dimensional case it may be obtained as one of the solutions of the equation

$$\det \mathscr{D}(\lambda, \omega) = 0 \tag{2.6}$$

provided σ_0 is real.

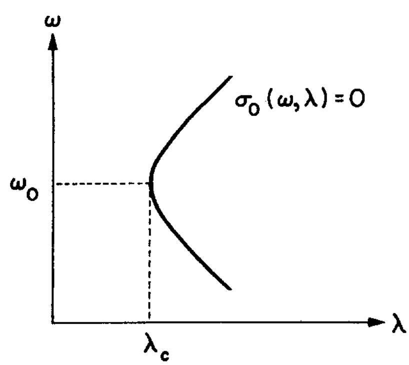

FIGURE 2.1

A typical neutral stability curve is shown in Fig. 2.1 (see Appendix for the neutral stability curve obtained for the Bénard problem). In such a case the critical parameter value λ_c and wave number ω_c are determined by minimizing λ over all values of ω on the curve $\operatorname{Re} \sigma_0(\omega, \lambda) = 0$.

Let us assume that in a given problem the onset of instability is characterized by the following features:

(i) $\sigma_0(\omega, \lambda)$ is real

(ii) For $\lambda < \lambda_c$, $\sigma_0(\omega, \lambda) < 0$ for all ω

(iii) For $\lambda = \lambda_c$, $\sigma_0(\omega, \lambda_c) < 0$ for all $\omega \neq \omega_c$ and $\sigma_0(\omega_c, \lambda_c) = 0$

(iv) $d/d\lambda\, \sigma_0(\omega_c, \lambda)|_{\lambda=\lambda_c} > 0$

(v) the eigenvalue branch $\sigma_0(\omega, \lambda)$ is simple (both algebraically and geometrically.)

Under these conditions we are going to discuss the bifurcation of spatially doubly periodic solutions of (1.1) in a neighborhood of $\lambda = \lambda_c$, $u = 0$. As in the case of bifurcation at a simple eigenvalue (see [29], Chapter 4) one can prove that

$$\frac{d\sigma_0}{d\lambda}(\omega_c, \lambda_c) = \langle \mathscr{D}_1(\omega_c, \lambda_c)\, v, v^* \rangle_{B_1}$$

where

$$\mathscr{D}_1(\omega_c, \lambda_c) = \frac{d}{d\lambda} \mathscr{D}(\omega_c, \lambda)|_{\lambda=\lambda_c}$$

and v^* is the adjoint eigenvector — that is,

$$\mathscr{D}^*(\omega_c, \lambda_c)\, v^* = 0.$$

We assume v and v^* are normalized so that $\langle v, v^* \rangle = 1$.

Because of the covariance of $A(\lambda, \omega)$ the kernel of $L(\lambda_c)$ consists of a continuous family of critical wave vectors

$$\{S_r v e^{i\langle r\boldsymbol{\omega}, \mathbf{x}\rangle}\}_{r \in 0(k)} \tag{2.7}$$

Thus, in $\mathscr{F}_1$ the kernel of $L(\lambda_c)$ is infinitely degenerate with the power of the continuum.

This infinite multiplicity reduces to a finite one if we restrict ourselves to the subclass of multiply periodic functions. (See [9]). Let Λ be a lattice of vectors in the plane : that is, if $\boldsymbol{\omega}_i$ and $\boldsymbol{\omega}_j$ belong to Λ then so do $n\boldsymbol{\omega}_i + m\boldsymbol{\omega}_j$ for any integers n and m. Let $H(\Lambda)$ denote the lattice subgroup of $T(2)$, the group of translations in the plane:

$$H(\Lambda) = \{T_\omega : \boldsymbol{\omega} \in \Lambda\}.$$

We consider all functions ψ which are invariant under $H(\Lambda)$:

$$(T_\omega \psi)(x) = \psi(x + \omega) = \psi(x) \tag{2.8}$$

for all $\omega \in \Lambda$. Such subclasses of $\mathscr{F}_1$ or $\mathscr{F}_2$ constitute Banach spaces of doubly periodic functions in the plane. We denote them by $\mathscr{F}_1(\Lambda)$ and $\mathscr{F}_2(\Lambda)$. We may abbreviate our notation somewhat by calling such functions Λ-periodic.

From the translational invariance of G it follows that any such subspace of Λ-periodic functions (2.8) is invariant under G. On such a subspace the dimension of the kernel of $L(\lambda_c)$ is finite, being equal to the number of wave functions in (2.7) which satisfy (2.8).

Now if ψ is Λ-periodic then $T_r\psi$ (where r is a rotation) is Λ-periodic if and only if r^{-1} leaves Λ invariant. We denote by $\mathscr{D}(\Lambda)$ the largest subgroup of $0(k)$ which leaves Λ invariant. In crystallography $\mathscr{D}(\Lambda)$ is called the *holohedry* of the lattice Λ. The holohedries are finite subgroups of $0(k)$. When we restrict ourselves to the bifurcation of doubly periodic solutions (2.8) the kernel (2.7) is spanned by the finite set of wave functions

$$\mathscr{N}(\Lambda) = \{(S_r v)\, e^{i\langle r\boldsymbol{\omega}, \mathbf{x}\rangle}\}_{r \in \mathscr{D}_\Lambda} \tag{2.9}$$

The set (2.9) is finite, though the number of linearly independent wave functions is not necessarily equal to the number of elements in $\mathscr{D}(\Lambda)$. For example, in the case of the hexagonal lattice where $\mathscr{D}(\Lambda) = D_6$ (the symmetry group of the hexagon), the order of D_6 is 12 while the number of linearly independent vectors in (2.9) may be only 6, each wave vector being covered twice.

We denote by $\mathscr{E}(\Lambda)$ the subgroup of $\mathscr{E}(k)$ which leaves $\mathscr{F}_1(\Lambda)$ invariant. $\mathscr{E}(\Lambda)$ is generated by the translations and by the rotations in the holohedry $\mathscr{D}(\Lambda)$. $\mathscr{E}(\Lambda)$ is the symmetry group of equations (1.1) restricted to the subspace $\mathscr{F}_1(\Lambda)$.

For our present purposes it will be more suitable to classify the various lattice types and their holohedries in a manner different from the usual procedure in crystallography. We suppose that at criticality ($\lambda = \lambda_c$) the dispersion relation $\sigma = \sigma(\omega, \lambda_c)$ is negative everywhere except at $\omega = \omega_c$, where it vanishes; and that for λ slightly greater than λ_c, $\sigma(\omega, \lambda) > 0$ for $\omega_1 < \omega < \omega_2$, where $\omega_1 < \omega_c < \omega_2$. Then all disturbances with wave numbers $\omega < \omega_1$ or $\omega > \omega_2$ will be damped out, while disturbances with wave numbers $\omega_1 < \omega < \omega_2$ will grow. An argument of Kirchgassner [9], [10] in connection with the Taylor problem suggests that only bifurcating disturbances with wave numbers $\omega = \omega_c$ will be stable. These considerations, which are admittedly only heuristic, suggest that we might be justified in restricting ourselves to lattices generated by basic vectors $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ with $|\boldsymbol{\omega}_1| = |\boldsymbol{\omega}_2| = \omega_c$. In that case there are three lattice types to be considered, as follows.

The Hexagonal Lattice Λ_6. This is the lattice generated by two vectors $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ making an angle of $\pi/3$ with one another. The number of critical wave vectors $\{r\boldsymbol{\omega}\}_{r\in D_6}$ in this case is 6. Putting $\psi_r(x) = (S_r v)\, e^{i\langle r\omega, x\rangle}$, the kernel of $L(\lambda_c)$ is given by $\{\psi_r\}_{r\in D_6}$.

This space may be 6 or 12 dimensional, depending on the representation S_r. In the case of the Bénard problem the kernel is six dimensional (see the Appendix and 3.1a).

The Square Lattice Λ_4. In this case the basic vectors make a right angle with each other. The holohedry is D_4, the symmetry group of the square, and there are four critical wave vectors.

The Rhombic Lattice Λ_2. Here the vectors $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ make an acute angle other than $\pi/3$ with one another. There are again four critical wave vectors. The holohedry is D_2 — the group containing a reflection in a line l and a rotation of 180r. The line l may be taken to be the one which bisects the angle between $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$. There is a one parameter family of such lattices, the parameter being the angle between $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$.

The configurations of critical wave vectors for each of the three lattices above are depicted in Fig. 2.2.

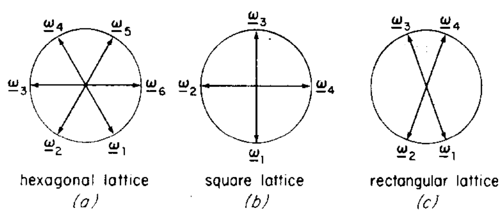

FIGURE 2.2

3. Group Action on the Kernel

Before discussing the structure of the bifurcation equations (1.2) we determine the action of $\mathscr{E}(\Lambda)$ on the kernel $\mathscr{N}(\Lambda)$ given in (2.9). We have seen that the elements in $\mathscr{N}(\Lambda)$ are of the form $\psi = ve^{i\langle\boldsymbol{\omega},\mathbf{x}\rangle}$ where $v \in B_1$ and $\boldsymbol{\omega}$ is some vector such that $T_a\psi = \psi$ for $a \in \Lambda$. We remark that *the function* $e^{i\langle\boldsymbol{\omega},\mathbf{x}\rangle}$ *is* Λ-*periodic if and only if* $\boldsymbol{\omega} \in \Lambda'$, *the dual lattice.*

The dual lattice is that lattice generated by a dual basis $\boldsymbol{\omega}_1'$ and $\boldsymbol{\omega}_2'$. If Λ is generated by $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$, let $\boldsymbol{\omega}_1'$ and $\boldsymbol{\omega}_2'$ be chosen to satisfy

$$\langle \omega_j\, , \boldsymbol{\omega}_k'\rangle = 2\pi\, \delta_{jk}\, .$$

Then Λ' consists of all integer linear combinations of $\boldsymbol{\omega}_1'$ and $\boldsymbol{\omega}_2'$. To prove the remark above note first that $e^{i\langle \boldsymbol{\omega}, \mathbf{x}\rangle}$ is Λ-periodic if and only if $e^{i\langle \boldsymbol{\omega}, s\boldsymbol{\omega}_1 + t\boldsymbol{\omega}_2\rangle}$ is periodic in s and t with period 1. Therefore

$$\langle \boldsymbol{\omega}, \boldsymbol{\omega}_1\rangle = 2\pi n \qquad \text{and} \qquad \langle \boldsymbol{\omega}, \boldsymbol{\omega}_2\rangle = 2\pi m$$

where m and n are integers. Now expand $\boldsymbol{\omega} = a\boldsymbol{\omega}_1' + b\boldsymbol{\omega}_2'$. It is clear that a and b are uniquely determined since $\boldsymbol{\omega}_1'$ and $\boldsymbol{\omega}_2'$ are linearly independent, and moreover, that $a = n$ and $b = m$. Therefore $\boldsymbol{\omega} \in \Lambda'$.

Now choose a wave vector $\psi \in \mathscr{N}(\Lambda)$ and denote it by $\psi = ve^{i\langle \boldsymbol{\omega}_1, \mathbf{x}\rangle}$. (We now drop the distinction between Λ and Λ' since it will not be needed). Other elements of $\mathscr{N}(\Lambda)$ are obtained by operating on ψ by the rotations and reflections in $\mathscr{D}(\Lambda) = \mathscr{D}(\Lambda')$. If we denote these group operations by $g_1, \ldots, g_{2n}$, with g_1 the identity, we obtain the wave vectors

$$\psi_j(\mathbf{x}) = (S_{g_j} v)\, e^{i\langle \boldsymbol{\omega}_j, \mathbf{x}\rangle}$$

where

$$\boldsymbol{\omega}_j = g_j \boldsymbol{\omega}_1 .$$

As g_j ranges over $\mathscr{D}(\Lambda)$ the vectors $\boldsymbol{\omega}_j$ range over the vertices of a polyhedron (or polygon if the underlying space is $\mathbb{R}^2$), which we shall call the fundamental polyhedron.

