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## On the free surface of a viscous fluid motion

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We consider a container of fluid with a rod inserted in the centre. As the rod rotates the surface of the fluid forms a curved surface whose shape correctly balances the forces of gravity, internal stress, atmospheric pressure, and surface tension. The surface of the fluid is depressed in the neighbourhood of the rod in the case of a Newtonian fluid, but the fluid may climb along the rod in the case of non-Newtonian fluids. In recent work by D. D. Joseph and R. S. Fosdick this free surface problem has been treated quantitatively by virtue of a formal perturbation series in which the solution is developed in powers of the angular velocity of the rod. The purpose of the present paper is to give a rigorous proof of convergence in the case of a Newtonian fluid. The method here may possibly be applied to other free surface problems – for example, surface waves of a viscous fluid.

The convergence proof provides, to our knowledge, the first rigorous existence theorem for a free surface problem in the theory of viscous fluids. The proof also raises a novel problem in the theory of elliptic systems of partial differential equations. By means of the implicit function theorem, the question of convergence is reduced to that of obtaining *a priori* estimates for an elliptic boundary value problem. That problem is formulated on a domain with a ridge where the fluid surface meets the rod. In addition, the type of boundary conditions prescribed differ on the free surface and on the rod. In general, the solution to such a mixed boundary value problem is not smooth at the ridge, even if the boundary data is smooth. The solution will be smooth, however, if the boundary data satisfy certain consistency conditions which make it compatible with the given set of partial differential equations. The consistency conditions in question here are a set of linear relations between various derivatives of the boundary and inhomogeneous data. It is shown that if one makes the assumption that the wetting angle of the surface is zero, the consistency conditions are invariant under the full nonlinear equations of the free surface problem. This makes it possible to consider the original problem on a smaller function space – namely the subclass of functions satisfying the appropriate consistency conditions – and in this subclass one can apply the implicit function theorem and obtain the required *a priori* estimates. The solution thus obtained is regular up to the ridge.

## 1. INTRODUCTION

In two recent papers Joseph, Beavers & Fosdick (1973) and Joseph & Fosdick (1973) have considered the following free surface problem both theoretically and experimentally. A fluid rests in a container with a rod inserted in the centre (figures 1, 2) of radius  $a$  which rotates slowly with angular velocity  $\epsilon$ . The surface of the fluid is free to adopt the appropriate shape which correctly balances out the forces of gravity, centripetal accelerations, internal stress, atmospheric pressure, and surface tension. A perturbation scheme is developed which expresses the solution to the problem in powers of  $\epsilon$ .

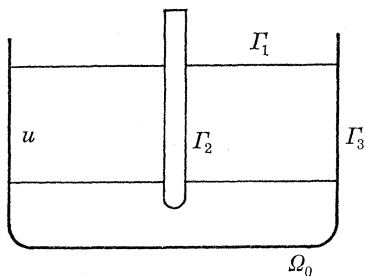


FIGURE 1

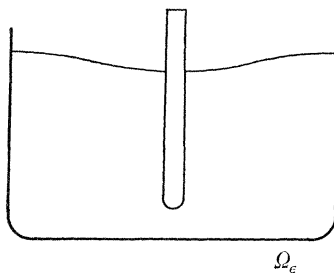


FIGURE 2

Although their perturbation analysis is purely formal, a quite general class of fluids is considered, including non-Newtonian fluids (specifically, simple fluids). Their work provides important theoretical and experimental tools for the study of non-Newtonian fluids.

The purpose of the present paper is to provide a mathematical proof of the convergence of the perturbation series for the case of a Newtonian fluid – that is, a fluid whose stress tensor is given by

$$T = -pI + 2D, \quad (1.1)$$

where  $D$  is the deformation tensor. We thus provide a rigorous existence theorem for a free surface problem for a viscous fluid motion.

The mathematical problem at hand presents a number of novel features. In the first place, we have here a ‘domain perturbation’ problem. That is, the domain on which the equations of fluid mechanics must hold varies with the parameter  $\epsilon$ . In the second place, the domain in question has a ridge where the fluid surface meets the rod, so that we are forced to consider boundary value problems for partial differential equations in which the boundary of the domain is not smooth.

The problem of treating the free surface problem for simple fluids presents mathematical difficulties of a much more profound nature, and we shall not deal with that aspect of the problem here.

A second fundamental assumption in the present work, in addition to the constitutive law (1.1), is that the wetting angle is zero. The wetting angle is the angle

of the normal to the free surface with the wall of the rod at the line of contact. This assumption not only leads to a considerable simplification of the mathematical problem, but may also fundamentally affect the regularity of the solution at the line of contact. In the case of zero wetting angle the solutions obtained are regular everywhere, even at the contact line: the fluid velocities are of Hölder class  $C_{2+\alpha}$ , the pressure is of class  $C_{1+\alpha}$ , and the height function  $z = h(r)$  of the free surface is of class  $C_{3+\alpha}$ .

In the case of non-zero wetting angle the velocities may suffer discontinuities in the first derivatives at the line of contact. A heuristic argument to this effect, due to D. Joseph, will be presented in § 8. The assumption of zero wetting angle, though restrictive, does not exclude cases of physical interest; and, in fact, Joseph and his co-workers have treated the walls of the rod chemically in order to obtain a zero wetting angle in their experiments.

The perturbation scheme we use is essentially that used by Joseph *et al.*, although it does differ in some respects. Since the region occupied by the fluid must be determined as part of the solution to the problem, the region occupied by the fluid is mapped on to a domain which remains fixed throughout the perturbation scheme. The equations are then expressed in tensor notation on the fixed domain; and we arrive at a system of partial differential equations which depends on the parameter  $\epsilon$ , but which is defined on a fixed domain. The same method was employed by Garaebian & Schiffer (1952/3) in computing Hadamard's variational formulae.

The fixed domain in question is  $\Omega_0$  shown in figure 1. By the implicit function theorem the question of the convergence of the perturbation series may be reduced to that of obtaining suitable *a priori* estimates for the solution of a linear boundary value problem on  $\Omega_0$ . This problem is complicated by the fact that the domain contains a ridge at the lines of contact  $z = 0, r = a$  and  $z = 0, r = R$ , and that the boundary conditions to be prescribed are different on the rod and on the free surface. In general the solution to such a boundary value problem will not be smooth up to the ridge unless the boundary data and inhomogeneous term satisfy a certain set of consistency conditions.

Consider, for example, the Dirichlet problem

$$\begin{aligned} \Delta u &= f, \quad \text{on } x \geq 0, \quad y \geq 0, \\ u(x, 0) &= g_1(x), \\ u(0, y) &= g_2(y). \end{aligned}$$

Let us assume that  $f \in C_\alpha(\bar{Q})$  ( $\bar{Q} = \{(x, y): x \geq 0, y \geq 0\}$ ) and that  $g_1, g_2 \in C_{2+\alpha}(R^+)$ . If  $u \in C_{2+\alpha}(\bar{Q})$  then necessarily at the corner  $x = 0, y = 0$  we must have

$$g_1(0+) = g_2(0+) \tag{1.2}$$

and 
$$g_1''(0+) + g_2''(0+) = f(0, 0). \tag{1.3}$$

It has been shown in Volkov (1965) that the consistency conditions (1.2) and (1.3) are also sufficient for  $u$  to belong to  $C_{2+\alpha}(\bar{Q})$ . Boundary value problems for elliptic

equations on domains with corners, and with mixed boundary conditions, have been investigated by Volkov (1965), Shamir (1963) and Peetre (1961).

The essential facts to be used in the present case are that (i) for the boundary value problem to be considered, consistency conditions analogous to (1.2), (1.3) exist which, if satisfied, ensure that the solution will be regular up to the ridge; and (ii) these consistency conditions are invariant under the *nonlinear* operations represented by the equations and boundary conditions. These two features are present in the case of zero wetting angle, and it is for this reason that the solutions obtained are regular at the line of contact.

2. MATHEMATICAL FORMULATION OF THE PHYSICAL PROBLEM

Let  $\Omega_0$  and  $\Omega_\epsilon$  be the domains shown in figures 1 and 2 respectively. The regions  $\Omega_0$  and  $\Omega_\epsilon$  differ in that the top boundary of  $\Omega_0$  is given by

$$\Gamma_1 = \{(r, z): z = 0, a \leq r \leq R\}$$

while that of  $\Omega_\epsilon$  is given by

$$\Gamma_1^\epsilon = \{(r, z): a \leq r \leq R, z = h(r, \epsilon)\}.$$

The function  $h(r, \epsilon)$  is the level of the free surface formed by the fluid when the inner rod rotates with angular velocity  $\epsilon$ . The function  $h$  is subject to the condition

$$\int_a^R r h(r, \epsilon) dr = 0, \tag{2.1}$$

which is a consequence of the conservation of mass and the incompressibility of the fluid.

