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CALOGERO–FRANÇOISE FLOWS AND PERIODIC PEAKONS

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The completely integrable Hamiltonian systems discovered by Calogero and Françoise contain the finite-dimensional reductions of the Camassa–Holm and Hunter–Saxton equations. We show that the associated spectral problem has the same form as that of the periodic discrete Camassa–Holm equation. The flow is linearized by the Abel map on a hyperelliptic curve. For two-particle systems, which correspond to genus-1 curves, explicit solutions are obtained in terms of the Weierstrass elliptic functions.

Keywords: finite-dimensional Hamiltonians, elliptic and hyperelliptic curves, Abel maps

1. Introduction

Calogero [1] introduced a class of Hamiltonian systems of the form

$$H(x,m)=\sum_{j,k=1}^{d} m_j m_k F(x_j-x_k), \tag{1.1}$$

motivated by the multipeakon Hamiltonian system discovered by Camassa and Holm [2]. He found that system (1.1) is completely integrable when $F(x)=\lambda+\mu\cos\nu x$. Subsequently, Calogero and Françoise [3] proved complete integrability in the more general case

$$F(x)=\lambda+\mu\cos\nu x+\mu'\sin\nu|x| \tag{1.2}$$

and in the limiting case

$$F(x)=\alpha+\beta|x|+\gamma x^2 \tag{1.3}$$

(see [4] for a full treatment).

In the normalization used in this paper, the multipeakon Camassa–Holm flow corresponds to the case where $\lambda=0$, $\mu=1/2$, $\mu'=1/2i$, and $\nu=-2i$ in (1.2). It was shown in [5] that the case of (1.3) where $\alpha=\gamma=0$ and $\beta=1$ corresponds to discrete solutions of the Hunter–Saxton equation [6], which is a limiting case of the Camassa–Holm equation. The question arises whether the inverse scattering methods used to construct explicit solutions of the multipeakon equations [7] and the discrete Hunter–Saxton flow [5] can be extended to the more general Calogero–Françoise (CF) equations.

The answer to this question is a qualified yes. The relevant spectral problem has the same form as that arising for the periodic multipeakon system. (This is so, even though there is no periodic structure in the continuum model associated with the CF system.) The inverse spectral problem is therefore analogous to the inverse problem for the Hill equation in the theory of the periodic solutions of the KdV equation or the periodic Toda lattice.

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The connection between the CF system and finite-dimensional reductions of the Camassa–Holm partial differential equation arises as follows. The Camassa–Holm equation is best represented as a coupled system of partial differential equations given in (2.4) (or (4.2)) below. The equation connecting m and u can be solved for u using a Green's function. If we impose the boundary conditions $u \to 0$ as $|x| \to \infty$, the Green's function is $e^{-2|x|}/2$ and

$$u(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-2|x-y|} m(y)\, dy.$$

If m is taken to be a discrete measure $\sum m_j(t)\delta(x - x_j(t))$, the Camassa–Holm evolution is equivalent to the Hamilton equations for x_j and m_j with respect to the Hamiltonian above with $F(x) = e^{-2|x|}/2$.

More generally, for F, we can take any even fundamental solution for $(4\nu^2 - D^2)Du = Dm$, the second equation in (4.2) below. The general form of such a fundamental solution is

$$\begin{aligned} &\frac{\beta}{2\nu} e^{2\nu|x|} + \frac{1+\beta}{2\nu}\, e^{-2\nu|x|} + \gamma, && \nu \neq 0, \\ &\frac{1}{2}\,|x| + \beta x^2 + \delta, && \nu = 0. \end{aligned}$$

Functions (1.2) and (1.3) are obtained up to a multiplicative constant by rescaling $\nu \to i\nu/2$. Because a multiplicative constant can be absorbed in the time scale, we are exactly led to the family of Hamiltonians discovered by Calogero and Françoise. But in the absence of zero boundary conditions at $\pm\infty$, the associated spectral problem is more complicated than in the multipeakon case.

In this note, we derive some fundamental properties of the spectral problems in the general case and obtain solutions in the case of two particles. In this case, the flows can be explicitly obtained in terms of the Weierstrass elliptic functions (more precisely, in terms of the associated σ functions). For d particles, the associated invariant curve is hyperelliptic of genus $d - 1$; we defer a full discussion of the general case to a future paper.