We shall make the assumption that

$$S_{g_j} v = S_{g_k} v \qquad \text{whenever} \quad g_j \boldsymbol{\omega}_1 = g_k \boldsymbol{\omega}_1 \tag{3.1a}$$

and

$$S_g v = \bar{v} \qquad \text{whenever} \quad g\boldsymbol{\omega}_1 = -\boldsymbol{\omega}_1 . \tag{3.1b}$$

These assumptions are valid in the case of the Bénard problem. (see Appendix). Then there are precisely as many linearly independent wave functions in $\mathscr{N}(\Lambda)$ as vertices in the fundamental polyhedron. Furthermore, these wave functions transform under the operations T_{g_j} just as the vertices of the fundamental polyhedron transform under $\mathscr{D}(\Lambda)$. To put it another way, the action of $\mathscr{D}(\Lambda)$ on the basis vectors of $\mathscr{N}(\Lambda)$ is isomorphic to the action of the symmetry group of the fundamental polyhedron.

We summarize the preceding discussion as

THEOREM 3.1. *Let T_σ be the representation* (2.1) *restricted to the kernel $\mathscr{N}(\Lambda)$. Then the action of T_σ on the basis vectors $\{\psi_1 \cdots \psi_n\}$ is given as follows:*

(i) *If $\sigma = a$ is a translation,*

$$T_{\mathbf{a}} \psi_j = e^{i\langle \boldsymbol{\omega}_j, \mathbf{a}\rangle} \psi_j$$

(ii) *If $\sigma = r$ is a rotation or reflection in $\mathscr{D}(\Lambda)$ then T_r acts as a permutation on the $\{\psi_j\}$. Specifically, let $r(j)$ be the permutation of the vertices $\boldsymbol{\omega}_j$ of the fundamental polyhedron induced by r.* Then

$$T_r\psi_j = \psi_{r(j)}$$

(iii) $T_r\psi_j = \bar{\psi}_j$ whenever $r\boldsymbol{\omega}_j = -\omega_j$.

It seems to be a generally accepted principle among physicists that the finite dimensional invariant subspaces of an invariant operator L which occur in nature are in most cases irreducible. (relative to the full symmetry group of L). Reducibility is considered to be an "uncommon situation" and is referred to as "accidental degeneracy". (See [33], p. 120). In the present case we can prove

THEOREM 3.2. *The subspace $\mathscr{N}(\Lambda)$ constructed above is irreducible under $\mathscr{E}(\Lambda)$.*

Proof. In §5 we shall show that if A is any linear operator on $\mathscr{N}(\Lambda)$ which is covariant under T_σ (that is, $T_\sigma A = AT_\sigma$ for any $\sigma \in \mathscr{G}(\Lambda)$), then A is a scalar multiple of the identity. It follows from Schur's lemma (see [33], p. 75) that $\mathscr{N}(\Lambda)$ is irreducible under $\mathscr{E}(\Lambda)$.

The finite symmetry group $\mathscr{D}(\Lambda)$ is a subgroup of $\mathscr{E}(\Lambda)$ and $\mathscr{N}(\Lambda)$ is reducible under $\mathscr{D}(\Lambda)$. The decomposition of $\mathscr{N}(\Lambda)$ into invariant irreducible subspaces of $\mathscr{D}(\Lambda)$ will be discussed in §4.

We close this section with a brief discussion of $\mathscr{N}(\Lambda)$ and Theorem 3.1 in the case of the hexagonal lattice. Consider the hexagon in Fig. 2.2 with vertices $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_6$. Its symmetry group D_6 is generated by the permutations

$$\alpha = (1\ 2\ 3\ 4\ 5\ 6) \quad \text{and} \quad \beta = (26)(35).$$

The corresponding operations in $\mathscr{D}(\Lambda)$ are, respectively, a rotation through 60° and a reflection through the axis joining $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_4$. The resulting transformations are

$$T(\alpha)\,\psi_1 = \psi_2\,, \qquad T(\alpha)\,\psi_2 = \psi_3\,,\ldots$$
$$T(\beta)\,\psi_1 = \psi_1\,, \qquad T(\beta)\,\psi_2 = \psi_6\,,\ldots$$

If w is a vector in $\mathscr{N}(\Lambda)$ we write

$$w = \sum_{j=1}^{6} z_j\psi_j$$

If w is to be real we must require that

$$z_j = \bar{z}_k \qquad \text{whenever} \qquad \psi_j = \bar{\psi}_k\,,$$

hence whenever $\boldsymbol{\omega}_j = -\boldsymbol{\omega}_k$. The group action on the components $(z_1, \ldots, z_6)$ is easily seen to be

$$T(\alpha)(z_1, \ldots, z_6) = (z_6, z_1, z_2, z_3, z_4, z_5)$$
$$T(\beta)(z_1, \ldots, z_6) = (z_1, z_6, z_5, z_4, z_3, z_2)$$
$$T_{\mathbf{a}}(z_1, \ldots, z_6) = (e^{i\langle\boldsymbol{\omega}_1, \mathbf{a}\rangle} z_1, \ldots, e^{i\langle\boldsymbol{\omega}_6, \mathbf{a}\rangle} z_6).$$

Furthermore, for real vectors we require

$$z_4 = \bar{z}_1, \qquad z_5 = \bar{z}_2, \qquad z_6 = \bar{z}_3. \tag{3.2}$$

Later it will be seen that the number of bifurcation equations is cut in half if we introduce a set of "action-angle" variables. We set

$$z_j = x_j e^{i\theta_j}, \qquad z_{j+3} = x_j e^{-i\theta_j}, \qquad j = 1, 2, 3, \tag{3.3}$$

where $x_j \geqslant 0$ and $0 \leqslant \theta_j < 2\pi$. Then condition (3.2) is satisfied automatically. The group action in these coordinates is given by

$$\begin{aligned} T(\alpha^{-1})(x_1, x_2, x_3; \theta_1, \theta_2, \theta_3) &= (x_2, x_3, x_1; \theta_2, \theta_3, -\theta_1) \\ T(\beta)(x_1, x_2, x_3; \theta_1, \theta_2, \theta_3) &= (x_1, x_3, x_2; \theta_1, -\theta_3, -\theta_2) \\ T_{\mathbf{a}}(x_1, x_2, x_3; \theta_1, \theta_2, \theta_3) &= (x_1, x_2, x_3; \theta_1 + \langle\boldsymbol{\omega}_1, \mathbf{a}\rangle, \theta_2 + \langle\boldsymbol{\omega}_2, \mathbf{a}\rangle, \\ &\qquad \theta_3 + \langle\boldsymbol{\omega}_3, \mathbf{a}\rangle). \end{aligned} \tag{3.4}$$

Similar coordinate systems can be introduced in the rhombic and square lattices as well.

4. Planform functions of Hydrodynamics

As a simple application of the above group-theoretic machinery we shall decompose the kernel $\mathscr{N}(\Lambda_6)$ into irreducible invariant subspaces of the subgroup $\mathscr{D}(\Lambda_6)$. In this way we obtain a constructive method for determining the complete set of planform functions of hydrodynamics. (See Schwiderski [31]). Similar constructions can be carried out for the other lattices as well. The results in the hexagonal case will be of interest in our discussion of the stability of hexagonal patterns in §9.

We begin by listing the basic facts about the symmetry group D_6, its character table, and its irreducible representations. Consider the hexagon in Fig. 2.2 The elements of its symmetry group are

$$
\begin{array}{ll}
e = \text{identity} & p = (12)(36)(45) \\
\alpha = (1\,2\,3\,4\,5\,6) & q = (14)(23)(56) \\
\alpha^2 = (135)(246) & r = (16)(25)(34) \\
\alpha^3 = (14)(25)(36) & \sigma = (46)(13) \\
\alpha^4 = (153)(264) & \beta = (35)(26) \\
\alpha^5 = (654321) & \tau = (24)(15).
\end{array}
$$

There are six conjugacy classes for D_6 and, accordingly, six irreducible representations. The dimensions of these irreducible representations must satisfy ([16], p. 73)

$$n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 + n_6^2 = 12.$$

The only solution of this equation is $n_1 = \cdots = n_4 = 1$, $n_5 = n_6 = 2$. The character table for D_6 is given below:

	$\{e\}$	$\{\alpha^2, \alpha^4\}$	$\{p, q, r\}$	$\{\alpha^3\}$	$\{\alpha, \alpha^5\}$	$\{\sigma, \beta, \tau\}$
$\chi^{(1)}$	1	1	1	1	1	1
$\chi^{(2)}$	1	1	-1	1	1	-1
$\chi^{(3)}$	1	1	1	-1	-1	-1
$\chi^{(4)}$	1	1	-1	-1	-1	1
$\chi^{(5)}$	2	-1	0	2	-1	0
$\chi^{(6)}$	2	-1	0	-2	1	0

This table was constructed by noting that D_6 is the direct product of D_3 and C_2, and by using the method described on page 82 of [11].

The two dimensional representation $T^{(6)}$ is equivalent to the 2×2 matrix representation one obtains by considering D_6 as a transformation group in the plane. The standard matrix representation for $T^{(6)}$ is generated by the following matrices :

$$T^{(6)}(\alpha) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \qquad T^{(6)}(p) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.1}$$

The other two dimensional representation $T^{(5)}$ is obtained as a tensor product of $T^{(4)}$ and $T^{(6)}$. Note from the character table that $\chi^{(5)} = \chi^{(4)}\chi^{(6)}$. Therefore the matrix representation for $T^{(5)}$ can be obtained as $T^{(5)} = \chi^{(4)}T^{(6)}$; its generators are given by

$$T^{(5)}(\alpha) = \begin{bmatrix} -\frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \qquad T^{(5)}(p) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4.2}$$

THEOREM 4.1. *Let T_g be the representation* (2.1) *restricted to the kernel $\mathcal{N}^{(n)}$ in the case of the hexagonal lattice. Then T_g may be decomposed into irreducible representations of D_6 as*

$$T_g = T^{(1)} \oplus T^{(4)} \oplus T^{(5)} \oplus T^{(6)} \tag{4.3}$$

Proof. As we saw in Theorem (3.1) the representation T_g acts as a permutation group on the wave vectors so each T_g can be represented as a permutation matrix. For example

$$T(\alpha) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad T(\sigma) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

From inspection of the cycle representations for α, α^2,..., σ,... we can immediately compute the character of our representation $T : \chi(g) = \operatorname{Tr} T(g)$. We get

	$\{e\}$	$\{\alpha^2, \alpha^4\}$	$\{p, q, r\}$	$\{\alpha^3\}$	$\{\alpha, \alpha^5\}$	$\{\sigma, \beta, \tau\}$
χ	6	0	0	0	0	2

Since each matrix T is a permutation matrix its trace is numerically equal to the number of elements left fixed by T, that is, to the number of vertices of the fundamental hexagon left invariant by the corresponding permutation. The only permutations which leave any vertices fixed are σ, β, and τ, each of which fix precisely two vertices.

We now write T as a direct sum of irreducible representations

$$T = \sum a_\mu T^{(\mu)}$$

where

$$a_\mu = \langle \chi, \chi^{(\mu)} \rangle = \frac{1}{12} \sum_{i=1}^{6} m_i \chi_i \overline{\chi_i^{(\mu)}}.$$

(see [16], p. 75). Here m_i is the number of elements in the i^{th} conjugacy class, and $\chi^{(\mu)}$ is the character of the μ^{th} irreducible representation. The multiplicities a_μ are easily computed, and one obtains $a_1 = a_4 = a_5 = a_6 = 1$, and $a_2 = a_3 = 0$, so equation (4.3) follows.