The fluid is acted upon by gravity, the gravitational potential being given by

$$\Phi = gz.$$

The equations of motion for the viscous fluid are

$$\left. \begin{aligned} \nu \Delta v_1 - (1/\rho) \partial P / \partial x^i &= v_j \partial v_i / \partial x^j, \\ \partial v_i / \partial x^i &= 0, \end{aligned} \right\} \text{ in } \Omega_\epsilon \tag{2.2}$$

where  $P = p + \Phi$  and  $p$  is the hydrodynamic pressure. The boundary conditions satisfied by the fluid velocities on the side of the rod are

$$\mathbf{v} \cdot \hat{r} = 0, \quad \mathbf{v} \cdot \hat{k} = 0, \quad \mathbf{v} \cdot \hat{\theta} = \epsilon \tag{2.3}$$

and on the container, 
$$\mathbf{v} = 0. \tag{2.4}$$

In (2.3) the vectors  $\hat{r}$ ,  $\hat{k}$  and  $\hat{\theta}$  denote the unit vectors in the radial, vertical, and tangential directions. The boundary conditions on the free surface  $z = h(r, \epsilon)$  are

$$\begin{aligned} \mathbf{v} \cdot \hat{n} &= 0, \\ (-pI + 2\nu D) \cdot \hat{n} &= -p_a \hat{n} + \sigma J \hat{n}. \end{aligned} \tag{2.5}$$

The quantities appearing above are defined as follows:

$v_i$  = velocity components in Cartesian coordinates,

$\hat{n}$  = outward normal to the free surface,

$D$  = deformation tensor =  $\frac{1}{2}\partial v_i/\partial x^j + \partial v_j/\partial x^i$  (in Cartesian coordinates),

$\nu$  = coefficient of viscosity (taken to be unity from now on),

$\rho$  = density (also taken to be unity),

$p_a$  = atmospheric pressure,

$p$  = hydrodynamic pressure,

$\sigma$  = coefficient of surface tension,

$J$  = mean curvature of free surface =  $\frac{1}{r} \left( \frac{rh'}{\sqrt{(1+(h')^2)}} \right)'$ ,

*The hydrostatic case*

Let us consider the case where  $\epsilon = 0$ . When  $\epsilon = 0$  the fluid is motionless, so  $D = 0$  and the equations (2.2) reduce to

$$\partial P/\partial x^i = 0,$$

hence

$$P = p + \Phi = A = \text{const}$$

and

$$p = A - gz.$$

The boundary condition (2.5) in this case is

$$\begin{aligned} -p &= -p_a + \sigma J, \\ -A + gh &= -p_a + \sigma \frac{1}{r} \left( \frac{rh'}{\sqrt{(1+(h')^2)}} \right). \end{aligned}$$

Applying condition (2.1) together with the boundary conditions

$$h'(a) = h'(R) = 0 \quad (\text{zero wetting angle})$$

we see that  $A = p_a$ . Thus  $A$  is uniquely determined and the free surface in the hydrostatic case is given by  $z = h(r)$ , where

$$\begin{aligned} \frac{\sigma}{r} \left( \frac{rh'}{\sqrt{(1+(h')^2)}} \right)' - gh &= 0, \\ h'(a) = h'(R) &= 0. \end{aligned}$$

The unique solution of this boundary value problem is  $h \equiv 0$ , so when  $\epsilon = 0$  the fluid occupies the region  $\Omega_0$ . Since  $h = 0$  when  $\epsilon = 0$  we replace  $h$  by  $\epsilon h$  in the remaining calculations.

3. DOMAIN PERTURBATIONS

Let  $\Omega_0$  and  $\Omega_\epsilon$  be the domains described in § 2. We construct a transformation  $W$  from  $\Omega_0$  to  $\Omega_\epsilon$  which, in a neighbourhood of  $z = 0$ , takes the form

$$W(r, \theta, z) = (r \cos \theta, r \sin \theta, z + \epsilon h(r, \epsilon)). \tag{3.1}$$

The cylindrical coordinates  $(r, \theta, z)$  assume the ranges  $a \leq r \leq R$ ,  $0 \leq \theta \leq 2\pi$ , and  $z_0 \leq z \leq 0$ , where  $z_0$  is some negative quantity, but is greater than the depth of the

rod. Although it is not necessary, it will simplify later arguments if we choose  $W$  so that it is volume preserving. The construction of such a transformation  $W$  is obtained by integration of an appropriate divergence – free vector field. The details are given in the appendix.

In carrying out the tensor calculations below it will be convenient to use the notation  $x^1 = r, x^2 = \theta, x^3 = z$  in the base domain  $\Omega_0$ . The fluid motion takes place in the region  $\Omega_\epsilon$ , but via the transformation  $W$  we can write the equations on  $\Omega_0$ , using that region as a fixed coordinate domain.

The metric tensor on  $\Omega_0$  is given by

$$ds^2 = g_{ij} dx^i dx^j,$$

where

$$g_{ij} = \frac{\partial W}{\partial x^i} \frac{\partial W}{\partial x^j}.$$

Let  $\mathcal{U}$  be the region  $\Omega_0 \cap \{z_0 < z \leq 0\}$ . In  $\mathcal{U}$  the metric tensor is given by

$$ds^2 = (1 + \epsilon^2 h'^2) dr^2 + r^2 d\theta^2 + dz^2 + \epsilon h' dr dz,$$

so that

$$g_{ij} = \begin{pmatrix} 1 + \epsilon^2 h'^2 & 0 & \epsilon h' \\ 0 & r^2 & 0 \\ \epsilon h' & 0 & 1 \end{pmatrix}. \tag{3.2}$$

The tensor  $g^{ij}$  is given in  $\mathcal{U}$  by

$$g^{ij} = \begin{pmatrix} 1 & 0 & -\epsilon h' \\ 0 & r^{-2} & 0 \\ -\epsilon h' & 0 & 1 + \epsilon^2 h'^2 \end{pmatrix}. \tag{3.3}$$

Throughout  $\Omega_0$  we have

$$g = \det \|g_{ij}\| = r^2 \tag{3.4}$$

since the transformation  $W$  is volume preserving.

If  $V$  is a vector field on  $\Omega_\epsilon$  then the components  $v^i$  of  $V$  form a contravariant quantity on  $\Omega_0$ . These are given by

$$v^i = g^{ij} \left( V \frac{\partial W}{\partial x^j} \right).$$

For example, let us compute the components  $n^i$  of the normal vector  $\mathbf{n}$  to the free surface. In physical space  $\mathbf{n}$  is given by

$$\mathbf{n} = \frac{-\epsilon \cos \theta \hat{i} - \epsilon \sin \theta \hat{j} + \hat{k}}{\sqrt{(1 + \epsilon^2 h'^2)}},$$

while

$$\left. \begin{aligned} W_1 &= \partial W / \partial r = \cos \theta \hat{i} + \sin \theta \hat{j} + \epsilon h' \hat{k}, \\ W_2 &= \partial W / \partial \theta = -r \sin \theta \hat{i} + r \cos \theta \hat{j}, \\ W_3 &= \partial W / \partial z = \hat{k}. \end{aligned} \right\} \tag{3.5}$$

Then

$$n_1 = \mathbf{n} \cdot W_1 \quad \text{and} \quad n^i = g^{ij} \mathbf{n} \cdot W_j;$$

so

$$n_1 = 0, \quad n_2 = 0, \quad n_3 = \frac{1}{\sqrt{(1 + \epsilon^2 h'^2)}}, \tag{3.6}$$

and

$$n^1 = -\frac{\epsilon h'}{\sqrt{(1 + \epsilon^2 h'^2)}}, \quad n^2 = 0, \quad n^3 = \sqrt{(1 + \epsilon^2 h'^2)}. \tag{3.7}$$

We now write the boundary value problem (2.2), (2.3), (2.4) in tensor notation on the coordinate domain  $\Omega_0$ . Letting  $P = p + \Phi$ , where  $\Phi = g(x^3 + \epsilon h)$ , we obtain

$$v^j v^i_{,j} = -g^{ij} \frac{\partial P}{\partial x^j} + 2g^{jk} (D_j^i)_{,k}, \tag{3.8}$$

$$v^i_{,i} = 0, \tag{3.9}$$

where

$$v_i = g_{ij} v^j, \quad D_j^i = g^{im} D_{mj}, \\ D_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}).$$

The notation  $( )_{,k}$  denotes covariant differentiation.

The boundary conditions (2.3) on the rod become

$$\left. \begin{aligned} g_{ij} v^i r^j &= 0, \\ g_{ij} v^i \theta^j &= \epsilon, \\ g_{ij} v^i k^u &= 0, \end{aligned} \right\} \tag{3.10}$$

where  $r^j$ ,  $\theta^j$ , and  $k^j$  are the components of the vectors  $\hat{r}$ ,  $\hat{\theta}$ , and  $\hat{k}$ . Finally, the boundary conditions on the free surface in tensor notation are

$$v^i n_i = g_{ij} v^i n^j = 0 \tag{3.11}$$

and

$$(-pg_{ij} + 2D_{ij}) n^j = -p_a n_i + \sigma J n_i. \tag{3.12}$$

Conditions (3.11) and (3.12) hold on  $\Gamma_1$ .