As mentioned above, there is a close relation between the CF flows and multipeakon flows with a discrete periodic potential m. We consider the periodic Camassa–Holm equation for d particles in Sec. 2. The case where $d = 2$, for which the associated curve is elliptic, is considered in detail in Sec. 3. Solutions of this problem for the analogues of the finite-gap solutions of the KdV equation (i.e., smooth periodic m in (2.1) below) were obtained using spectral methods by Constantin and McKean [8]. Alber et al. discussed billiard solutions of the Camassa–Holm equations with periodic m [9]. In Sec. 4, we consider the CF flows, with detailed solutions again given in the genus-1 case.

2. Periodic multipeakons/antipeakons

The linear spectral problem associated with the Camassa–Holm evolution is given by

$$L(\lambda)\varphi \equiv D^2\varphi - \varphi - \lambda m(x)\varphi = 0, \qquad D = \frac{d}{dx}. \tag{2.1}$$

An operator of this form is compatible with a generalized Lax evolution

$$-\lambda m_t = \frac{d}{dt}L(\lambda) = \big[L(\lambda), A(\lambda)\big] + 2u_x L(\lambda), \tag{2.2}$$

where

$$A(\lambda) = \left\{\frac{1}{\lambda} - u(x)\right\}D + \frac{1}{2}u_x(x). \tag{2.3}$$

In fact, Eq. (2.2) is equivalent to the conditions

$$m_t = u_x m + (um)_x, \qquad 2m_x = 4u_x - u_{xxx} \tag{2.4}$$

(cf. the argument in [10]). If we replace the second equation with $2m = 4u - u_{xx}$ and substitute in the first equation, we obtain the Camassa–Holm evolution

$$4u_t - u_{xxt} = 2u_x(4u - u_{xx}) + u(4u_x - u_{xxx}). \tag{2.5}$$

Multipeakon/antipeakon solutions of (2.5) on the line correspond to discrete measures

$$m(x,t) = \sum_{j=1}^{d} m_j(t)\, \delta(x - x_j(t)); \tag{2.6}$$

here, Eq. (2.4) and spectral problem (2.1) must be interpreted in the sense of distributions (cf. [7]). The associated function u is

$$u(x,t) = \frac{1}{2}\sum_{j=1}^{d} m_j(t)\, e^{-2|x-x_j(t)|}. \tag{2.7}$$

The Camassa–Holm evolution in this discrete case is equivalent to the Hamiltonian system with variables x_j, the dual variables m_j, and the Hamiltonian

$$H(x_1,\ldots,x_d,m_1,\ldots,m_d) = \frac{1}{2}\sum_{j,k=1}^{d} m_j m_k\, e^{-2|x_j-x_k|}. \tag{2.8}$$

(See [7], [11] for this problem and its relations to the classical moment problem, Toda flows, and string problems.)

We here consider the periodic version of the above problem. The formalism is the same, but m is taken to be a periodic discrete measure with period 1. We begin by studying the spectral problem with no time dependence. We assume that

$$m(x) = \sum_{n=-\infty}^{\infty}\sum_{j=1}^{d} m_j \delta(x - x_j - n), \tag{2.9}$$

where

$$x_1 < x_2 < \cdots < x_d < x_1 + 1. \tag{2.10}$$

The associated function u is

$$u(x) = \sum_{j=1}^{d} m_j G(x - x_j), \qquad G(x) = \frac{1}{2}\sum_{n=-\infty}^{\infty} e^{-2|x-n|}. \tag{2.11}$$

In what follows, we need the identities

$$\begin{aligned} G(x) + \frac{1}{2}G'(x) &= \sum_{n=1}^{\infty} e^{2x-2n} = \frac{e^{2x-1}}{e - e^{-1}}, \\ G(x) - \frac{1}{2}G'(x) &= \sum_{n=-\infty}^{-1} e^{2n-2x} = \frac{e^{1-2x}}{e - e^{-1}} \end{aligned} \tag{2.12}$$

for $0 < x < 1$. Because G is even, it follows that

$$G(x) = \frac{\cosh(2|x| - 1)}{2\sinh 1}, \quad |x| < 1.$$

The periodic version of (2.5) is again a Hamiltonian system with the Hamiltonian

$$H(x_1, \ldots, x_d, m_1, \ldots, x_d) = \frac{1}{2} \sum_{j,k=1}^{d} m_j m_k G(x_j - x_k). \tag{2.13}$$