We now decompose the kernel $\mathcal{N}$ into irreducible invariant subspaces. This computation leads to a complete set of the so-called "planform functions" of hydrodynamics. The irreducible invariant subspaces of $\mathcal{N}$ can be obtained by operating on any element of $\mathcal{N}$, say ψ_1, by the projections

$$P_\mu = \frac{n_\mu}{N} \sum_{g\in\mathscr{D}_6} \overline{\chi^{(\mu)}(g)}\, T_g$$

where T is the given representation, N is the order of the (finite) group, and n_μ is the dimension and $\chi^{(\mu)}$ the character of the μ^{th} representation (see [16], p. 95).

In the case of the two dimensional representations a further decomposition must be made in order to obtain both basis vectors. Consider the subgroup $H = \{e, p\}$, where p is a reflection in the y-axis. Corresponding to the subgroup H are the two projections.

$$P_H = \frac{e+p}{2} \qquad Q_H = \frac{e-p}{2}\,.$$

For $\mu = 5, 6$ we compute $P_H P_\mu \psi_1$ and $Q_H P_\mu \psi_1$ to get the two independent basis vectors up to a sign factor. (Note : From now on the difference in notation between group elements and their representation may be neglected. This should cause the reader no confusion.)

We now begin the computations:

$$P_1\psi_1 = \frac{e\psi_1 + \alpha\psi_1 + \alpha^2\psi_1 + \cdots + p\psi_1 + \sigma\psi_1 + \cdots}{12}\,.$$

Using the cycle notation for the group elements we have

$$\alpha\psi_1 = \psi_2\,, \qquad \alpha^2\psi_1 = \psi_3\,,\ldots\, p\psi_1 = \psi_2\,,\ldots\, \sigma\psi_1 = \psi_3\,,$$

so

$$P_1\psi = \frac{\psi_1 + \psi_2 + \cdots + \psi_6}{6}\,.$$

In some cases it is possible to introduce a Hilbert Space structure in which the wave functions ψ_j form an orthonormal set. In that case the above element is not normalized. The normalized function is

$$\varphi_1 = \frac{\psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6}{\sqrt{6}}\,.$$

The function φ_1 is completely invariant under D_6. The subspace $[\varphi_1]$ is the one dimensional subspace of $\mathcal{N}$ which transforms according to the identity representation.

The projections $P_2\psi_1$ and $P_3\psi_1$ both vanish, as one may check by direct computation. This is to be expected since, by Theorem 4.1, the representation T does not contain the irreducible representations $T^{(2)}$, $T^{(3)}$.

Next one computes $P_4\psi_1$ and normalizes the resulting wave function, obtaining

$$\varphi_2 = i\left(\frac{\psi_1 - \psi_2 + \psi_3 - \psi_4 + \psi_5 - \psi_6}{\sqrt{6}}\right).$$

The factor i is introduced in order to obtain a real wave function. The function φ_2 transforms according to the representation $T^{(4)}$. That is, $T(g)\,\varphi_2 = \chi^{(4)}(g)\,\varphi_2$ for all $g \in D_6$.

Next we compute

$$P_5\psi_1 = \tfrac{1}{6}(2(\psi_1 + \psi_4) - (\psi_2 + \psi_5) - (\psi_3 + \psi_6))$$

and

$$P_H P_5\psi_1 = \tfrac{1}{12}((\psi_1 + \psi_4) + (\psi_2 + \psi_5) - 2(\psi_3 + \psi_6)).$$

The normalized vector is

$$\varphi_3 = \frac{1}{2\sqrt{3}}(\psi_1 + \psi_2 - 2\psi_3 + \psi_4 + \psi_5 - 2\psi_6).$$

The vector φ_4 is given by $-Q_H P_5\psi_1$, the minus sign being chosen so that $\{\varphi_3, \varphi_4\}$ transform according to the standard matrix representation given for $T^{(5)}$ in (4.2). That is, we choose φ_3 and φ_4 so that

$$\begin{aligned}
\alpha\varphi_3 &= -\frac{1}{2}\varphi_3 + \frac{\sqrt{3}}{2}\varphi_4 \\
\alpha\varphi_4 &= -\frac{\sqrt{3}}{2}\varphi_3 - \frac{1}{2}\varphi_4 \\
p\varphi_3 &= \varphi_3 \\
p\varphi_4 &= -\varphi_4 .
\end{aligned}$$

The vector φ_4 is given by

$$\varphi_4 = -\tfrac{1}{2}(\psi_1 - \psi_2 + \psi_4 - \psi_5).$$

The basis vectors for the two dimensional subspace of $\mathcal{N}$ which transforms according to $T^{(6)}$ are given by

$$\begin{aligned}
\varphi_5 &= \frac{i}{2\sqrt{3}}(-\psi_1 + \psi_2 + 2\psi_3 + \psi_4 - \psi_5 - 2\psi_6) \\
\varphi_6 &= \frac{i}{2}(\psi_1 + \psi_2 - \psi_4 - \psi_5).
\end{aligned}$$

The vectors φ_5 and φ_6 transform according to the standard matrix representation for $T^{(6)}$ given in (4.1).

We denote the irreducible subspaces as follows

$$V^{(1)} = [\varphi_1] \qquad V^{(5)} = [\varphi_3, \varphi_4]$$
$$V^{(4)} = [\varphi_2] \qquad V^{(6)} = [\varphi_5, \varphi_6].$$

The matrix for the change of basis from the wave functions $\{\psi_1, \ldots, \psi_6\}$ to $\{\varphi_1, \ldots, \varphi_6\}$ is defined by the relation

$$\varphi_i = P_{ji}\psi_j .$$

The matrix P is given by

$$\begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{i}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & -\frac{1}{2} & -\frac{i}{2\sqrt{3}} & \frac{i}{2} \\ \frac{1}{\sqrt{6}} & -\frac{i}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & \frac{1}{2} & \frac{i}{2\sqrt{3}} & \frac{i}{2} \\ \frac{1}{\sqrt{6}} & \frac{i}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & 0 & \frac{i}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{i}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & -\frac{1}{2} & \frac{i}{2\sqrt{3}} & -\frac{i}{2} \\ \frac{1}{\sqrt{6}} & \frac{i}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & \frac{1}{2} & -\frac{i}{2\sqrt{3}} & -\frac{i}{2} \\ \frac{1}{\sqrt{6}} & -\frac{i}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & 0 & -\frac{i}{\sqrt{3}} & 0 \end{bmatrix} \tag{4.4}$$

The complete set of normalized plan functions is (see Schwiderski [31])

$$\begin{aligned} \varphi_1 &= \frac{2}{\sqrt{6}}\left(\cos x + 2\cos\frac{x}{2}\cos\frac{\sqrt{3}y}{2}\right) & V^{(1)} \\ \varphi_2 &= \frac{2}{\sqrt{6}}\left(\sin x - 2\sin\frac{x}{2}\cos\frac{\sqrt{3}y}{2}\right) & V^{(2)} \\ \left.\begin{aligned} \varphi_3 &= \frac{2}{\sqrt{3}}\left(-\cos x + \cos\frac{x}{2}\cos\frac{\sqrt{3}y}{2}\right) \\ \varphi_4 &= -2\sin\frac{x}{2}\sin\frac{\sqrt{3}y}{2} \end{aligned}\right\} & V^{(5)} \\ \left.\begin{aligned} \varphi_5 &= \frac{2}{\sqrt{3}}\left(\sin x + \sin\frac{x}{2}\cos\frac{\sqrt{3}y}{2}\right) \\ \varphi_6 &= 2\cos\frac{x}{2}\sin\frac{\sqrt{3}}{2}y \end{aligned}\right\} & V^{(6)} \end{aligned} \tag{4.5}$$

5. Computation of the Invariant Bifurcation Equations

In this section we develop an algorithm for computing the general structure of the bifurcation equations for the rectangular, square, and hexagonal lattices in the plane. The details are carried out explicitly for the hexagonal case while the square and rectangular cases are left to the reader.

In [28], Theorem 4.1, we proved that the bifurcation equations are covariant under the group representation restricted to the kernel. Let the mapping be denoted by $F = (F_1, F_2, \ldots F_6)$, and let the coordinates on the null space $\mathcal{N}$ be $z = (z_1, \ldots z_6)$. F is covariant with respect to the following operations

$$\begin{aligned} \alpha(z_1, \ldots, z_6) &= (z_2, z_3, \ldots, z_6, z_1) \\ \beta(z_1, \ldots, z_6) &= (z_1, z_6, z_5, z_4, z_3, z_2) \\ J(z_1, \ldots, z_6) &= (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_6) \\ T_a(z_1, \ldots, z_6) &= (e^{i\langle \omega_1, a\rangle} z_1, \ldots, e^{i\langle \omega_6, a\rangle} z_6). \end{aligned} \tag{5.1}$$

From $\alpha F = F\alpha$ we obtain

$$F_2(z_1, \ldots z_6) = F_1(z_2, z_3, \ldots z_6, z_1). \tag{5.2}$$

Thus, knowing the component F_1 the other components of F may be obtained by cyclic permutation of the variables. From $\beta F = F\beta$ we get

$$F_1(z_1, \ldots z_6) = F_1(z_1, z_6, z_5, z_4, z_3, z_2). \tag{5.3}$$

Similarly $JF = FJ$ implies

$$\overline{F_1(z_1, z_2, z_3, z_4, z_5, z_6)} = F_1(\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4, \bar{z}_5, \bar{z}_6). \tag{5.4}$$

(That F commutes with complex conjugation is a consequence of our assumption that equations (1.1) are real). Finally, from the translational invariance of F we get

$$e^{i\langle \omega_1, a\rangle} F_1(z_1, \ldots, z_6) = F_1(e^{i\langle \omega_1, a\rangle} z_1, \ldots, e^{i\langle \omega_1, a\rangle} z_6) \tag{5.5}$$

for all a. Using (5.3), (5.4) and (5.5) we may compute the general scalar invariant F_1.

We decompose F_1 into linear, quadratic, cubic terms, etc. First, let us suppose F_1 is homogeneous of degree 1, and write F_1 as a sum

$$\sum_{k=1}^{6} a_k z_k$$

where the a_k are constants. From (5.5)

$$e^{i\langle \omega_1, \mathbf{a} \rangle} \sum_{k=1}^{6} a_k z_k = \sum_{k=1}^{6} a_k e^{i\langle \omega_1, \mathbf{a} \rangle} z_k .$$

Since this relation must hold identically in $\mathbf{a}$, $z_1 ,\dots z_6$ it follows that $a_k = 0$ unless $k = 1$. Therefore the only translationally invariant linear term is of the form

$$az_1 ,$$

where a is a scalar. Furthermore a is real by (5.4). From (5.2) $F_2 = az_2$, $F_3 = az_3 ,\dots$ etc. So the only linear mapping which is invariant under our group is a scalar multiple of the identity. This shows, as we said earlier (Theorem 3.2), that $\mathscr{N}$ is irreducible.

Now let us proceed to terms of degree two. We may write

$$F_1 = \sum a_{jk} z_j z_k$$

for a general quadratic term. However, as the reader may easily check for himself, the sum is covariant if and only if each of the terms are separately covariant. Thus it suffices to consider a term $z_j z_k$. In order for (5.5) to hold we must have

$$e^{i\langle \omega_1, \mathbf{a} \rangle} z_j z_k = e^{i\langle \omega_j + \omega_k, \mathbf{a} \rangle} z_j z_k$$

for all $\mathbf{a}$, z_j , z_k . This can be so if and only if

$$\omega_1 = \omega_j + \omega_k , \tag{5.6}$$

which is satisfied only by taking ω_j , ω_k to be ω_2 and ω_6 . Thus the only covariant quadratic term is of the form

$$bz_2 z_6$$

with b real. Then $F_2 = bz_3 z_1$, $F_3 = bz_4 z_2$, etc.