The system of equations (3.8)–(3.12) constitute a perturbation problem on the fixed domain  $\Omega_0$ . In the next section we calculate the explicit form of these equations in  $\alpha$ .

#### 4. EXPLICIT CALCULATION OF THE PERTURBATION SCHEME

As we mentioned in the introduction, the nonlinear equations (3.8)–(3.12) possess certain invariance properties which are of crucial importance in obtaining the appropriate *a priori* estimates and regularity results. In order to demonstrate these invariance properties we require an explicit evaluation of the equations and boundary conditions in a neighbourhood of the ridges  $z = 0$ ,  $r = a, R$ .

For the convenience of the reader we list below the definitions of the Christoffel symbols and covariant derivatives.

*Christoffel symbols*

$$\Gamma_{ijm} = \frac{1}{2} \left( \frac{\partial g_{jm}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m} \right), \\ \Gamma_{ij}^l = g^{lm} \Gamma_{ijm}.$$

*Covariant derivatives*

$$u^i_{,j} = \frac{\partial u^i}{\partial x^j} + \Gamma_{mj}^i u^m, \tag{4.1}$$

$$(\sigma^i_j)_{,k} = \frac{\partial \sigma^i_j}{\partial x^k} - \Gamma_{jk}^r \sigma_r^i + \Gamma_{sk}^i \sigma_j^s. \tag{4.2}$$

Given the metric tensor  $ds^2$  in  $\mathcal{U}$  in (3.2) we get the following Christoffel symbols in  $\mathcal{U}$

$$\Gamma_{ij}^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (4.3a)$$

$$\Gamma_{ij}^2 = \begin{pmatrix} 0 & 1/r & 0 \\ 1/r & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (4.3b)$$

$$\Gamma_{ij}^3 = \begin{pmatrix} \epsilon h'' & 0 & 0 \\ 0 & \epsilon r h' & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (4.3c)$$

Below we list explicitly the equations and boundary conditions in the region  $\mathcal{U}$ . Since the invariance properties in question occur at the contact lines, we only need the explicit form of the equations in a neighbourhood of the contact line.

$$\begin{aligned} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v^1}{\partial r} \right) + \frac{\partial^2 v^1}{\partial z^2} - \frac{v^1}{r^2} - \epsilon \left( 2h' \frac{\partial^2 v^1}{\partial r \partial z} + h'' \frac{\partial v^1}{\partial z} - \frac{h'}{r} \frac{\partial v^1}{\partial z} \right) \\ + \epsilon^2 h'^2 \frac{\partial^2 v^1}{\partial v^2} - \frac{\partial p}{\partial r} + \epsilon h' \frac{\partial p}{\partial z} - v^1 \frac{\partial v^1}{\partial r} - v^3 \frac{\partial v^1}{\partial z} - r(v^2)^2 = 0, \end{aligned} \quad (4.4a)$$

$$\begin{aligned} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v^2}{\partial r} \right) + \frac{\partial^2 v^2}{\partial z^2} + \frac{2}{r} \frac{\partial v^2}{\partial r} - \epsilon h' \left( \frac{\partial^2 v^2}{\partial r \partial z} + \frac{3}{r} \frac{\partial v^2}{\partial z} \right) - \epsilon h'' \frac{\partial v^2}{\partial z} \\ + \epsilon^2 h'^2 \frac{\partial^2 v^2}{\partial z^2} - v^1 \frac{\partial v^2}{\partial r} - v^3 \frac{\partial v^2}{\partial z} - \frac{2}{r} v^1 v^2 = 0, \end{aligned} \quad (4.4b)$$

$$\begin{aligned} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v^3}{\partial r} \right) + \frac{\partial^2 v^3}{\partial z^2} + \epsilon \left( 2h'' \frac{\partial v^1}{\partial r} - h'' \frac{\partial v^3}{\partial z} + h''' v^1 - 2h' \frac{\partial^2 v^3}{\partial r \partial z} \right. \\ \left. - \frac{h'}{r} \frac{\partial v^3}{\partial z} + \frac{h'}{r^2} v^1 + \frac{h''}{r} v^1 \right) + \epsilon^2 \left( h'^2 \frac{\partial^2 v^3}{\partial z^2} - 2h' h'' \frac{\partial v^1}{\partial z} \right) - v^1 \frac{\partial v^3}{\partial r} \\ - v^3 \frac{\partial v^3}{\partial z} - \epsilon h'' (v^1)^2 - \epsilon r h' (v^2)^2 - \frac{\partial p}{\partial z} + \epsilon h' \frac{\partial p}{\partial r} - \epsilon^2 h'^2 \frac{\partial p}{\partial z} = 0, \end{aligned} \quad (4.4c)$$

$$\frac{1}{r} \frac{\partial}{\partial r} (r v^1) + \frac{\partial v^3}{\partial z} = 0. \quad (4.4d)$$

The boundary conditions are

$$v^1 = 0, \quad v^2 = \frac{\epsilon}{\alpha}, \quad v^3 = 0 \quad (4.5a)$$

on  $I_2$ , the lateral surface of the inner rod; and

$$v^1 = v^2 = v^3 = 0 \quad (4.5b)$$

on  $\Gamma_3$ , the surface of the container. On the free surface,  $\Gamma_1 = \{(r, 0) : a \leq r \leq R\}$ , the boundary conditions are

$$\frac{1}{2} \left[ \frac{\partial v^1}{\partial z} + \epsilon h' \left( \frac{\partial v^3}{\partial z} + \frac{\partial v^1}{\partial r} \right) + \epsilon^2 h'^2 \frac{\partial v^1}{\partial z} + \epsilon h'' v^1 \right] - \epsilon h' D_{11} + \epsilon^2 (h')^2 D_{13} = 0, \quad (4.6a)$$

$$\frac{1}{2} \frac{\partial v^3}{\partial z} - \frac{\epsilon h'}{r^2} D_{21} + \frac{\epsilon^2 h'^2}{r^2} D_{23} = 0, \quad (4.6b)$$

$$v^3 = 0, \quad (4.6c)$$

$$\sigma \frac{1}{r} \left( \frac{\epsilon r h'}{\sqrt{1 + \epsilon^2 h'^2}} \right)' - \epsilon g h = p_a - P - 2\epsilon h' D_{31} + 2 \frac{\partial v^3}{\partial z} + 2\epsilon h' \frac{\partial v^1}{\partial z} + 2\epsilon^2 h'^2 D_{33}. \quad (4.7)$$

Let us sketch the derivation of equations (4.4)–(4.7). The continuity relation  $v^i_{,i} = 0$ , expressed by (4.4d), can also be written

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{g} v^i) = 0.$$

Since the coordinate transformation is volume preserving,  $\sqrt{g} = r$  everywhere, and the continuity equation is invariant under the transformation.

On the rod  $r = a$  we have the boundary conditions (3.10). It is easily seen that on  $r = a$

$$k^j = \delta_3^j, \quad r^j = \delta_1^j \quad \text{and} \quad \theta^j = \delta_2^j/a$$

so that the boundary conditions (4.5d) follow immediately (recall that  $h'(a) = 0$ ). Equations (4.5b) follow in a similar way. The condition (3.11) reduces immediately to (4.6c) in view of the fact that  $n_1 = n_2 = 0$ . Conditions (4.6a, b) are obtained from (3.12) by setting  $i = 1, 2$  respectively. We first lower the index  $i$  in (3.12) to obtain

$$-p n_i + 2D_{ij} n^j = -p_a n_i + \sigma J n_i. \quad (4.8)$$

When  $i = 1$  we get

$$D_{1j} n^j = 0,$$

$$\sqrt{(1 + \epsilon^2 h'^2)} D_{13} - \frac{\epsilon h'}{\sqrt{(1 + \epsilon^2 h'^2)}} D_{11} = 0,$$

$$D_{13} + \epsilon^2 h'^2 D_{13} - \epsilon h' D_{11} = 0. \quad (4.9)$$

The deformation tensor  $D_{ij}$  is given by

$$D_{ij} = \frac{1}{2} \begin{pmatrix} 2 \frac{\partial v_1}{\partial r} - 2\epsilon h'' v_3 & \frac{\partial v_2}{\partial r} - \frac{2}{r} v_2 & \frac{\partial v_1}{\partial z} + \frac{\partial v_3}{\partial r} \\ \frac{\partial v_2}{\partial r} - \frac{2}{r} v_2 & -2\epsilon r h' v_3 & \frac{\partial v_2}{\partial z} \\ \frac{\partial v_1}{\partial z} + \frac{\partial v_3}{\partial r} & \frac{\partial v_2}{\partial z} & 2 \frac{\partial v_3}{\partial z} \end{pmatrix}. \quad (4.10)$$

Furthermore

$$\left. \begin{aligned} v_1 &= (1 + \epsilon^2 h'^2) v^1 + \epsilon h' v^3, \\ v_2 &= r^2 v^2, \\ v_3 &= v^3 + \epsilon h' v^1. \end{aligned} \right\} \quad (4.11)$$

Using these relations we get

$$\begin{aligned} D_{13} &= \frac{1}{2} \left[ \frac{\partial v_1}{\partial z} + \frac{\partial v_1}{\partial r} \right] = \frac{1}{2} \left[ \frac{\partial}{\partial z} ((1 + \epsilon^2 h'^2) v^1 + \epsilon h' v^3) + \frac{\partial}{\partial r} (v^3 + \epsilon h' v^1) \right] \\ &= \frac{1}{2} \left[ \frac{\partial v^1}{\partial z} + \frac{\partial v^3}{\partial r} + \epsilon^2 h'^2 \frac{\partial v^1}{\partial z} + \epsilon h' \frac{\partial v^3}{\partial z} + \epsilon h' \frac{\partial v^1}{\partial r} + \epsilon h'' v^1 \right]. \end{aligned}$$

The boundary condition (4.6*a*) now follows from the above expression for  $D_{13}$  and from the fact that  $v^3 \equiv 0$  on  $z = 0$ . Condition (4.6*b*) is obtained in a similar fashion.