We suppose that φ is a solution of (2.1). On each interval $I_j = (x_{j-1}, x_j)$, φ is a linear combination $a_j e^x + b_j e^{-x}$. Equation (2.1) itself translates into a continuity equation and a jump condition

$$\begin{aligned} a_{j+1}e^{x_j} + b_{j+1}e^{-x_j} &= a_j e^{x_j} + b_j e^{-x_j}, \\ a_{j+1}e^{x_j} - b_{j+1}e^{-x_j} &= a_j e^{x_j} - b_j e^{-x_j} + \lambda m_j (a_j e^{x_j} + b_j e^{-x_j}). \end{aligned} \tag{2.14}$$

The transition is therefore given by

$$\begin{bmatrix} a_{j+1} \\ b_{j+1} \end{bmatrix} = \begin{bmatrix} 1 + \frac{\lambda}{2} m_j & \frac{\lambda}{2} m_j e^{-2x_j} \\ -\frac{\lambda}{2} m_j e^{2x_j} & 1 - \frac{\lambda}{2} m_j \end{bmatrix} \begin{bmatrix} a_j \\ b_j \end{bmatrix}. \tag{2.15}$$

With the transition matrix in (2.15) denoted by $T_j(\lambda)$, the Floquet matrix for this problem is the product

$$\Phi(\lambda) = T_d(\lambda) T_{d-1}(\lambda) \cdots T_1(\lambda). \tag{2.16}$$

It follows by induction that

$$\Phi(\lambda) = I + \frac{\lambda}{2} \begin{bmatrix} M & M_- \\ -M_+ & -M \end{bmatrix} + O(\lambda^2), \tag{2.17}$$

where

$$M = \sum_{j=1}^{d} m_j, \qquad M_\pm = \sum_{j=1}^{d} m_j e^{\pm 2x_j}. \tag{2.18}$$

The entries $\Phi_{ij}(\lambda)$ are polynomials of degree d in λ. We note that $\det T_j = 1$ and therefore $\det \Phi = 1$.

We note that $\varphi(x+1) \equiv \varphi(x)$ if and only if the coefficients of φ with respect to the basis e^x, e^{-x} on the interval $(x_d, x_1 + 1]$ are the same as the coefficients with respect to the basis e^{x+1}, e^{-x-1} on the interval $(x_d - 1, x_1]$. Therefore, to compare wave functions φ near $x = x_1$ and near $x = x_1 + 1$ with periodicity in mind, we renormalize on the subinterval $(x_d - 1, x_1]$ by writing φ as a linear combination of e^{x+1} and e^{-x-1} rather than e^x and e^{-x}. The associated transition matrix that relates the coefficients of $\varphi(x_1)$ with respect to this basis to coefficients of $\varphi(x_1 + 1)$ with respect to the basis e^x, e^{-x} is given by

$$\Psi(\lambda) = \Phi(\lambda) E = \Phi(\lambda) \begin{bmatrix} e & 0 \\ 0 & e^{-1} \end{bmatrix}. \tag{2.19}$$

Then φ is periodic with period 1 if and only if $\Psi\varphi = \varphi$. We note that $\det \Psi = 1$.

We now consider evolution under the flow. Equation (2.2) implies

$$(D^2 - \lambda m - 1)(\dot{\varphi} + A(\lambda)\varphi) = 0, \tag{2.20}$$

where the dot denotes differentiation with respect to time. By (2.3),

$$A(\lambda)(a\,e^x + b\,e^{-x}) = \frac{1}{\lambda}(a\,e^x - b\,e^{-x}) - ae^x \sum_{j=1}^{d} m_j \left[G(x-x_j) - \frac{1}{2}G'(x-x_j)\right] + + be^{-x}\sum_{j=1}^{d} m_j \left[G(x-x_j) + \frac{1}{2}G'(x-x_j)\right]. \tag{2.21}$$