The general procedure is now clear, though it becomes somewhat more complicated as the degree of homogeneity increases. The cubic term $z_j z_k z_l$ is covariant if and only if

$$\omega_1 = \omega_j + \omega_k + \omega_l .$$

This condition is met only if one vector on the right is ω_1 and the sum of the other two is zero. There are three possibilities, and the corresponding covariant terms are

$$z_1{}^2 z_4 , \quad z_2 z_5 z_1 , \qquad \text{and} \qquad z_3 z_6 z_1 .$$

From (5.3) we see that F_1 must be symmetric in the variables z_2, z_6 and z_3, z_5. Therefore we must symmetrize in these variables. The general covariant scalar F_1 thereby obtained is

$$F_1 = a(z_3 z_6 + z_2 z_5)\, z_1 + b z_1 z_4 z_1$$

where, again, a and b are real parameters.

By similar arguments mappings of all orders may be obtained. For future reference we note the generator for fourth degree mappings :

$$a(z_2 z_5 + z_3 z_6)\, z_2 z_6 + b(z_1 z_4)\, z_2 z_6 \tag{5.7}$$

Moreover, the same procedures apply to the lattices Λ_2 and Λ_4. In the table below we list the generators F_1 for the covariant mappings homogeneous of degree k, $k = 1, 2, 3$ for the three lattices under consideration.

TABLE 5.1

k	Λ_2	Λ_4	Λ_6
1	az_1	az_1	az_1
2	0	0	$b(z_2 z_6)$
3	$c(z_1{}^2 z_3) + d(z_1 z_2 z_4)$	$c(z_1{}^2 z_3) + d(z_1 z_2 z_4)$	$c(z_2 z_5 + z_3 z_6) z_1 + d(z_1{}^2 z_4)$

Remark. The procedure for obtaining the components F_2, F_3, and F_4 in the case Λ_2 differs slightly from that for Λ_4 and Λ_6. The symmetries of D_2 are e, $\alpha = (13)(24)$, $\beta = (12)(34)$, and $\gamma = (23)(14)$. The components F_2, F_3, F_4 are respectively obtained by applying the symmetries $\beta F = F\beta$, $\alpha F = F\alpha$, and $\gamma F = F\gamma$. In the cases Λ_4 and Λ_6 the other components are obtained by cyclic permutation of the arguments.

6. Analysis of the Bifurcation Equations

We turn now to an explicit analysis of the branching equations. We recall ([28], § 3) that by examining the Newton diagram for a mapping it is possible to determine an appropriate scaling parameter for the solutions. Suppose we wish to construct solutions of a system of branching equations (1.2) where $F(\lambda, z)$ has the form

$$F(\lambda, z) = \lambda z + Q_k(z) + \cdots$$

where $Q_k(tz) = t^k Q_k(z)$. Putting $\lambda = \epsilon^{k-1}$ and $z = \epsilon\xi$ (Here $z = (z_1, \ldots z_n)$ and $\xi = (\xi_1, \ldots \xi_n)$) we get

$$F(\epsilon^{k-1}, \epsilon\xi) = \epsilon^k[\xi + Q_k(\xi)] + 0(\epsilon^{k+1}).$$

Dividing by ϵ^k we obtain a system of equations of the form

$$[\xi + Q_k(\xi)] + \epsilon R(\epsilon, \xi) = 0 \tag{6.1}$$

Letting $\epsilon \to 0$ we obtain the *reduced bifurcation equations*

$$\xi + Q_k(\xi) = 0. \tag{6.2}$$

If ξ_0 is a solution of (6.2) and if the Jacobian of these equations with respect to ξ is non-singular at ξ_0, then by the implicit function theorem ξ_0 may be extended to a solution $\xi(\epsilon) = \xi_0 + \epsilon\xi_1 + \cdots$ of the full equations (6.1).

As we have seen the bifurcation equations $F(\lambda, z)$ are covariant with respect to a two parameter transformation group. This same covariance is inherited by the reduced bifurcation equations (6.2), whose solutions therefore appear in two-parameter sheets; and so the Jacobian always has a null space of dimension at least two. (It is actually three dimensional in one case, as we shall see).

All these difficulties can be circumvented by introducing the action-angle variables (3.3). We have

LEMMA 6.1. *In action-angle variables a mapping $F: \mathcal{N}(\Lambda) \to \mathcal{N}(\Lambda)$, covariant with respect to $\mathscr{E}(\Lambda)$, has the forms below*

1. $\Lambda = \Lambda_6$, the hexagonal lattice

$$F_1 = \overline{F_4} = e^{i\theta_1} W(x_1, x_2, x_3; \gamma)$$

$$F_2 = \overline{F_5} = e^{i\theta_2} W(x_2, x_3, x_1; -\gamma)$$

$$F_3 = F_6 = e^{i\theta_3} W(x_3, x_1, x_2; \gamma)$$

where $\gamma = \theta_2 - \theta_1 - \theta_3$. *Furthermore*

$$W(x_1, x_2, x_3; \gamma) = W(x_1, x_3, x_2; \gamma)$$

and

$$\overline{W(x_1, x_2, x_3; \gamma)} = W(x_1, x_2, x_3; -\gamma)$$

2. $\Lambda = \Lambda_2$ *or* Λ_4, *the square or rhombic lattice*

$$F_1 = \overline{F_3} = e^{i\theta_1} W(x_1, x_2)$$

$$F_2 = \overline{F_4} = e^{i\theta_2} W(x_2, x_1).$$

Proof. We shall carry out the details for the hexagonal case and leave the other cases to the reader. Let $\omega_1, \ldots \omega_6$ denote the vertices of the hexagon in

Fig. 2.2. Let $\boldsymbol{\omega}_1'$ and $\boldsymbol{\omega}_3'$ form a basis dual to $\boldsymbol{\omega}_1, \boldsymbol{\omega}_3$. That is, choose $\boldsymbol{\omega}_1'$, $\boldsymbol{\omega}_3'$ so that

$$\langle \boldsymbol{\omega}_i, \boldsymbol{\omega}_j' \rangle = 2\pi\, \delta_{ij}\, i, j = 1, 3.$$

For a vector $\mathbf{a} \in \mathbb{R}^2$ we write $\mathbf{a} = s\boldsymbol{\omega}_1' + t\boldsymbol{\omega}_3'$. Then

$$\langle \boldsymbol{\omega}_1, \mathbf{a} \rangle = s, \qquad \langle \boldsymbol{\omega}_3, \mathbf{a} \rangle = t, \qquad \langle \boldsymbol{\omega}_2, \mathbf{a} \rangle = \langle \boldsymbol{\omega}_1 + \boldsymbol{\omega}_3, \mathbf{a} \rangle = s + t.$$

From (3.4) we have

$$\begin{aligned} T_{\mathbf{a}} F_1(x_1, x_2, x_3; \theta_1, \theta_2, \theta_3) &= e^{i\langle \boldsymbol{\omega}_1, \mathbf{a} \rangle} F_1(x_1, x_2, x_3; \theta_1, \theta_2, \theta_3) \\ &= F_1(x_1, x_2, x_3; \theta_1 + s, \theta_2 + s + t, \theta_3 + t). \end{aligned}$$

Putting $s = -\theta_1$, $t = -\theta_3$ we get

$$F_1(x_1, x_2, x_3; \theta_1, \theta_2, \theta_3) = e^{i\theta_1} F_1(x_1, x_2, x_3; 0, \gamma, 0).$$

where $\gamma = \theta_2 - \theta_1 - \theta_3$. Now set

$$W(x_1, x_2, x_3; \gamma) = F_1(x_1, x_2, x_3; 0, \gamma, 0).$$

From the symmetry $\beta = (35)(26)$ we get (see (3.4))

$$W(x_1, x_2, x_3; \gamma) = W(x_1, x_3, x_2; \gamma).$$

Furthermore, (5.4) implies

$$\overline{F_1(x_1, x_2, x_3; \theta_1, \theta_2, \theta_3)} = F_1(x_1, x_2, x_3; -\theta_1, -\theta_2, -\theta_3);$$

consequently $W(x_1, x_2, x_3, \gamma) = W(x_1, x_2, x_3; -\gamma)$.

To get $F_2(x_1, x_2, x_3; \theta_1, \theta_2, \theta_3)$ we apply the symmetry $\alpha = (1\,2\,3\,4\,5\,6)$ (see (3.4)), obtaining

$$\begin{aligned} F_2(x_1, x_2, x_3; \theta_1, \theta_2, \theta_3) &= F_1(x_2, x_3, x_1; \theta_2, \theta_3, -\theta_1) \\ &= e^{i\theta_2} W(x_2, x_3, x_1; 0, \theta_3 - \theta_2 + \theta_1, 0) \\ &= e^{i\theta_2} W(x_2, x_3, x_1; -\gamma). \end{aligned}$$

And so it goes.

As a consequence of Lemma 6.1 the bifurcation equations may be reduced to the form below.

LEMMA 6.2. *The bifurcation equations have the form*

1. Λ = *hexagonal lattice*

$$\begin{aligned} &W(x_1, x_2, x_3; \gamma) = 0 \\ &W(x_2, x_3, x_1; -\gamma) = 0 \\ &W(x_3, x_1, x_2; \gamma) = 0 \end{aligned} \tag{6.3}$$

2. Λ = *square or rectangular lattice*

$$\begin{aligned} &W(x_1, x_2) = 0 \\ &W(x_2, x_1) = 0. \end{aligned} \tag{6.4}$$

(We have suppressed the dependence of W on the parameter λ.)

The number of equations is cut in half in each case since the second half is always obtained by complex conjugation. Recall that we are interested only in real positive solutions x_i of equations (6.3) (or (6.4)). Since W is in general complex when $\gamma \neq 0$ we get real positive solutions only when $\gamma = 0$ in (6.3).

THEOREM 6.3. *The reduced bifurcation equations for the bifurcation of Λ-periodic disturbances of a Euclidean-covariant system* (1.1) *have the forms below*

1. Λ = *rectangular or square lattice* ($k = 3$)

$$\begin{aligned} x_1(\lambda + ax_1^2 + bx_2^2) = 0 \\ x_2(\lambda + ax_2^2 + bx_1^2) = 0 \end{aligned} \tag{6.5}$$

2. Λ = *hexagonal lattice* ($k = 2$)

$$\begin{aligned} \lambda x_1 - e^{i\gamma} x_2 x_3 = 0 \\ \lambda x_2 - e^{i\gamma} x_3 x_1 = 0 \\ \lambda x_3 - e^{i\gamma} x_1 x_2 = 0 \end{aligned} \tag{6.6}$$

3. Λ = *hexagonal lattice* ($k = 3$)

$$\begin{aligned} x_1(\lambda + a(x_2^2 + x_3^2) + bx_1^2) = 0 \\ x_2(\lambda + a(x_3^2 + x_1^2) + bx_2^2) = 0 \\ x_3(\lambda + a(x_1^2 + x_2^2) + bx_3^2) = 0 \end{aligned} \tag{6.7}$$

In particular, the γ-dependence factors out of equations (6.7) *and these equations are invariant under the three parameter group $\theta_i \to \theta_i + \alpha_i$, $i = 1, 2, 3$.*

The reduced bifurcation equations (6.7) thus have a greater symmetry than the full bifurcation equations. The above forms of the reduced bifurcation equations follow directly from Table 5.1 upon converting to action-angle variables. The coefficient of the linear term may always (assuming it does not vanish) be reduced to λ by a suitable rescaling of the variables. Similarly, the coefficients of the quadratic term in (6.6) may be taken to be ± 1 provided a correct choice of scaling of the variables λ and z is made. The fact that equations (6.7) are invariant under a three parameter group has important consequences for the stability of the bifurcating hexagonal solutions as we shall see in the next section. This extra symmetry is destroyed by the higher order terms (for example, by terms of degree four), as the reader may verify for himself.