To obtain (4.7) we set  $p = P - g(z + \epsilon g h)$  in (4.8) and then put  $i = 3$  and  $z = 0$ .

A complete derivation of the momentum equations (4.4) would require too much space. The third equation (4.4*c*) is the critical one for our work, so we shall sketch its derivation.

Let us begin by recalling two general rules of tensor analysis:

(i) the operations of raising or lowering indices commute with covariant differentiation

$$g_{kj}(u^j)_{,i} = (g_{kj}u^j)_{,i} = u_{k,i}.$$

(ii)  $u_{i,jk} = u_{i,kj}$  since the metric tensor in Euclidean space has zero curvature.

Using these two facts we show that

$$2g^{jk}(D_j^i)_{,k} = g^{jk}v^i_{,jk}.$$

In fact,  $2g^{jk}(D_j^i)_{,k} = 2g^{jk}g^{im}(D_{mj})_{,k} = g^{jk}g^{im}v_{m,jk} + g^{jk}g^{im}v_{j,mk}$ .

However,  $g^{jk}g^{im}v_{j,mk} = g^{jk}g^{im}v_{j,km} = g^{im}(g^{jk}v_{j,k})_{,m} = 0$ ,

since  $g^{jk}v_{j,k} = g^{kj}v_{j,k} = \delta_m^k g^{mj}v_{j,k} = \delta_m^k v^m_{,k} = v^k_{,k} = 0$ .

Therefore, equations (3.7) may be written

$$g^{jk}v^i_{,jk} - g^{ij} \frac{\partial F}{\partial x^j} = v^j v^i_{,j}.$$

The most difficult term to calculate is the first. The second covariant derivative of  $v^i$  is given by

$$v^i_{,jk} = \frac{\partial^2 v^i}{\partial x^j \partial x^k} + \Gamma_{mj}^i \frac{\partial v^m}{\partial x^k} - \Gamma_{jk}^m \frac{\partial v^i}{\partial x^m} + \Gamma_{mk}^i \frac{\partial v^m}{\partial x^j} + \left( \frac{\partial \Gamma_{mj}^i}{\partial x^k} - \Gamma_{jk}^s \Gamma_{ms}^i + \Gamma_{sk}^i \Gamma_{mj}^s \right) v^m.$$

We represent  $v^3_{,jk}$  as a matrix:

$$v^3_{,jk} = \begin{bmatrix} \frac{\partial^2 v^3}{\partial r^2} + 2\epsilon h'' \frac{\partial v^1}{\partial r} - \epsilon h'' \frac{\partial v^3}{\partial z} + \epsilon h''' v^1 & \epsilon h' r \frac{\partial v^2}{\partial r} & \frac{\partial^2 v^3}{\partial r \partial z} + \epsilon h'' \frac{\partial v^1}{\partial z} \\ \epsilon h' \frac{\partial v^2}{\partial r} & r \frac{\partial v^3}{\partial r} - \epsilon r h' \frac{\partial v^3}{\partial z} + \epsilon h' v^1 + \epsilon r h'' v^1 & \epsilon r h' \frac{\partial v^2}{\partial z} \\ \frac{\partial^2 v^3}{\partial r \partial z} + \epsilon h'' \frac{\partial v^1}{\partial z} & \epsilon r h' \frac{\partial v^3}{\partial z} & \frac{\partial^2 v^3}{\partial z^2} \end{bmatrix}.$$

We leave it to the reader to complete the verification of equation (4.4*c*) if his scepticism so dictates.

5. THE IMPLICIT FUNCTION THEOREM

Our aim in this paper is to prove the following theorem:

**THEOREM 5.1.** *The nonlinear boundary value problem (4.4)–(4.7) has a solution  $(v^i, P, h)$  which may be expanded in a convergent power series in  $\epsilon$  for sufficiently small  $\epsilon$ . The vector field  $v^i$  is of class  $C_{2+\alpha}(\Omega_0)$ ,  $P$  is of class  $C_{1+\alpha}(\Omega_0)$ , and  $h$  is of class  $C_{3+\alpha}(\Gamma_1)$ .*

To prove theorem 5.1 we shall apply the implicit function theorem in a Banach space to reduce the nonlinear problem to one of attaining *a priori* estimates of a certain linear boundary value problem.

**IMPLICIT FUNCTION THEOREM.** *Let  $\mathcal{E}$  and  $\mathcal{F}$  be complex Banach spaces and let  $F: \mathcal{E} \times \mathbb{C} \rightarrow \mathcal{F}$ , where  $\mathbb{C}$  denotes the complex numbers. Suppose that  $F$  is Frechet differentiable in  $(u, \epsilon) \in \mathcal{E} \times \mathbb{C}$ . Suppose in addition that  $F(0, 0) = 0$  and  $F_u(0, 0)$  is an isomorphism from  $\mathcal{E}$  to  $\mathcal{F}$ . Then there exists an analytic  $\mathcal{E}$ -valued function  $u(\epsilon)$  defined for sufficiently small  $\epsilon$  such that  $F(u(\epsilon), \epsilon) \equiv 0$ .*

For a proof and discussion of this theorem, see Agmon, Douglis & Nirenberg (1964).

Let us now describe the appropriate Banach spaces  $\mathcal{E}$  and  $\mathcal{F}$  for the problem at hand. As before,  $\Omega_0$  denotes the region given in § 2. A function  $f$  of Hölder class  $C_\alpha$  on  $\Omega_0$  is a continuous function on  $\bar{\Omega}_0$  for which  $\|f\|_\alpha < +\infty$ , where

$$\|f\|_\alpha = \sup_{\Omega_0} |f(x)| + \sup_{x, y \in \Omega_0} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

A function of class  $k + \alpha$  ( $k = 0, 1, \dots$ ) is one which, together with all derivatives up to order  $k$ , belongs to class  $C_\alpha$ . The norm  $\|f\|_{k+\alpha}$  is obtained by summing the  $C_\alpha$  norms of  $f$  and all its derivatives up to order  $k$ . The spaces  $C_{k+\alpha}$  are Banach algebras: that is, they are Banach spaces for which  $\|fg\|_{k+\alpha} \leq \|f\|_{k+\alpha} \|g\|_{k+\alpha}$ .

Let  $\hat{\mathcal{E}}_1$  denote the Banach space of divergence-free axisymmetric vector fields of class  $C_{2+\alpha}(\Omega_0)$ . The norm of a vector field in  $\hat{\mathcal{E}}_1$  is obtained by summing the  $2 + \alpha$  norms of each of the components. We define  $\mathcal{E}_1$  to be the subspace of  $\hat{\mathcal{E}}_1$  consisting of vector fields  $v^i$  such that

$$v^3 \equiv 0 \quad \text{on} \quad \Gamma_1,$$

$$\frac{\partial v^1}{\partial z} = \frac{\partial v^2}{\partial z} = \frac{\partial^2 v^1}{\partial r \partial z} = 0 \quad \text{at} \quad z = 0, \quad r = a, R, \tag{5.1}$$

and 
$$v^1 = v^3 = \partial v^2 / \partial z = 0 \quad \text{on} \quad \Gamma_2 \cap \mathcal{A}, \tag{5.2}$$

$$v^1 = v^2 = v^3 = 0 \quad \text{on} \quad \Gamma_3 \cap \mathcal{A}. \tag{5.3}$$

Clearly  $\mathcal{E}_1$  is a Banach space.

Let us next derive some consequences of the above conditions on the vector fields  $v$  in  $\mathcal{E}_1$ . We work in cylindrical coordinates in the domain  $\mathcal{A}$ .

From the continuity equation (4.4d) and (5.1)–(5.3) we get

$$\partial v^1 / \partial r = 0, \quad \partial^2 v^3 / \partial r^2 = 0, \quad \partial v^3 / \partial r = 0 \tag{5.4}$$

at the corners  $(a, 0)$  and  $(R, 0)$ . Furthermore, differentiating (4.4d) with respect to  $z$  and noting that  $\partial^2 v^3 / \partial z^2 = \partial v^1 / \partial z = 0$  at the corners we get

$$\partial^2 v^1 / \partial r \partial z = 0 \tag{5.5}$$

at the corners.

Let  $\mathcal{E}_2$  be the Banach space of functions  $P$  in  $C_{1+\alpha}(\bar{D}_0)$  which vanish at some pre-assigned point, and which satisfy

$$\partial P / \partial z = 0 \quad \text{at} \quad (a, 0) \quad \text{and} \quad (R, 0). \tag{5.6}$$

Finally, let  $\mathcal{E}_3$  be the Banach space of functions  $h(r)$  in  $C_{3+\alpha}(I_1)$  which satisfy the conditions

$$h'(a) = h'(R) = 0, \tag{5.7}$$

$$\int_a^R h(r) r \, dr = 0. \tag{5.8}$$

The Banach space  $\mathcal{E}$  is the Cartesian product  $\mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3$ , and an element of  $\mathcal{E}$  will be denoted by the triple  $(v, P, h)$ .

Let  $\mathcal{F}_1$  be the Banach space of vector fields  $f$  in the class  $C_\alpha(\bar{D}_0)$  which satisfy the condition

$$f \cdot \hat{k} = 0 \quad \text{at} \quad r = a, R.$$

In cylindrical coordinates this condition may be written

$$f^3(r, 0) = 0, \quad r = a, R. \tag{5.9}$$

Let  $\mathcal{F}_2$  be the Banach space of function triples  $\{g^1, g^2, g^3\}$  with  $g^1, g^2$  in  $C_{1+\alpha}(I_1)$  and  $g^3 \equiv 0$  which satisfy the conditions

$$\left. \begin{aligned} g^i(a) = g^i(R) = 0 \quad (i = 1, 2), \\ dg^1/dr|_{r=a, R} = 0. \end{aligned} \right\} \tag{5.10}$$

Let  $\mathcal{F}_3$  denote the Banach space of function triples  $\{h^1, h^2, h^3\}$  in  $C_{2+\alpha}(I_2 \cup I_3)$  which satisfy the conditions

$$\left. \begin{aligned} h^1 = h^3 = dh^2/dz = 0 \quad \text{on} \quad I_2 \cap \mathcal{U}, \\ h^1 = h^2 = h^3 = 0 \quad \text{on} \quad I_3 \cap \mathcal{U}. \end{aligned} \right\} \tag{5.11}$$

Finally, let  $\mathcal{F}_4$  denote the Banach space  $C_{1+\alpha}(I_1)$ . Then the Banach space  $\mathcal{F}$  is the Cartesian product  $\mathcal{F}_1 \times \mathcal{F}_2 \times \mathcal{F}_3 \times \mathcal{F}_4$ .

The nonlinear operation  $F: \mathcal{E} \times \mathbb{C} \rightarrow \mathcal{F}$  is defined to be

$$F(v, P, h, \epsilon) = (F_1(v, P, h, \epsilon), \quad F_2(v, P, h, \epsilon), \quad F_3(v, \epsilon), \quad F_4(v, P, h, \epsilon)),$$

where the components  $F_1, F_2, F_3, F_4$  are given as follows:

(1)\*  $F_1$  is a mapping from  $\mathcal{E} \times \mathbb{C}$  to  $\mathcal{F}_1$  defined to be

$$F_1(v, P, h, \epsilon) = f,$$

where  $f$  is the vector field whose components are given by

$$f^i = g^{jk} v_{,jk}^i - g^{ij} \partial P / \partial x^j - v^j v_{,j}^i.$$

Since all operations are algebraic,  $F_1$  is an analytic mapping from  $\mathcal{E} \times \mathbb{C}$  to  $\mathcal{F}_1$  provided  $1 + \epsilon^2 h^{12}$  does not vanish – that is, for sufficiently small values of  $\epsilon$  and  $h'$ .

(2)\*  $F_2$  is given by

$$F_2(v, P, h, \epsilon) = (D_{1j}n^j, D_{2j}n^j, v^3)|_{\Gamma_1}.$$

That is,  $F_2$  is a mapping of  $(v, P, h)$ , quantities defined on  $\bar{D}_0$  or  $\Gamma_1$ , to function triples  $\{g^1, g^2, g^3\}$  defined on  $\Gamma_1$ .  $F_2$  is also analytic in its variables in a sufficiently small neighbourhood of the origin in  $\mathcal{E} \times \mathbb{C}$ .

(3)\*  $F_3$  is given by

$$F_3(v, \epsilon) = v|_{r_2 \cup r_3}.$$

Thus,  $F_3$  is a linear trace operator.

(4)\*  $F_4$  is given by

$$F_4(v, P, h, \epsilon) = \epsilon gh - P + \frac{2\partial v_3}{\partial z} (1 + \epsilon^2 h'^2) - 2\epsilon h' D_{31} + p_a - \sigma \frac{\epsilon}{r} \left( \frac{r h'}{\sqrt{(1 + \epsilon^2 h'^2)}} \right)'.$$

We now prove

**THEOREM 5.2.** *The nonlinear mapping  $F$  defined above is an analytic mapping from  $\mathcal{E} \times \mathbb{C}$  to  $\mathcal{F}$ . In particular, if  $(v, P, h)$  satisfy conditions (5.1)–(5.3), (5.6)–(5.8), then  $f = F_1(v, P, h, \epsilon)$ ,  $g = \{g^1, g^2, g^3\} = F_2(v, P, h, \epsilon)$  and  $\{h^1, h^2, h^3\} = F_3(v, P, h, \epsilon)$  satisfy the conditions (5.9), (5.10), and (5.11) respectively.*

Conditions (5.1–5.3), (5.6–5.8) and (5.9–5.11) are the consistency relations we discussed in the introduction. The content of theorem 5.2 is that the consistency conditions are invariant under the nonlinear equations.

*Proof.* We have already observed the analyticity of the mapping  $F$ , at least in a suitably small neighbourhood of the origin in  $\mathcal{E} \times \mathbb{C}$ . It remains to show that the consistency conditions are preserved. First we note that if  $v$  satisfies (5.2), (5.3) in  $\mathcal{U}$  then  $h^1, h^2, h^3$  satisfy (5.11) in  $\mathcal{U}$ , since  $h^i = v^i|_{r_2 \cup r_3}$ .

Now let us check that  $g^1, g^2$  and  $g^3$  satisfy the conditions (5.9). From (2)\* above we see that

$$g^1 = D_{1j}n^j, \quad g^2 = D_{2j}n^j, \quad g^3 = v^3.$$

It follows immediately that  $g^3 = 0$  since  $v^3 \equiv 0$  on  $\Gamma_1$ . Now let us examine the term  $g^1(r) = D_{1j}n^j$ . Referring to (4.6a) we see that

$$D_{1j}n^j|_{r=a, R} = \frac{1}{2}[\partial v^1/\partial z + \epsilon h''v^1],$$

since  $h'(a) = h'(R) = 0$ . But since  $v \in \mathcal{E}_1$ ,  $\partial v^1/\partial z = v^1 = 0$  at the corners by (5.1–5.3). Therefore  $g^1(a) = g^1(R) = 0$ . Similarly one can check that  $g^2(r) = D_{2j}n^j$  vanishes at the corners, since  $\partial v^2/\partial z = 0$  in  $(\Gamma_2 \cup \Gamma_3) \cup \mathcal{U}$ .

We now check to see that

$$\frac{d}{dr} D_{1j}n^j = 0$$

at the corners. First note that

$$\frac{d}{dr} D_{1j}n^j = \frac{d}{dr} (D_{11}n^1 + D_{13}n^3) = 0$$

at  $r = a, R$  is equivalent to

$$\frac{d}{dr} (D_{13} + \epsilon^2 h'^2 D_{13} - \epsilon h' D_{11}) = 0. \tag{5.12}$$

This follows from the fact that the quantity

$$\frac{d}{dr} \frac{1}{\sqrt{(1 + \epsilon^2 h'^2)}} = \frac{-\epsilon^2 h' h''}{(1 + \epsilon^2 h'^2)^{\frac{3}{2}}}$$

vanishes at  $r = a, R$ .

Differentiating the expression in (5.12) with respect to  $r$  and setting  $r = a$  or  $r = R$ , we see that it must be shown that

$$\frac{d}{dr} D_{13} - \epsilon h'' D_{11} = 0$$

at the corners. But at the corners,

$$D_{11} = \partial v_1 / \partial r - \epsilon h'' v_3 = \frac{\partial}{\partial r} [(1 + \epsilon^2 h'^2) v^1 + \epsilon h' v^3] - \epsilon h'' [v^3 + \epsilon h' v^1] = \partial v^1 / \partial r,$$

since  $h' = 0$  there by (5.7). But from (5.4)  $\partial v^1 / \partial r = 0$  at the corners, so  $D_{11} = 0$  there.