The parameters a, b in (6.5) are not numerically equal for the different lattices, and they differ also from the parameters a, b in (6.7). Further information concerning the relative sizes of the parameters for the different lattices may also be obtained by group theoretic methods. See the forthcoming article "Selection Mechanisms for Pattern Formation" to appear in the *Archives for Rational Mechanics and Analysis.*

A complete analysis of the reduced equations (6.5)–(6.7) is a matter of straightforward algebra. Consider, for example, equations (6.7). There are three cases : $x_3 = 0$ and $x_1x_2 \neq 0$; $x_3 = x_2 = 0$, $x_1 \neq 0$; $x_1x_2x_3 \neq 0$. Let us take the last case. Then equations (6.7) reduce to

$$\lambda + a(x_2^2 + x_3^2) + bx_1^2 = 0, \ldots.$$

These equations are *linear* in x_1^2, x_2^2, x_3^2 and so their solution poses no difficulties whatsoever.

In the case (6.6) we see that x_1, x_2, x_3 all vanish if any one of them does. Furthermore, equations (6.6) imply that $\lambda^3 x_1x_2x_3 = (x_1x_2x_3)^2 e^{3i\alpha}$, hence that $x_1x_2x_3 = (\lambda e^{-i\alpha})^3$. Choose $\alpha = 0$ or π so that $\lambda e^{-i\alpha} > 0$. Then the unique solution is $x_1 = x_2 = x_3 = \lambda e^{-i\alpha} = |\lambda|$.

The analysis of (6.5) follows the lines of the analysis of (6.7). The solutions of equations (6.5)–(6.7) are summarized in Table 6.1 below.

THEOREM 6.4. *When all eigenvalues of the Jacobian of the reduced equations are non-zero, the solutions listed in Table* 6.1 *may be extended to solutions of the full equations* 1.1. *The solutions are real and positive if* $\gamma = 0$.

Theorem 6.4 is an immediate consequence of the classical implicit function theorem. In the next section we show that the eigenvalues of the Jacobian of the reduced equations determine the stability of the bifurcating solutions in a neighborhood of the branch point.

The fact that equations (6.7) are independent of γ means that they are invariant under the three parameter group $\theta_i \to \theta_i + \gamma_i$, $i = 1, 2, 3$. Thus the reduced equations possess a higher symmetry than the original problem. This

TABLE 6.1

	Solutions	Eigenvalues of the Jacobian	Remarks
(6.5)	$x_2 = 0, x_1 = \sqrt{\frac{-\lambda}{a}}$	$-2\lambda, \frac{-\lambda}{a}(b-a)$	$a < 0$ for supercritical bifurcation; stable if $a < 0, b-a < 0$
	$x_1 = x_2 = \sqrt{\frac{-\lambda}{(a+b)}}$	$-2\lambda, -2\lambda\left(\frac{a-b}{a+b}\right)$	$a+b <$ for supercritical bifurcation; stable if $a+b < 0$, $b-a > 0$
(6.6)	$x_1 = x_2 = x_3 = 1$	$2\lambda, 2\lambda, -\lambda$	unstable on both sides of criticality
(6.7)	$x_2 = x_3 = 0, x_1 = \sqrt{\frac{-\lambda}{b}}$	$-2\lambda, \frac{-\lambda}{b}(a-b), \frac{-\lambda}{b}(a-b)$	$b < 0$ for supercritical bifurcation; stable if $b < 0, b-a > 0$
	$x_3 = 0, x_1 = x_2 = \sqrt{\frac{-\lambda}{a+b}}$	$\lambda\frac{b-a}{a+b}, -2\lambda, -2\lambda\left(\frac{b-a}{b+a}\right)$	$a+b < 0$; unstable
	$x_1 = x_2 = x_3 = \sqrt{\frac{-\lambda}{2a+b}}$	$-2\lambda, \frac{-2\lambda(b-a)}{b+2a}, \frac{-2\lambda(b-a)}{2a+b}$	$2a+b < 0$ for supercritical bifurcation; stable (?) if $2a+b < 0$, $b-a < 0$

additional symmetry is already destroyed at fourth order, as the reader may check for himself using (5.7). Since the Jacobian of the reduced equations is independent of γ the solutions of the reduced problem extend to a three parameter family of the full nonlinear problem : two parameters are due to the translational invariance, and the third is γ. These solutions will not be real, however, unless $\gamma = 0$. Since we require that the solutions (x_1, x_2, x_3) be real and nonnegative, all except those for which $\gamma = 0$ are to be excluded on physical grounds. One serious consequence of the additional invariance of the reduced equations, as we shall see in the next section, is that the stability of the hexagonal solutions cannot be determined at lowest order.

7. Stability of the Bifurcating Solutions

In this section we relate the stability of the bifurcating solutions to the eigenvalues of the Jacobian of the reduced bifurcation equations. In one respect the results of this section can be viewed as an extension to multiple eigenvalues of previous results of Sattinger [27], McLeod and Sattinger [13], and Crandall and Rabinowitz [3]. (See also L. Nirenberg [17], pp. 102-110). The problem is to relate the stability of the bifurcating solutions to an analysis of the reduced bifurcation equations. The present discussion is limited to an analysis of the stability relative to disturbances within the same class of Λ-periodic solutions. If $\{\lambda(\epsilon), u(\epsilon)\}$ is a one-parameter family of solutions of (1.1), the stability of the solution is formally determined by the eigenvalues of the linear operator $L(\epsilon) = G_u(\lambda(\epsilon), u(\epsilon))$. When $\epsilon = 0$, $L_0 = G_u(0, 0)$ (recall that we have taken $\lambda_c = 0$ for convenience) has an n-dimensional null space. If $\lambda_c = 0$ is the critical parameter value at which $u = 0$ loses stability then by hypothesis all other eigenvalues of L_0 lie in the left half plane. Therefore the stability of the bifurcating solutions is determined by the behavior of the n-fold eigenvalue at the origin under the perturbation.

Before beginning let us note that if G is invariant under a k-parameter Lie group then $G_u(\lambda, u)$ has in general a k-dimensional null space. In fact, letting $T(g_1, \dots g_k)$ be the k-parameter group representation we have

$$G(\lambda, T(g_1, \dots g_k)u) = 0.$$

Differentiating with respect to g_j and setting $g_1 = \cdots = g_k = 0$ we have

$$\frac{d}{dg_j} G(\lambda, T(g_1, \dots, g_k)\, u)\Big|_{g_j = \cdots g_k = 0} = G_u(\lambda, u)\, L_j u = 0$$

where L_j is the Lie derivative of T with respect to g_j at the origin. Thus the vectors $L_j u$ are null functions of the Frechet derivative $G_u(\lambda, u)$.

The L_j are the infinitesimal generators of the group. In the case of the translation group the infinitesimal generators are $\partial/\partial x$ and $\partial/\partial y$, so if $u(x, y, \lambda)$ is the solution, the null functions are u_x and u_y. The rotation group is excluded here since it does not leave the lattice invariant. We therefore have, in the present situation, a two dimensional null space, so at best we can expect only an orbital type of stability.

We first prove a preliminary result in the perturbation theory of an isolated n-fold eigenvalue.

Lemma 7.1. *Let $L(\epsilon) = L_0 + \epsilon L_1 + \cdots$ be an analytic family of bounded operators from B_1 to B_2 (complex Banach spaces with $B_1 \subseteq B_2$) and suppose that L_0 has an isolated eigenvalue at the origin of multiplicity n and Riesz index one.*

Let the null space $\mathcal{N}$ be spanned by $\{\varphi_1, \dots \varphi_n\}$. Then there is an analytic projection-valued operator $E(\epsilon)$ and an analytic matrix $B_{ij}(\epsilon)$ such that

$$L(\epsilon)\, E(\epsilon)\, \varphi_i = \sum_{j=1}^{n} B_{ij}(\epsilon)\, E(\epsilon)\, \varphi_j\,. \tag{7.1}$$

Let $P_0 = E(0)$ be the projection onto $\mathcal{N}$ which commutes with L_0, and suppose $P_0L_0 = \cdots = P_0L_{k-1} = 0$. Then

$$B_{ij}(\epsilon) = \epsilon^k \langle L_k \varphi_i\,, \varphi_j{}^* \rangle + 0(\epsilon^{k+1}) \tag{7.2}$$

where the vectors $\varphi_j{}^* \in B_2{}^*$ form a dual basis to the vectors $\{\varphi_1, \dots \varphi_n\}$. ($\langle \varphi_i\,, \varphi_j{}^* \rangle = \delta_{ij}$ and $L_0{}^* \varphi_j{}^* = 0$).

Proof. We define the projections $E(\epsilon)$ for small ϵ by

$$E(\epsilon) = \frac{1}{2\pi i} \int_C (z - L(\epsilon))^{-1}\, dz$$

where C encloses the origin and contains no other points in the spectrum of L_0. For small ϵ the eigenvalues vary continuously, so the above expression is well defined. (See Dunford and Schwartz, vol. I lemma 6, p. 586; $E(\epsilon)$ is a projection by Theorem 10, p. 568). Since $\{E(\epsilon)\, \varphi_1, \dots E(\epsilon)\, \varphi_n\}$ is an invariant finite dimensional subspace of $L(\epsilon)$, the existence of $B_{ij}(\epsilon)$ and equation (7.1) follow immediately. Moreover, $B_{ij}(\epsilon)$ may be determined by solving

$$\langle L(\epsilon)\, E(\epsilon)\, \varphi_i\,, \varphi_k{}^* \rangle = \sum_{j=1}^{n} B_{ij}(\epsilon) \langle E(\epsilon)\, \varphi_j\,, \varphi_k{}^* \rangle \tag{7.3}$$

These equations are invertible for small ϵ since

$$\det \| \langle E(\epsilon)\, \varphi_j\,, \varphi_k{}^* \rangle \| = 1 + 0(\epsilon).$$

The spectrum of $L(\epsilon)$ restricted to the invariant subspace $E(\epsilon)\mathcal{N}$ consists precisely of the eigenvalues of $B_{ij}(\epsilon)$.

Now let us show that $P_0L_0 = \cdots = P_0L_{k-1} = 0$ implies $P_0E_1 = \cdots = P_0E_{k-1} = 0$. We have ($R_z(\epsilon) = (z - L(\epsilon))^{-1}$)

$$E(\epsilon) = \frac{1}{2\pi i} \int_C R_z(\epsilon)\, dz$$

$$E^{[1]}(\epsilon) = \frac{1}{2\pi i} \int_C R_z{}^2(\epsilon)\, L^{[1]}(\epsilon)\, dz$$

$$E^{[k-1]}(\epsilon) = \frac{1}{2\pi i} \int_C R_z{}^2(\epsilon)\, L^{[k-1]}(\epsilon) + \cdots + (k-1)!\, R_z{}^k(\epsilon) (L^{[1]})^{(k-1)}\, dz.$$

Therefore

$$P_0 E^{[l]}(0) = \frac{1}{2\pi i} \int_C R_z^2(0)\, P_0 L_l + \cdots + (l-1)!\, R_z^l(0)\, P_0 (L_1)^l \, dz$$
$$= 0 \qquad \text{for} \quad l = 1, 2,\ldots, k-1.$$

Substituting these results in (7.3) we get (Note that $P_0{}^* \varphi_m{}^* = \varphi_m{}^*$)

$$\langle P_0 L(\epsilon)\, E(\epsilon)\, \varphi_i\,, \varphi_m{}^* \rangle = \sum_{j=1}^{n} B_{ij}(\epsilon) \langle P_0 E(\epsilon)\, \varphi_j\,, \varphi_m{}^* \rangle,$$

$$\epsilon^k \langle L_k \varphi_i\,, \varphi_m{}^* \rangle + O(\epsilon^{k+1})$$
$$= \sum_{j=1}^{n} [(B_0)_{ij} + \epsilon(B_1)_{ij} + \cdots + \epsilon^k (B_k)_{ij} + \cdots][\delta_{jm} + O(\epsilon^k)].$$

This equation implies $(B_0)_{ij} = 0$, so $(B_1)_{ij} = \cdots = (B_{k-1})_{ij} = 0$ and

$$\langle L_k \varphi_i\,, \varphi_m{}^* \rangle = (B_k)_{ij}\,;$$

and this is the content of equation (7.2).