Finally, at  $r = a$  or  $r = R$ ,

$$\begin{aligned} 2 \frac{d}{dr} D_{13} &= \frac{\partial}{\partial r} \left( \frac{\partial v_1}{\partial z} + \frac{\partial v_3}{\partial r} \right) \\ &= \frac{\partial}{\partial r} \left( \frac{\partial}{\partial z} \{1 + \epsilon^2 h'^2 v^1 + \epsilon h' v^3\} + \frac{\partial}{\partial r} \{v^3 + \epsilon h' v^1\} \right) \\ &= \frac{\partial}{\partial r} (1 + \epsilon^2 h'^2) \frac{\partial v^1}{\partial z} + \frac{\partial}{\partial r} \epsilon h' \frac{\partial v^3}{\partial z} + \frac{\partial^2 v^3}{\partial r^2} + \frac{\partial}{\partial r} \left( \epsilon h' \frac{\partial v^1}{\partial r} + \epsilon h'' v^1 \right) \\ &= \frac{\partial^2 v^1}{\partial r \partial z} + \epsilon h'' \frac{\partial v^3}{\partial z} + \frac{\partial^2 v^3}{\partial r^2} + \epsilon h'' \frac{\partial v^1}{\partial r} + \epsilon h''' v^1 + \epsilon h'' \frac{\partial v^1}{\partial r}. \end{aligned}$$

But at the corners,

$$\partial^2 v^1 / \partial r \partial z = 0 \quad \text{by (5.5),}$$

$$\partial v^1 / \partial r = \partial^2 v^3 / \partial r^2 = 0 \quad \text{by (5.4),}$$

and

$$v^1 = \partial v^3 / \partial z = 0 \quad \text{by (5.2), (5.3).}$$

In order to verify the condition (5.9) one must show that the expression on the left side of (4.4c) vanishes at  $r = a, R$  under the sole assumption that  $(v, P, h)$  satisfy conditions (5.1)–(5.7). Any term containing  $h'$  may be neglected, since  $h' = 0$  at the corners. Now

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v^3}{\partial r} \right) = \frac{\partial^2 v^3}{\partial r^2} + \frac{1}{r} \frac{\partial v^3}{\partial r} = 0 \quad \text{on } \Gamma_1$$

since  $v^3$  vanishes identically on  $\Gamma_1$ . Similarly,

$$\partial^2 v^3 / \partial z^2 = 0,$$

since  $v^3$  vanishes identically on  $(\Gamma_2 \cup \Gamma_3) \cap \mathcal{A}$ . The term  $h'' \partial v^1 / \partial r$  vanishes as a consequence of (5.4), and so on. The remaining terms may be checked by the reader. This concludes the proof of theorem 5.2.

We now evaluate the Frechet derivative of  $F$  at  $\epsilon = 0$ . For a nonlinear operator  $F$  the Frechet derivative at  $u_0$  may be computed directly by

$$F'(u_0) = \lim_{\epsilon \rightarrow 0} \frac{F(u_0 + \epsilon v) - F(u_0)}{\epsilon}.$$

The Frechet derivatives of the components of  $F$  are

$$\left. \begin{aligned} F'_1(v, P, h, 0) &= \Delta v_1 - \partial P / \partial x^i \quad \text{on } \Omega_0, \\ F'_2(v, P, h) &= [\tfrac{1}{2}(\partial v^1 / \partial x^3 + \partial v^3 / \partial x^1), \quad \tfrac{1}{2}(\partial v^2 / \partial x^3 + \partial v^3 / \partial x^2), v^3] |_{\Gamma_1}, \\ F'_3(v, P, h, 0) &= (v^1, v^2, v^3) |_{\Gamma_2 \cup \Gamma_3}, \\ F'_4(v, P, h, 0) &= gh - P + 2 \partial v^3 / \partial x^3 - \sigma(1/r)(rh)'. \end{aligned} \right\} \quad (5.13)$$

For convenience we have written the linearized equations in Cartesian coordinates. We have seen that  $F$  is a differentiable mapping from  $\mathcal{E} \times \mathbb{C}$  to  $\mathcal{F}$ . To apply the implicit function theorem we must show that  $F'$  is an isomorphism between  $\mathcal{E}$  and  $\mathcal{F}$ . This will be done in the next section.

### 6. SOLUTION OF THE LINEAR BOUNDARY VALUE PROBLEM

The problem of proving that  $F'$  is an isomorphism from  $\mathcal{E}$  to  $\mathcal{F}$  amounts to solving and obtaining *a priori* estimates for the following boundary value problem

$$\left. \begin{aligned} \Delta v_i - \partial p / \partial x^i &= f_i \\ \partial v_i / \partial x^i &= 0 \end{aligned} \right\} \quad \text{in } \Omega_0,$$

$$\left. \begin{aligned} \partial v_i / \partial x^3 &= g_i \quad (i = 1, 2) \\ v_3 &= 0 \end{aligned} \right\} \quad \text{on } \Gamma_1,$$

$$v_i = h_i \quad \text{on } \Gamma_2 \cup \Gamma_3.$$

For the following analysis it will be convenient to rewrite the above equations in cylindrical coordinates:

$$\left. \begin{aligned} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v^1}{\partial r} \right) + \frac{\partial^2 v^1}{\partial z^2} - \frac{v^1}{r^2} - \frac{\partial p}{\partial r} &= f^1, \\ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v^2}{\partial r} \right) + \frac{\partial^2 v^2}{\partial z^2} + \frac{2}{r} \frac{\partial v^2}{\partial r} &= f^2, \\ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v^3}{\partial r} \right) + \frac{\partial^2 v^3}{\partial z^2} - \frac{\partial p}{\partial z} &= f^3, \\ \frac{1}{r} \frac{\partial}{\partial r} (rv^1) + \frac{\partial v^3}{\partial z} &= 0, \\ \frac{\partial v^1}{\partial z} = g^1(r), \quad \frac{\partial v^2}{\partial z} = g^2(r), \quad v^3 = 0 &\quad \text{on } \Gamma_1, \\ v^1 = v^3 = 0, \quad \frac{\partial v^2}{\partial z} = 0 &\quad \text{on } \Gamma_2 \cap \mathcal{A}, \\ v^1 = v^2 = v^3 = 0 &\quad \text{on } \Gamma_3 \cap \mathcal{A}. \end{aligned} \right\} \quad (6.1)$$

This is the form of the equations in local coordinates in the region  $\mathcal{A}$ .

THEOREM 6.1. Let  $f \in \mathcal{F}_1, \{g^1, g^2, 0\} \in \mathcal{F}_2$ ; then the boundary value problem (6.1) has a solution  $(v, p)$  in  $\mathcal{E}_1 \times \mathcal{E}_2$  and the following a priori estimates hold

$$(\|v\|_{2+\alpha} + \|p\|_{1+\alpha}) \leq C(\|f\|_\alpha + \|g^1\|_{1+\alpha} + \|g^2\|_{1+\alpha}).$$

(Note: the pressure  $p$  is determined only up to an additive constant; recall that we are fixing  $p$  uniquely by requiring  $p$  to vanish at some preassigned point in  $\bar{\Omega}_0$ .)

*Proof.* We extend the data  $f^i$  smoothly to the entire half space  $z < 0$  and the data  $g^i$  to the entire  $x$ - $y$  plane in such a way that the extensions  $C_\alpha$  and  $C_{1+\alpha}$  respectively, are axisymmetric, and have compact support. We now solve the half space problem consisting of equations (6.1) together with the boundary conditions on  $z = 0$ . The problem (6.1) is elliptic in the sense of Agmon *et al.* (1964), and so the following estimates hold

$$\|v\|_{2+\alpha} + \|p\|_{1+\alpha} \leq C(\|f\|_\alpha + \|g^1\|_{1+\alpha} + \|g^2\|_{1+\alpha}).$$

Furthermore, at the points  $r = a, R \quad z = 0$  we get

$$\partial v^i / \partial z = g_i(r) = 0 \quad (i = 1, 2), \tag{6.2}$$

$$v^3 = \partial^2 v^3 / \partial z^2 = 0. \tag{6.3}$$

The second equality in (6.3) follows by differentiating the continuity relation with respect to  $z$ ,

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v^1}{\partial z} \right) + \frac{\partial^2 v^3}{\partial z^2} = 0,$$

and then noting that

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial v^1}{\partial z} = \frac{1}{r} \frac{\partial}{\partial r} r g^1(r) = g^1(r) + \frac{1}{r} \frac{d g^1}{d r} = 0$$

at  $r = a, R$ .