We have already remarked that if ξ_0 is a solution of (6.2) and if the eigenvalues of the Jacobian of (6.2) at ξ_0 are non-zero, then ξ_0 can be extended to a solution of the full bifurcation equations. What we shall prove here is that the stability of the bifurcating solutions is determined, to lowest nonvanishing order, by the eigenvalues of the Frechet derivative of equations (6.2).

THEOREM 7.2. *Consider a general bifurcation problem $G(\lambda, u) = 0$ where $G_u(0, 0)$ is a Fredholm operator with an l dimensional kernel and Riesz index* 1. *Let the corresponding bifurcation equations, obtained by the Lyapounov-Schmidt procedure (see below), be given by $F(\lambda, v) = 0$, where $v \in R^l$. Suppose that these equations have a one-parameter family of solutions*

$$\lambda = \epsilon^m \sigma_0 (\sigma_0 = \text{const.})$$
$$v = \epsilon^n w, \qquad w = w_0 + \epsilon w_1 + \epsilon^2 w_2 + \cdots$$

and that

$$F(\epsilon^m \sigma_0\,, \epsilon^n w) = \epsilon^k Q(\sigma_0\,, w_0) + O(\epsilon^{k+1})$$

where $k > \max\{m, n\}$. The reduced equations are therefore

$$Q(\sigma_0\,, w_0) = 0$$

where σ_0 is to be chosen conveniently.

Then the linear operator $B(\epsilon) = L(\epsilon) E(\epsilon)$ *given by Lemma* 7.1 *has the form*

$$B(\epsilon) = \epsilon^{k-n} Q_w(\sigma_0, w_0) + O(\epsilon^{k-n+1}) \tag{7.4}$$

where Q_w *denotes the Frechet derivative of the mapping* Q. *The stability of the bifurcating solutions is therefore determined at lowest order by the eigenvalues of the Jacobian of the reduced equations.*

Proof. We begin with a quick review of alternative methods for bifurcation problems. Let $G(\lambda, u) : \mathbb{C} \times \mathscr{E} \to \mathscr{F}$.

We shall assume that $\mathscr{E} \subseteq \mathscr{F}$, with the injection $\mathscr{E} \to \mathscr{F}$ continuous in the $\mathscr{F}$-topology. This set-up is particularly convenient when dealing with nonlinear systems of partial differential equations . Let $L_0 = G_u(0, 0)$ be a Fredholm operator with a finite dimensional kernel $\mathscr{N}_0 \subseteq \mathscr{E}$. The projection onto $\mathscr{N}_0$ is given by

$$P_0 = \frac{1}{2\pi i} \int_C (z - L_0)^{-1} \, dz$$

where C encloses the origin and no other points in the spectrum of L_0. Clearly $P_0 : \mathscr{F} \to \mathscr{E}$, and so we may regard P_0 as a bounded mapping from $\mathscr{E}$ to $\mathscr{E}$ or from $\mathscr{F}$ to $\mathscr{F}$. In the same way, $Q_0 = I - P_0$ is then a bounded mapping from $\mathscr{F}$ to $\mathscr{F}$ or from $\mathscr{E}$ to $\mathscr{E}$. Equation (1.1) can be decomposed into a coupled system

$$Q_0 G(\lambda, u) = 0$$
$$P_0 G(\lambda, u) = 0$$

and we may write $u = P_0 u + Q_0 u = v + \Psi$. We choose Ψ to satisfy the functional equation

$$Q_0 G(\lambda, v + \Psi(\lambda, v)) = 0. \tag{7.5}$$

This equation is easily solved by virtue of the implicit function theorem in a Banach space. In fact, the Frechet derivative at $\lambda = 0$, $v = 0$ is simply $Q_0 L_0$ which, by our assumption that L_0 is a Fredholm operator, is an isomorphism between $Q_0\mathscr{E}$ and $Q_0\mathscr{F}$. Therefore there exists a unique function $\Psi(\lambda, v)$, analytic in λ and v, with values in $Q_0\mathscr{E}$. Furthermore, $\Psi(\lambda, v)$ is uniquely defined for small $|\lambda| + |v|$.

The bifurcation equations are

$$F(\lambda, v) = P_0 G(\lambda, v + \Psi(\lambda, v)) = 0 \tag{7.6}$$

These reduce to a system of l equations in l unknowns if we introduce a basis $\{\varphi_1, \dots, \varphi_l\}$ for $\mathscr{N} = P_0\mathscr{E}$. (We assume here that the Riesz index ν of $\mathscr{N}$ is one

– i.e. that $\mathscr{N} = \{\varphi : L_0\varphi = 0\} = \mathscr{N}_2 = \{\varphi : L_0^2\varphi = 0\}$. The result is still true if $\nu > 1$ – that is, we still get l equations in l unknowns; however a little more care is required in their derivation).

The projection P_0 has the form

$$P_0 u = \sum_{j=1}^{n} c_j(u)\, \varphi_j\, .$$

From the linearity of P_0 we may infer that the $c_j(u)$ are linear functionals on $\mathscr{F}$. They therefore have the form $c_j(u) = \langle u, \varphi_j^*\rangle$ where $\varphi_j^* \in \mathscr{F}^* \subseteq \mathscr{E}^*$. From the relationship $P_0L_0 = L_0P_0$ it is easily proved that $L_0^*\varphi_j^* = 0$; and from the relationship $P_0^2 = P_0$ one sees that $\langle \varphi_i, \varphi_j^*\rangle = \delta_{ij}$. Putting $v = z_1\varphi_1 + \cdots + z_l\varphi_l$ we find that (2.2) is equivalent to the system of equations

$$F_j(\lambda, z_1, ..., z_l) = \langle G(\lambda, v + \Psi(\lambda, v)), \varphi_j^*\rangle = 0 \qquad j = 1,..., l.$$

Now consider the solution curve $(\lambda(\epsilon), u(\epsilon))$. According to Lemma 7.1 it suffices to investigate the operator $P_0G_u(\lambda(\epsilon), u(\epsilon))\, P_0$. In fact, from (7.2) it follows that to lowest order

$$B_{ij}(\epsilon) \equiv \langle P_0G_u(\lambda(\epsilon), u(\epsilon))\, P_0\varphi_i\, , \varphi_j^*\rangle \text{ (lowest order)}$$

The gradient of the bifurcation equations is

$$\begin{aligned} F_v(\lambda, v) &= \frac{\partial}{\partial v} P_0G(\lambda, v + \Psi(\lambda, v)) \\ &= P_0G_u(\lambda, u + \Psi(\lambda, v))[P_0 + \Psi_v] \\ &= P_0G_u(\lambda, u)[P_0 + \Psi_v] \end{aligned}$$

where $u = v + \Psi(\lambda, v)$. Note that Ψ_v is a mapping from the kernel $\mathscr{N}$ to the subspace $Q_0\mathscr{E}$. The offending term here is $P_0G_u(\lambda, u)\, \Psi_v$, but we claim this term vanishes to higher order than the one we are interested in, namely $P_0G_u(\lambda, u)\, P_0$. In fact, it is enough to show that $\Psi_v(0, 0) = 0$, since v and λ both vanish as $\epsilon \to 0$. Differentiating (7.5) with respect to v we get

$$Q_0G_u(\lambda, u)[P_0 + \Psi_v] = 0,$$

and putting $\lambda = v = 0$ we get

$$Q_0L_0[P_0 + \Psi_v(0, 0)] = 0.$$

Since $L_0P_0 = 0$ we have $Q_0L_0\Psi_v(0, 0) = 0$; and, since the range of Ψ_v is the Banach space $Q_0\mathscr{E}$, we have $\Psi_v(0, 0) = 0$.

We have shown that along the solution curve $\{\lambda(\epsilon), u(\epsilon)\}$

$$F_v(\lambda, v) = P_0 G_u(\lambda, u) P_0 + \text{higher order terms.}$$

Let us now determine the lowest order terms of $F_v(\lambda, v)$. Upon introducing the scaling $\lambda = \epsilon^m \sigma$, $v = \epsilon^n w$ we get

$$H(\epsilon, \sigma, w) = F(\epsilon^m \sigma, \epsilon^n w).$$

By hypothesis H has the form

$$H(\epsilon, \sigma, w) = \epsilon^k Q(\sigma, w) + 0(\epsilon^{k+1}).$$

Therefore

$$H_w = \epsilon^n F_v(\epsilon^m \sigma, \epsilon^n w) = \epsilon^k Q_w(\sigma, w) + 0(\epsilon^{k+1}),$$

and

$$F_v(\epsilon^m \sigma, \epsilon^n w) = \epsilon^{k-n} Q_w(\sigma, w) + 0(\epsilon^{k-n+1}).$$

Equating lowest order terms of F_v and $P_0 G_u(\lambda, u) P_0$ we get

$$P_0 G_u(\lambda, u) P_0 = \epsilon^{k-n} Q_w(\sigma, w) + O(\epsilon^{k-n+1}),$$

which completes the proof of Theorem 7.2.

As an application of Theorem 7.2 let us consider the stability of bifurcating solutions at a simple branch point. Suppose that $G(\lambda, 0) \equiv 0$ but that $G(0, u)$ does not vanish in a punctured neighborhood of the origin : $0 < \|u\| < \delta$. Let $G_u(\lambda, 0)\, \varphi(\lambda) = \gamma(\lambda)\, \varphi(\lambda)$ and suppose further that $\gamma(0) = 0$, $\gamma'(0) > 0$. Then it can be shown that the bifurcation equation takes the form $f(\lambda, v) = 0$, where

$$f(\lambda, v) = \gamma'(0)\, \lambda - bv^n + \cdots$$

v being a scalar and b being constant. (See [28]) The reduced equations, obtained by scaling $\lambda = \epsilon^{n-1}\sigma$, $v = \epsilon w$, are

$$Q(\sigma, w) = \gamma'(0)\, \sigma w - bw^n = 0.$$

Taking $\sigma = b$, the solutions of the reduced equations are

$$w = \begin{cases} \pm(\gamma'(0))^{1/n-1} & \text{if } n \text{ is odd} \\ (\gamma'(0))^{1/n-1} & \text{if } n \text{ is even} \end{cases}$$

In either case,

$$Q_w(\sigma, w) = b\gamma'(0)(1 - n).$$

According to Theorem 7.2, the leading term in the critical eigenvalue is

$$\xi(\epsilon) = \epsilon^{n-1} b\gamma'(0)(1 - n) + \cdots$$

while

$$\lambda(\epsilon) = \epsilon^{n-1} b.$$

Therefore

$$\frac{\xi(\epsilon)}{\lambda(\epsilon)} = \gamma'(0)(1 - n) + O(\epsilon)$$

and, for small ϵ, ξ and λ opposite signs. Therefore, $\lambda > 0$ implies $\xi < 0$, etc. Since $\gamma'(0) > 0$ the zero solution is unstable for $\lambda > 0$. We call this the supercritical case. If $\{\lambda(\epsilon), u(\epsilon)\}$ is a solution branch we say that the bifurcation is supercritical when $\lambda(\epsilon) > 0$ and subcritical when $\lambda(\epsilon) < 0$. The stability of the bifurcating solution is determined by the sign of ξ, being stable if ξ is negative and unstable if ξ is positive. Thus, we have shown that supercritical solutions are stable and subcritical solutions are unstable. (See [27], [3], [17]). The reader may go on to the more general case when several of the derivatives $\gamma'(0), \ldots \gamma^{[k]}(0)$ vanish. (A more thorough account of these matters appears in the survey artice. "Recent Progress in Bifurcation Theory," to appear in the memorial volume for E. Rothe).