Now let us check that  $\partial p / \partial z = 0$  at  $r = a, R$ . From the third equation (6.1) we have

$$\frac{\partial^2 v^3}{\partial r^2} + \frac{1}{r} \frac{\partial v^3}{\partial r} + \frac{\partial^2 v^3}{\partial z^2} - \frac{\partial p}{\partial z} = f^3.$$

But all derivatives of  $v^3$  vanish at the corner, since  $v^3 \equiv 0$  on  $\Gamma_1$  and on  $(\Gamma_2 \cup \Gamma_3) \cap \omega$ . Therefore

$$\partial p / \partial z = -f^3 = 0 \quad \text{at } r = a, R.$$

In order to construct a solution of the original boundary value problem in  $\Omega_0$  we write the solution  $v^i = u^i + w^i$  where  $u^i$  satisfies the half space problem and  $w^i$  satisfies equations (6.1) with homogeneous right hand side together with the boundary conditions

$$\left. \begin{aligned} \partial w^i / \partial z &= 0 \quad (i = 1, 2) \\ w^3 &= 0 \end{aligned} \right\} \text{ on } \Gamma_1,$$

and

$$w^i = h^i - u^i \quad \text{on } \Gamma_2 \cup \Gamma_3,$$

where the functions  $h^i$  are the values of  $v^i$  on  $\Gamma_2 \cup \Gamma_3$ .

Now let  $\Omega'$  be the domain consisting of the union of  $\Omega_0$  and its reflexion in the  $x$ - $y$  plane. In order to find the vector field  $w^i$  we reflect the boundary values of  $w^i$  across the plane  $z = 0$ . By conditions (6.2), (6.3) above, and since

$$h^1 = h^3 = dh^2/dz = 0 \quad \text{on} \quad (\Gamma_2 \cup \Gamma_3) \cap \mathcal{A},$$

we see that  $w^1$  and  $w^2$  have smooth even extensions across  $z = 0$  while  $w^3$  has a smooth odd extension.

The above problem on  $\Omega'$  is invariant under the transformation

$$(w^1(r, z), w^2(r, z), w^3(r, z), p(r, z)) \rightarrow (w^1(r, -z), w^2(r, -z), -w^3(r, -z), p(r, -z)).$$

Therefore the solution obtained satisfies the symmetry properties

$$\begin{aligned} w^i(r, z) &= w^i(r, -z) \quad (i = 1, 2) \\ w^3(r, z) &= -w^3(r, -z), \\ p(r, z) &= p(r, -z). \end{aligned}$$

Thus, the boundary conditions

$$\partial w^1/\partial z = \partial w^2/\partial z = w^3 = 0 \quad \text{on} \quad \Gamma_1$$

are satisfied; and, moreover,

$$\partial^2 w^3/\partial z^2 = 0 \quad \text{and} \quad \partial p/\partial z = 0 \quad \text{on} \quad \Gamma_1.$$

Therefore  $v^i = w^i + w^i$  and the corresponding pressure function satisfy the conditions (5.1)–(5.3) and (5.6). Also, since  $\Omega'$  is a smoothly bounded domain we obtain the classical Schauder estimates for  $(w^i, p)$ , hence also for the solution of the original problem (6.1).

**THEOREM 6.2.** *The Frechet derivative of the mapping  $F$ , given in (5.13), is an isomorphism from  $\mathcal{E}$  to  $\mathcal{F}$ .*

*Proof.* In view of theorem 6.1 it only remains to show that (a) the operator  $F'_4$  has a bounded inverse; and (b) the constant which must be added to the pressure term  $P$  in order to satisfy condition (5.8) is bounded by the norms of the given data.

To prove (a) we consider the boundary value problem

$$\begin{aligned} gh - P + 2\partial v^3/\partial x^3 - \sigma(1/r)(rh')' &= H, \\ h'(a) = h'(R) &= 0, \end{aligned} \tag{6.4}$$

where  $H$  belongs to the class  $C_{1+\alpha}(\Gamma_1)$ . We already have bounds on the norms  $\|P\|_{1+\alpha}$  and  $\|v^3\|_{2+\alpha}$  in the region  $\bar{\Omega}_0$ ; therefore  $\|P\|_{\Gamma_1^{1+\alpha}}$  and  $\|v^3\|_{\Gamma_1^{1+\alpha}}$  have the same bounds. The operator

$$\begin{aligned} Lh &= gh - \sigma(1/r)(rh')', \\ h'(a) = h'(R) &= 0, \end{aligned}$$

has a bounded inverse on  $a \leq r \leq R$ ; and if  $Lh = H$  then there is a constant  $C$  independent of  $H$  such that

$$\|h\|_{3+\alpha} \leq C\|H\|_{1+\alpha}.$$

The additive constant in  $P$  is determined as follows. The original function  $P$ , in the solution of the boundary value problem (6.1), was normalized by requiring it to vanish at a fixed point in  $\bar{\Omega}_0$ . We now replace  $P$  by  $P + \lambda$  and apply condition (5.8) to (6.4). Integrating (6.4) against  $r dr$  over  $a \leq r \leq R$ , we obtain

$$\int_a^R (-P - \lambda + 2 \partial v^3 / \partial x^3) r dr = \int_a^R H r dr,$$

$$\lambda \frac{1}{2}(R^2 - a^2) = \int_a^R (-P + 2 \partial v^3 / \partial x^3 - H) r dr.$$

The constant  $\lambda$  is therefore bounded by the quantity

$$\|P\|_{1+\alpha} + \|v^3\|_{2+\alpha} + \|H\|_{1+\alpha},$$

and these quantities are in turn bounded by the given inhomogeneous data in (6.1). This completes the proof of theorem 6.2, and the proof of the main result of the paper is now also complete.

### 7. REMARKS ON THE FLOW PROBLEM BETWEEN CONCENTRIC CYLINDERS

In the work by Joseph *et al.* (1973) the free surface problem was considered in the region between two concentric cylinders,

$$a \leq r \leq R, \quad z \leq h(r).$$

In this case the base domain  $\Omega_0$  is the unbounded region  $\{(r, z) | a \leq r \leq R, z \leq 0\}$  and additional problems arise in obtaining *a priori* estimates for the solution of the linear boundary value problem. By a reflexion argument similar to that used in § 6 we may replace the original problem by one on the infinite annulus

$$\{(r, z) : a \leq r \leq R, -\infty < z < \infty\}$$

with smooth inhomogeneous data.

Referring to equations (6.1) we note first that the second equation is not coupled to the other two and so may be separately solved by itself. Since it is a second order elliptic equation the maximum principle applies, and there is no difficulty in obtaining uniform estimates for  $v^2$  and its derivatives in the entire annulus.

To solve the system consisting of the first and third equations we introduce the bilinear form

$$B(v, \phi) = \int_a^R \int_{-\infty}^{\infty} \left\{ \frac{\partial v^1}{\partial r} \frac{\partial \phi^1}{\partial r} + \frac{\partial v^1}{\partial z} \frac{\partial \phi^1}{\partial z} + \frac{\partial v^3}{\partial r} \frac{\partial \phi^3}{\partial r} + \frac{\partial v^3}{\partial z} \frac{\partial \phi^3}{\partial z} + \frac{v^1 \phi^1}{r^2} \right\} r dr dz$$

defined on divergence free vector fields on the annulus.

By the usual arguments the original problem with inhomogeneous boundary data can be converted to one with homogeneous boundary data. Therefore we may work on the class of vector fields which vanish on the boundary. Let  $\hat{H}_1$  be the class of such vector fields  $u$  on the annular region for which  $B(u, u) < +\infty$ .  $\hat{H}_1$  is a Hilbert

space with inner product  $(u, \phi) = B(u, \phi)$ . The system of equations consisting of the first and third equations in (6.1) can be written in weak form as

$$B(u, \phi) = (f, \phi),$$

where 
$$(f, \phi) \stackrel{\text{def}}{=} \int_a^R \int_{-\infty}^{\infty} \{f^1 \phi^1 + f^3 \phi^3\} r \, dr.$$

Here  $\phi$  is a test function in  $\mathring{H}_1$  while  $f$  is an arbitrary vector field in

$$L_2\{(r, z) | a \leq r \leq R, -\infty < z < \infty\}.$$

Since  $v^1 = v^3 = 0$  on  $r = a$  and  $r = R$  we easily obtain the estimates.

$$\int_{-\infty}^{\infty} \int_a^R (v^i)^2 r \, dr \, dz \leq \text{const} \int_{-\infty}^{\infty} \int_a^R (\partial v^i / \partial r)^2 r \, dr \, dz$$

for  $i = 1, 3$ . Therefore  $|(u, \phi)| \leq B(u, \phi)$  and so  $\phi \rightarrow (f, \phi)$  is a linear functional on  $\mathring{H}_1$ . By the Riesz representation theorem, given  $f$  in  $L_2$  there exists an element  $u$  in  $\mathring{H}_1$  such that  $B(u, \phi) = (f, \phi)$  for all test functions  $\phi$  in  $\mathring{H}_1$ . The vector field  $u$  is the weak solution of the problem and satisfies

$$\|u\| \leq \text{const} |f|,$$

where  $|f|$  denotes the  $L_2$  norm of  $f$  and  $\|u\| = \sqrt{B(u, u)}$ .