8. Stability of the Square and Rectangular Solutions

8. *Stability of the Square and Rectangular Solutions.*

Let us consider the stability of the solutions bifurcating in the square and rectangular lattices. We denote the reduced bifurcation equations by

$$Q(\lambda, z) = 0, \tag{8.1}$$

where $Q = (Q_1, Q_2, Q_3, Q_4)$ and $z = (z_1, z_2, z_3, z_4)$.

THEOREM 8.1. *Let* $W(x_1, x_2) = x_1 + x_1(ax_1^2 + bx_2^2)$ *be the function described in Theorem* 6.3; *and put* $W_1(x_1, x_2) = W(x_1, x_2)$, $W_2(x_1, x_2) = W(x_2, x_1)$. *In the case of the square or rectangular lattices the eigenvalues of the Jacobian of* (8.1) *are*

$$0, 0 \textit{ and those of } \frac{\partial(W_1, W_2)}{\partial(x_1, x_2)}.$$

Proof. We have

$$Q_z = \frac{\partial(Q_1, Q_2, Q_3, Q_4)}{\partial(z_1, z_2, z_3, z_4)} = \frac{\partial(Q_1, Q_2, Q_3, Q_4)}{\partial(x_1, x_2; \theta_1, \theta_2)} \frac{\partial(x_1, x_2; \theta_1, \theta_2)}{\partial(z_1, z_2, z_3, z_4)}.$$

Since the bifurcation equations are invariant under the two parameter group $(\theta_1, \theta_2) \to (\theta_1 + \alpha\theta_2 + \beta)$ it suffices to evaluate the Jacobian at $\theta_1 = \theta_2 = 0$. (The eigenvalues are constant along trajectories of the group action). We have (Cf Lemma 6.1)

$$\frac{\partial(Q_1, Q_2, Q_3, Q_4)}{\partial(x_1, x_2; \theta_1, \theta_2)} = \begin{bmatrix} \frac{\partial W_1}{\partial x_1} & \frac{\partial W_1}{\partial x_2} & iW_1 & 0 \\ \frac{\partial W_2}{\partial x_1} & \frac{\partial W_2}{\partial x_2} & 0 & iW_2 \\ \frac{\partial W_1}{\partial x_1} & \frac{\partial W_1}{\partial x_2} & -iW_1 & 0 \\ \frac{\partial W_2}{\partial x_1} & \frac{\partial W_2}{\partial x_2} & 0 & -iW_2 \end{bmatrix}$$

At a solution of the bifurcation equations, $W_1 = W_2 = 0$, so the above reduces to

$$\frac{\partial(Q_1, Q_2, Q_3, Q_4)}{\partial(x_1, x_2, \theta_1, \theta_2)} = \left[\begin{array}{c|cc} J & 0 & 0 \\ & 0 & 0 \\ \hline J & 0 & 0 \\ & 0 & 0 \end{array}\right]$$

where J denotes the Jacobian $\partial(W_1, W_2)/\partial(x_1, x_2)$.

On the other hand,

$$\frac{\partial(x_1, x_2, \theta_1, \theta_2)}{\partial(z_1, z_2, z_3, z_4)} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \frac{-i}{x_1} & 0 & \frac{i}{x_1} & 0 \\ 0 & \frac{-i}{x_2} & 0 & \frac{i}{x_2} \end{bmatrix}.$$

The reader may verify that

$$\frac{\partial(Q_1, Q_2, Q_3, Q_4)}{\partial(z_1, z_2, z_3, z_4)} = \frac{1}{2}\left[\begin{array}{c|c} J & J \\ \hline J & J \end{array}\right] \tag{8.2}$$

Two null vectors of any such matrix are

$$\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}.$$

Moreover, if $\binom{a}{b}$ is an eigenvector of J then

$$\begin{pmatrix} a \\ b \\ a \\ b \end{pmatrix}$$

is an eigenvector of (8.2) with the same eigenvalue.

COROLLARY 8.2. *In the rectangular or square lattice rolls* ($x_1 = 0$ *or* $x_2 = 0$) *are stable* iff $b < a < 0$ *and square or rectangular solutions* ($x_1 = x_2$) *are stable if* $a + b < 0$ *and* $b - a > 0$.

Corollary 8.2 follows immediately from Theorem 8.1, the first row of Table 6.1, and Theorem 7.2. From corollary 8.2 we see that within a fixed lattice, the values of the parameters a and b uniquely select a pattern : either rolls or squares (or rectangles).

9. STABILITY OF SOLUTIONS IN Λ_6

9. *Stability of Solutions in* Λ_6.

The stability analysis of the Λ_6 solutions is a bit more delicate. Let the reduced bifurcation equations be denoted by $Q(\lambda, z) = (Q_1, ..., Q_6)$, where $z = (z_1, ... z_6)$. The quadratic case (6.6) is easily dealt with directly. The equations are generated by cyclic permutation of $Q_1(z) = z_1 - z_2 z_6$ (λ is scaled out). As we have seen,the unique (real) solution is $x_1 = x_2 = x_3 = 1$, or $z_1 = \cdots = z_6 = 1$ (taking $\theta_1 = \theta_2 = \theta_3 = 0$). The Jacobian is

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & -1 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 \\ -1 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \tag{9.1}$$

Do not try to find the eigenvalues of (9.1) by finding the roots of the characteristic polynomial! There is an easier way. First note that the invariant subspaces of (9.1) must be precisely the invariant irreducible subspaces of the symmetry group D_6 which were calculated in §4. This may be proved as follows. If $F(z)$ is covariant with respect to a group representation T_g then it is easily seen that $T_g F'(z) = F'(T_g z)\, T_g$. If now $F(z_0) = 0$ and z_0 is invariant under $\mathscr{G}$, i.e. if $T_g z_0 = z_0$ for all $g \in \mathscr{G}$ then $T_g F'(z_0) = F'(z_0)\, T_g$ for all g. Consequently the invariant irreducible subspaces of $\mathscr{G}$ are precisely the invariant subspaces of $F'(z_0)$.

In the case at hand our hexagonal solutions $z_1 = \cdots = z_6$ are invariant under the full symmetry group D_6 of the lattice Λ_6. The invariant subspaces of D_6 are $V^{(1)}$, $V^{(4)}$, $V^{(5)}$, and $V^{(6)}$, and are given respectively by the first, the second, the third and fourth, and the fifth and sixth columns of the matrix P of (4.4). Actually, it turns out that eigenvectors of (9.1) are precisely the column vectors of P. The eigenvalues are

$$-1, \quad 3, \quad 2, \quad 2, \quad 0, \quad 0. \tag{9.2}$$

THEOREM 9.1. *The hexagonal solutions which appear in the quadratic case* (6.6) *are unstable on both sides of criticality.*

This follows from the fact that there are both positive and negative eigenvalues of the Jacobian of the reduced equations.

Finally, for the case (6.7) we have

THEOREM 9.2. *When the reduced bifurcation equations are of the form* (6.7) *the eigenvalues of the Jacobian of the bifurcation equations are*

$$0, 0, 0 \text{ and those of } \frac{\partial(W_1, W_2, W_3)}{\partial(x_1, x_2, x_3)},$$

where

$$W_1(x_1, x_2, x_3) = x_1 + a(x_2^2 + x_3^2)\, x_1 + bx_1^3$$

and W_2 and W_3 are obtained from W_1 by cyclic permutation of x_1, x_2, x_3.

As we have already pointed out, two of the zero eigenvalues occur as a result of the translational invariance of the original problem (1.1), and they remain identically zero along the branching curve (λ, u). The third eigenvalue above is due to the additional symmetry of the reduced bifurcation equations. Since this symmetry is destroyed by higher terms in the bifurcation equations the last eigenvalue does not remain zero but perturbs to the right or left along the branch curve. Consequently, *the stability of the hexagonal patterns cannot be determined at lowest order in the case* (6.7).

From Table 6.1 we see that a necessary condition for purely hexagonal solutions ($x_1 = x_2 = x_3 \neq 0$) to be stable is that $2a + b < 0$ and $b - a < 0$. The second class of solutions in Λ_6 (*viz.* $x_3 = 0$, $x_1 = x_2$) can never be stable, since the first and third eigenvalues have opposite signs. The first class, the so-called "rolls" can only be stable in Λ_6 if $a < b < 0$.

In closing let us extract one final piece of information about the stability of the purely hexagonal solutions.

THEOREM 9.3. *Let $u \in \Lambda_6$ be the solution of $G(\lambda, u)$ which is invariant under the full symmetry group D_6. Then the (two dimensional) kernel of $G_u(\lambda, u)$ transforms like irreducible representation $T^{(6)}$ (Cf. (4.1)) under the action of D_6. The third eigenvalue of $G_u(\lambda, u)$ which tends to zero at the branch point corresponds to a one dimensional invariant subspace which transforms according to $T^{(4)}$ under the action of D_6.*

Proof. The kernel of $G_u(\lambda, u)$ is spanned by $\partial u/\partial x$ and $\partial u/\partial y$ since $\partial/\partial x$ and $\partial/\partial y$ are the generators of the translation group. Now if r is a rotation in the plane and $\mathbf{x} = (x, y)$, then

$$\nabla[u(r^{-1}\mathbf{x})] = r(\nabla u)(r^{-1}\mathbf{x}).$$

The representation $T^{(6)}(r)$ (Cf (4.1)) is the two dimensional representation of D_6 as the ordinary rotation-reflections in the plane. Now suppose u is invariant under D_6 : that is

$$u(r^{-1}\mathbf{x}) = u(\mathbf{x}) \quad \mathbf{x} \in R^2, \quad r \in D_6 .$$

Then

$$\nabla u(\mathbf{x}) = \nabla[u(r^{-1}\mathbf{x})] = r(\nabla u)(r^{-1}\mathbf{x}),$$

or $(\nabla u)(r^{-1}\mathbf{x}) = r^{-1}(\nabla u)(\mathbf{x})$. This shows that ∇u transforms like $T^{(6)}$.

The reader may show that the Jacobian of the reduced bifurcation equations evaluated at the purely hexagonal solution is

$$F' = -\frac{1}{b+2a}\left[\begin{array}{c|c} J & J \\ \hline J & J \end{array}\right] \qquad \text{where} \quad J = \begin{bmatrix} b & a & a \\ a & b & a \\ a & a & b \end{bmatrix}.$$

If $\{\varphi_1, \dots \varphi_6\}$ denote the basis vectors given by (4.4) (i.e. the column vectors of the matrix P in (4.3)) then

$$F'\varphi_1 = \varphi_1 \qquad F'\varphi_2 = 0 \qquad F'\varphi_3 = 2\left(\frac{a-b}{b+2a}\right)\varphi_3$$

$$F'\varphi_4 = 2\left(\frac{a-b}{b+2a}\right)\varphi_4 \qquad F'\varphi_5 = F'\varphi_6 = 0.$$

The critical mode is thus φ_2. The subspaces $V^{(1)}$, $V^{(4)}$, $V^{(5)}$, $V^{(6)}$ remain invariant along the branch curve, so the critical eigenvalue λ_2 may be treated as a simple eigenvalue associated with the one-dimensional invariant subspace which transforms according to the representation $T^{(4)}$.

10. Gradient Structure of the Bifurcation Equations

As we remarked in the introduction, Busse [1] has shown that the reduced bifurcation equations possess a gradient structure – that is, that they are gradients of a nonlinear function $\mathscr{F}$ on the kernel. Busse was thus able to relate the extrema of $\mathscr{F}$ to stable bifurcating solutions. This same line has been pursued by Sather [25] in connection with stable bifurcating solutions of buckling problems in elasticity. Busse's discovery is interesting because the Boussinesq equations do not possess a variational structure. We show here that the variational structure of the reduced bifurcation equations is a consequence of the group invariance of the original problem and so does not depend on the structure of the equations.

Theorem 10.1. *Let $F_i(z_1,...)$ be the invariant bifurcation equations up to order $k = 1$, 2, or 3 for the lattices Λ_2, Λ_4 or Λ_6. Then on the real submanifold $z_3 = \bar{z}_1$, $z_4 = \bar{z}_2$ (or $z_4 = \bar{z}_1$, $z_5 = \bar{z}_2$, $z_6 = \bar{z}_3$ in the case Λ_6), the F_i are gradients of a function $\mathscr{F}$. In the case Λ_2 or Λ_4*

$$F_1 = \frac{\partial \mathscr{F}}{\partial z_1}, \qquad F_2 = \frac{\partial \mathscr{F}}{\partial z_2}, \qquad F_3 = \frac{\partial \mathscr{F}}{\partial \bar{z}_1}, \qquad F_4 = \frac{\partial \mathscr{F}}{\partial \bar{z}_2},$$

and similar equations hold in the case Λ_6.

Proof. It suffices to show that the Jacobian

$$\frac{\partial F_i}{\partial z_j}$$

is Hermitian symmetric when $z_3 = \bar{z}_1$, $z_4 = \bar{z}_2$. This is clear for the case $k = 1$ since then $F_i = \lambda z_i$, and $\partial F_i/\partial z_j = \lambda\, \delta_{ij}$.

In the case $k = 3$ the Jacobian is

$$\begin{bmatrix} 2az_1z_3 + bz_2z_4 & bz_1z_4 & az_1^2 & bz_1z_2 \\ bz_2z_3 & 2az_2z_4 + bz_1z_3 & bz_2z_1 & az_2^2 \\ az_3^2 & bz_3z_4 & 2az_3z_1 + z_2z_4 & bz_3z_2 \\ bz_4z_3 & az_4^2 & bz_4z_1 & 2az_4z_2 + bz_3z_1 \end{bmatrix}$$

and it is easily seen by inspection that this matrix is Hermitian symmetric when $z_3 = \bar{z}_1$, $z_4 = \bar{z}_2$. (Recall that a and b are real as a consequence of the reality of the original problem). The same proof applies in the case Λ_6 when $k = 2$ or 3. The details are left to the reader.

We remark, in closing, that the result is not true for all orders of the bifurcation equations. The term of degree 4 in the case of the hexagonal lattice, for example, does not possess a gradient structure. In fact, (Cf. (5.7)) the term of 4th degree contains the term

$$F_1 = z_2{}^2 z_5 z_6 + z_2 z_3 z_6{}^2,$$

hence F_2 would be

$$F_2 = z_3{}^2 z_6 z_1 + z_3 z_4 z_1{}^2.$$

We have accordingly

$$\frac{\partial F_1}{\partial z_2} = 2 z_2 \bar{z}_2 \bar{z}_3 + z_3 (\bar{z}_3)^2$$

whereas

$$\frac{\partial F_2}{\partial z_1} = z_3{}^2 \bar{z}_3 + 2 z_1 z_3 \bar{z}_1 .$$

Clearly these terms are not complex conjugates of each other.

11. Concluding Remarks

Group representation theory has proved itself an indispensable tool in modern physics. In quantum mechanics the theories of electron spin, selection rules, and elementary particles are intimately bound up with group representation theory. Mathematically one can think of bifurcation problems as non-linear analogues of the perturbation problems which arise in quantum mechanics. Furthermore, the applicability of group representation theory is not diminished by the non-linearities (Cf. [28]).

In the present paper we have considered a specific class of problems of physical significance to which group-theoretic methods can be applied. Using only the very natural and general hypothesis that the original equations are covariant we have derived a considerable amount of specific information about the structure of the bifurcation equations, and about the possible solutions which may form and their stability.

Our analysis of the stability of the various cellular patterns which may bifurcate at the onset of instability underscores the importance of differentiating between the lattice structure of a solution (i.e. its invariance under a lattice subgroup of translations) and its symmetry within that lattice structure (i.e. its invariance with respect to symmetries of the lattice).

For example, in the case (6.6) the null space is 6 dimensional. The problem can be reduced to a one dimensional null space if one considers the subclass of Λ_6 – periodic solutions which are also invariant under D_6. With respect to this subclass of disturbances of full hexagonal symmetry, the completely hexagonal solution ($z_1 = \cdots = z_6$) is stable above criticality and unstable below criticality. But when tested against all possible disturbances in Λ_6 this hexagonal solution is unstable on both sides of criticality.

Thus reduction of the problem by symmetry arguments to bifurcation at a simple eigenvalue leads to incorrect conclusions. Not only are some solutions *a priori* excluded by this procedure, but the stability of the symmetric solutions is not correctly determined. *The multiplicity of the branch point is an intrinsic aspect of the problem of pattern formation.*

Yet, the problem of pattern formation is far from resolved. Still to be answered are the following questions:

1) Is one pattern uniquely selected on the basis of stability considerations?

2) Can a rigorous justification be given for the preference for solutions of a cellular structure?

3) Can the linearized stability analysis of Λ-periodic solutions be extended to a larger class of disturbances?

Other aspects of the role of group representation theory in the analysis of nonlinear problems have been discussed by Othmer and Scriven [18], [19] and D. Ruelle [24]. Ruelle has treated the Hopf bifurcation theorem from the point of view of symmetry considerations. Finally, L. Michel [15] of I.H.E.S. has done considerable work on the role of group theory in symmetry-breaking perturbations.

Appendix

The Boussinesq equations governing convection in an inviscid fluid are [1], [2], [5]

$$\begin{gathered} \Delta u_k + \delta_{k3}\theta - \frac{\partial p}{\partial x_k} = \frac{1}{\mathscr{P}_r} u_j \frac{\partial u_k}{\partial x_j} \\ \Delta\theta + \mathscr{R} u_3 = u_j \frac{\partial \theta}{\partial x_j} \\ \frac{\partial u_j}{\partial x_j} = 0 \end{gathered} \tag{B.1}$$

where $\mathscr{P}_r = \nu/\kappa$ is the Prandtl number and the Rayleigh number $\mathscr{R}$ is given by $\mathscr{R} = \alpha g h^3/\nu\kappa\,(T_0 - T_1)$. Here g is the acceleration due to gravity; h is the depth of the layer; α is coefficient of thermal expansion; T_0 and T_1 are the temperatures on the lower and upper boundaries; ν and κ are the coefficients of viscosity and thermal conductivity; and is θ the perturbed temperature profile.

If θ is replaced by $\sqrt{R}\theta$ the equations assume the following symmetric structure (where $\lambda = \sqrt{R}$)

$$\begin{vmatrix} \Delta & 0 & 0 & 0 & -\dfrac{\partial}{\partial x_1} \\ 0 & \Delta & 0 & 0 & -\dfrac{\partial}{\partial x_2} \\ 0 & 0 & \Delta & \lambda & -\dfrac{\partial}{\partial x_3} \\ 0 & 0 & \lambda & \Delta & 0 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} & 0 & 0 \end{vmatrix} \begin{vmatrix} u_1 \\ u_2 \\ u_3 \\ \theta \\ p \end{vmatrix} + \begin{vmatrix} -\dfrac{1}{P_r} u_j \dfrac{\partial u_1}{\partial x_j} \\ -\dfrac{1}{P_r} u_j \dfrac{\partial u_2}{\partial x_j} \\ -\dfrac{1}{P_r} u_j \dfrac{\partial u_3}{\partial x_j} \\ -u_j \dfrac{\partial \theta}{\partial x_j} \\ 0 \end{vmatrix} = 0$$

The boundary conditions at $x_3 = 0, 1$ are

$$u_i = \theta = 0 \qquad \text{at a rigid boundary}$$

$$\theta = u_3 = \frac{\partial u_1}{\partial x_3} = \frac{\partial u_2}{\partial x_3} = 0 \qquad \text{at a free boundary.}$$

We now derive the dispersion operator $\mathscr{D}(\lambda, \omega)$ discussed in §2. Setting

$$\begin{aligned} u_k(x, y, z) &= U_k(z)\, e^{i\omega x} \\ \theta(x, y, z) &= \theta(z)\, e^{i\omega x} \\ p(x, y, z) &= P(z)\, e^{i\omega x} \end{aligned}$$

we get the following system of second order system of ordinary differential operators by separation of variables :

$$\mathscr{D}(\lambda, \omega) = \begin{vmatrix} (D^2 - \omega^2) & 0 & 0 & 0 & -i\omega \\ 0 & (D^2 - \omega^2) & 0 & 0 & 0 \\ 0 & 0 & (D^2 - \omega^2) & \lambda & -D \\ 0 & 0 & \lambda & (D^2 - \omega^2) & 0 \\ i\omega & 0 & D & 0 & 0 \end{vmatrix}$$

where $D = d/dx^2$. The operator $\mathscr{D}$ operates on vector valued functions

$$\operatorname{col}(U_1, U_2, U_3, \theta, P)$$

defined on $0 \leqslant x_3 \leqslant 1$. The Hilbert space $\mathscr{H}$ in this case consists of such vector valued functions with components in $L_2(0, 1)$ with the obvious inner product.

In the case of free-free boundary conditions the eigenvalue problem (2.6) may be solved exactly and the neutral stability curve and dispersion relation explicitly computed. We put

$$\begin{aligned} U_1(x_3) &= U_1 \cos \pi x_3 \\ U_2(x_3) &= U_2 \cos \pi x_3 \\ U_3(x_3) &= U_3 \sin \pi x_3 \\ \theta(x_3) &= U_4 \sin \pi x_3 \\ P(x_3) &= U_5 \cos \pi x_3 \end{aligned}$$

Then the eigenvalue problem (2.4) reduces to the algebraic problem

$$\begin{vmatrix} -(\omega^2+\pi^2) & 0 & 0 & 0 & -i\omega \\ 0 & -(\omega^2+\pi^2) & 0 & 0 & 0 \\ 0 & 0 & -(\omega^2+\pi^2) & \lambda & \pi \\ 0 & 0 & \lambda & -(\omega^2+\pi^2) & 0 \\ i\omega & 0 & \pi & 0 & 0 \end{vmatrix} \begin{vmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{vmatrix} = \sigma \begin{vmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{vmatrix}$$

The eigenvalues of this problem are given by the roots of

$$\det \begin{vmatrix} -\sigma-(\omega^2+\pi^2) & 0 & 0 & 0 & -i\omega \\ 0 & -\sigma-(\omega^2+\pi^2) & 0 & 0 & 0 \\ 0 & 0 & -\sigma-(\omega^2+\pi^2) & \lambda & \pi \\ 0 & 0 & \lambda & -\sigma-(\omega^2+\pi^2) & 0 \\ i\omega & 0 & \pi & 0 & -\sigma \end{vmatrix} = 0.$$

This equation, the dispersion relation, reduces to

$$Q\{-\sigma Q^3 - \pi^2 Q^2 + \sigma\lambda^2 Q - \omega^2 Q^2 + \omega^2\lambda^2\} = 0$$

where $Q = -(\sigma + \omega^2 + \pi^2)$. The neutral stability curve is obtained by setting $\sigma = 0$:

$$-(\pi^2 + \omega^2)^3 + \omega^2\lambda^2 = 0,$$

$$\lambda^2 = \frac{(\pi^2+\omega^2)^3}{\omega^2}.$$

The critical wave number and Rayleigh number for the onset of convection are then determined by minimizing λ as a function of ω. We get

$$\omega = \frac{\pi}{\sqrt{2}}, \qquad \lambda^2 = \frac{27\pi^4}{4} = \mathscr{R}.$$

The critical wave vector is given by

$$\begin{bmatrix} i \cos \pi z \\ 0 \\ -\sin \pi z/\sqrt{6} \\ -\sin \pi z/\sqrt{2} \\ 3\pi \cos \pi z/\sqrt{2} \end{bmatrix} e^{i\omega x}$$

This wave vector transforms as $\binom{1}{0}$ under the representation (2.3) and so the null space (2.9) is six dimensional. It also satisfies the transformation property $S_r v = \bar{v}$, where r is a rotation through 180°. Thus, the assumptions (3.1a) and (3.1b) are both satisfied.

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