The problem of obtaining global estimates for the  $L_2$  norms of the second derivatives of  $v$  is more difficult. That is, we should like to obtain estimates of the form

$$\int_{-\infty}^{\infty} \int_a^R \sum_{i=1}^3 \left( \frac{\partial^2 v^i}{\partial r^2} \right)^2 + \left( \frac{\partial^2 v^i}{\partial r \partial z} \right)^2 + \left( \frac{\partial^2 v^i}{\partial z^2} \right)^2 r \, dr \, dz \leq \text{const} |f|^2.$$

One method of doing this is to introduce a stream function  $\psi$ :

$$rv^1 = \psi_z, \quad rv^3 = -\psi_r.$$

By the usual procedures one obtains a fourth order elliptic equation (the biharmonic equation) for  $\psi$ . In the simpler case of rectangular coordinates in a strip

$$(-\infty < z < 0, -1 < x < 1)$$

the corresponding equation may be solved by separation of variables. The solution may be expanded in a biorthogonal sequence of functions. Specifically, one obtains for  $\check{\psi}$  the fourth order boundary value problem

$$(D^2 - \omega^2)^2 \check{\psi} = \check{\phi}, \quad \check{\psi}(\pm 1, \omega) = \frac{\partial \check{\psi}}{\partial x}(\pm 1, \omega) = 0,$$

where  $\check{\psi}(x, \omega)$  denotes the Fourier transform of  $\psi(x, z)$  in the  $z$  variable. The associated sequence of functions is given by the solutions of

$$(D^2 - \omega_n^2)^2 \psi_n(x, \omega_n) = 0,$$

$$\psi_n(\pm 1, \omega_n) = \frac{d\psi_n}{dx}(\pm 1, \omega_n) = 0.$$

Here  $D = \partial/\partial x$ . See Joseph & Fosdick (1973) and R. C. T. Smith (1952). The global estimates on the second derivatives of  $v^1$  and  $v^3$  can be obtained by using the representation of  $v^1$  and  $v^3$  in a biorthogonal sequence of the  $\{\psi_n\}$ .

One possible method for obtaining global  $L_2$  estimates in the case of the annular region is to develop the corresponding theory for the biharmonic equation obtained for  $\psi$  in the case of cylindrical coordinates.

Having obtained global  $L_2$  estimates for  $v$  and its first and second derivatives we can then obtain  $L_\infty$  estimates on  $v$  by applying the usual Sobolev inequalities. Finally, we obtain global Schauder estimates for  $v$  of the type obtained in theorem 6.1 by applying theorem 9.3 of Agmon, Douglis & Nirenberg (1964).

### 8. NON-ZERO WETTING ANGLE

We point out here that if the wetting angle is not zero or  $\frac{1}{4}\pi$  (that is, if

$$h'(a) \neq 0 \pm 1),$$

and if the velocity field is continuously differentiable up to the line of contact, then the deformation tensor  $D_{ij}$  vanishes identically along the line of contact. First, recall that  $\partial v^2/\partial z = 0$  at  $(a, 0)$  and

$$D_{2j}n^j = \sqrt{(1+h'^2)}D_{23} - (h'/\sqrt{(1+h'^2)})D_{21} = 0.$$

Since  $h$  no longer vanishes when  $\epsilon = 0$  we do not replace  $h$  by  $\epsilon h$  in the basic equations. Now

$$D_{23} = \frac{\partial v_2}{\partial z} = \frac{\partial}{\partial z}(r^2 v^2) = r^2 \frac{\partial v^2}{\partial z} = 0$$

and if  $h'(a) \neq 0$ , we get

$$0 = D_{21} = \frac{\partial}{\partial r}(r^2 v^2) - \frac{2}{a}a^2 v^2 = a^2 \frac{\partial v^2}{\partial r}.$$

Therefore  $\partial v^2/\partial r = 0$  at  $(a, 0)$ .

Now consider the boundary condition

$$\begin{aligned} 0 &= D_{1j}n^j = \sqrt{(1+h'^2)}D_{13} - \frac{h'}{\sqrt{(1+h'^2)}}D_{11} \\ &= \frac{1}{2}\sqrt{(1+h'^2)}\left(\frac{\partial v_1}{\partial z} + \frac{\partial v_3}{\partial r}\right) - \frac{h'}{\sqrt{(1+h'^2)}}\left(\frac{\partial v_1}{\partial r} - h''v_3\right) \\ &= 1\sqrt{(1+h'^2)}\left[(1+h'^2)\frac{\partial v^1}{\partial z} + h'\frac{\partial v^3}{\partial z} + \frac{\partial v^3}{\partial r} + h''v^1 + h'\frac{\partial v^1}{\partial r}\right] \\ &\quad - \frac{h'}{\sqrt{(1+h'^2)}}\left[(1+h'^2)\frac{\partial v^1}{\partial r} + h'h''v^1 + h'\frac{\partial v^3}{\partial r}\right]. \end{aligned}$$

At  $r = a, z = 0$  this condition becomes

$$\begin{aligned} 0 &= \frac{1}{2}\sqrt{(1+h'^2)}\left(\frac{\partial v^3}{\partial r} + h'\frac{\partial v^1}{\partial r}\right) - \frac{h'}{\sqrt{(1+h'^2)}}\left((1+h'^2)\frac{\partial v^1}{\partial r} + h'\frac{\partial v^3}{\partial r}\right) \\ &= \left(\frac{1-h'^2}{\sqrt{(1+h'^2)}}\right)\frac{\partial v^3}{\partial r} - \frac{1}{2}h'\sqrt{(1+h'^2)}\frac{\partial v^1}{\partial r}. \end{aligned}$$

(Recall that we still have  $\partial v^1/\partial z = \partial v^3/\partial z = 0$ , etc., at the ridge.) But we have already seen (from the mass continuity relation) that  $\partial v^1/\partial r = 0$ , so  $\partial v^3/\partial r = 0$  as well, provided  $h' \neq \pm 1$ .

Now consider  $D_{ij}$  as given by (4.10). We have already seen that

$$D_{12} = D_{21} = D_{23} = 0.$$

Furthermore, since  $\partial v^1/\partial r = \partial v^3/\partial r = 0$  we see that

$$D_{13} = D_{31} = 0; \quad D_{33} = \partial v_3/\partial z = \partial v^3/\partial z + h'\partial v^1/\partial z = 0;$$

$$D_{22} = -r h' v^3 - r h'^2 v^1 v^3 = 0.$$

Finally, since  $D_{13} = 0$  we also have  $D_{11} = 0$  by the preceding calculation. Thus all components of  $D_{ij}$  vanish identically along the line of contact. Physical considerations suggest that the deformation tensor of a viscous fluid should not vanish along a moving boundary, and Joseph's formal calculations show this not to be the case (see (5.2), p. 340; the quantity  $v^{(3)}$  is the third order coefficient in the expansion of the angular velocity  $v^2$ ). These considerations suggest that the fluid velocities may suffer singularities in the derivatives at the contact line.

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#### APPENDIX

We construct here a volume preserving transformation which maps  $\Omega_0$  to  $\Omega_\epsilon$  and which has the form (3.1) in the open set  $\mathcal{U}$ . The transformation  $W$  is obtained by integration of a divergence-free vector field  $X$ . Thus we solve

$$\left. \begin{aligned} dW^i/dt &= X^i(r, \theta, z), \\ W^1(r, \theta, z, 0) &= r \cos \theta, \\ W^2(r, \theta, z, 0) &= \sigma \sin \theta, \\ W^3(r, \theta, z, 0) &= z. \end{aligned} \right\} \quad (\text{A } 1)$$

Working in cylindrical coordinates, the vector field  $X$  must satisfy

$$\frac{1}{r} \frac{\partial}{\partial r} (r X^1) + \frac{\partial}{\partial \theta} X^2 + \frac{\partial}{\partial z} X^3 = 0.$$

We take  $X^2 \equiv 0$  and  $X^3(r, z) = -\xi(z)h(r, \epsilon)$ , where  $\xi \equiv 1$  in  $\mathcal{U}$  but  $\xi$  has compact support and vanishes identically below the bottom of the rod. Finally we let

$$X^1(r, z) = \xi'(z)\Phi(r, \epsilon).$$

Then  $X^1$  and  $X^3$  satisfy the continuity relation if  $\Phi$  satisfies the equation

$$\frac{1}{r} \frac{\partial}{\partial r} (r\Phi) + h(r, \epsilon) = 0.$$

Integrating this equation with respect to  $r$  and choosing  $\Phi$  to vanish at  $r = a$  we get

$$\Phi(r, \epsilon) = -\frac{1}{r} \int_a^r \tau h(\tau, \epsilon) d\tau.$$

Furthermore, by condition (2.1) we see that  $\Phi(R, \epsilon) = 0$  also. Finally, integration of equations (A 1) gives

$$\begin{aligned} W^1(r, \theta, z, \epsilon) &= r \cos \theta, \\ W^2(r, \theta, z, \epsilon) &= r \sin \theta \quad \text{in } u, \\ W^3(r, \theta, z, \epsilon) &= z + \epsilon h(r, \epsilon). \end{aligned}$$

Since  $\xi$  vanishes identically in the bottom of the container the transformation  $W$  reduces to the identity transformation there. Also, since  $\Phi(a, \epsilon) = \Phi(R, \epsilon) = 0$ , the lateral walls of the container and rod remain invariant under the transformation.

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