Equations (2.12) imply

$$G(x-x_j) + \frac{1}{2}G'(x-x_j) = \begin{cases} e^{2x-2x_j}\dfrac{e}{e-e^{-1}}, & x_d - 1 < x < x_1, \\ e^{2x-2x_j}\dfrac{e^{-1}}{e-e^{-1}}, & x_d < x < x_1 + 1, \end{cases}$$
$$G(x-x_j) - \frac{1}{2}G'(x-x_j) = \begin{cases} e^{2x_j-2x}\dfrac{e^{-1}}{e-e^{-1}}, & x_d - 1 < x < x_1, \\ e^{2x_j-2x}\dfrac{e}{e-e^{-1}}, & x_d < x < x_1 + 1. \end{cases} \tag{2.22}$$

Combining (2.21) and (2.22), we find that the vector representation of $A(\lambda)\varphi$ in terms of that of φ is given by

$$A(\lambda) : \begin{bmatrix} a \\ b \end{bmatrix} \to A_-(\lambda)\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\lambda} & \dfrac{M_- e}{e-e^{-1}} \\ -\dfrac{M_+ e^{-1}}{e-e^{-1}} & -\dfrac{1}{\lambda} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} \tag{2.23}$$

for $x_d - 1 < x < x_1$ and

$$A(\lambda) : \begin{bmatrix} a \\ b \end{bmatrix} \to A_+(\lambda)\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\lambda} & \dfrac{M_- e^{-1}}{e-e^{-1}} \\ -\dfrac{M_+ e}{e-e^{-1}} & -\dfrac{1}{\lambda} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} \tag{2.24}$$

for $x_d < x < x_1 + 1$.

We take solutions $\varphi_1 = e^x$ and $\varphi_2 = e^{-x}$ in the interval $x_d - 1 < x < x_1$ such that the two vector representations give the identity matrix on $(x_d - 1, x_1)$ and the matrix $\Phi(\lambda)$ on $(x_d, x_1 + 1)$. In view of the preceding, it follows from Eq. (2.20) on $(x_d, x_1 + 1)$ that

$$\dot{\Phi}(\lambda) + A_+(\lambda)\Phi(\lambda) = \Phi(\lambda)B(\lambda),$$

while (2.20) on $(x_d - 1, x_1)$ gives $A_-(\lambda) = B(\lambda)$.

We note that $A_-(\lambda) = E^{-1}A_+(\lambda)E$, and the preceding two equations can therefore be combined with (2.19) to give

$$\dot{\Psi}(\lambda) = \big[\Psi(\lambda), A_+(\lambda)\big] = \Psi(\lambda)A_+(\lambda) - A_+(\lambda)\Psi(\lambda). \tag{2.25}$$

One consequence is that the polynomial

$$P(\lambda) = \operatorname{Tr}\Psi(\lambda) = \Psi_{11}(\lambda) + \Psi_{22}(\lambda) \tag{2.26}$$

is an invariant of motion.

We write the entries of Ψ as Ψ_{jk} and, in particular, note the evolution of the off-diagonal entries,

$$\begin{aligned}\dot{\Psi}_{12} &= -\frac{2}{\lambda}\Psi_{12} + \frac{M_- e^{-1}}{e - e^{-1}}(\Psi_{11} - \Psi_{22}),\\ \dot{\Psi}_{21} &= \frac{2}{\lambda}\Psi_{21} + \frac{M_+ e}{e - e^{-1}}(\Psi_{11} - \Psi_{22}).\end{aligned} \tag{2.27}$$

We let λ_{1j} and λ_{2j}, $j = 1, \ldots, d-1$, be the respective nonzero roots of Ψ_{12} and of Ψ_{21}. Evaluating (2.27) at these roots, we obtain

$$\begin{aligned}\dot{\Psi}_{12}(\lambda_{1j}) &= \frac{M_- e^{-2}}{1 - e^{-2}}(\Psi_{11} - \Psi_{22}),\\ \dot{\Psi}_{21}(\lambda_{2j}) &= \frac{M_+}{1 - e^{-2}}(\Psi_{11} - \Psi_{22}).\end{aligned} \tag{2.28}$$

In view of (2.17) and (2.19),

$$\begin{aligned}\Psi_{12}(\lambda) &= \lambda\frac{M_-}{2e}\prod_{k=1}^{d-1}\left(1 - \frac{\lambda}{\lambda_{1k}}\right),\\ \Psi_{21}(\lambda) &= -\lambda\frac{M_+ e}{2}\prod_{k=1}^{d-1}\left(1 - \frac{\lambda}{\lambda_{2k}}\right).\end{aligned} \tag{2.29}$$

Evaluating at the roots, we obtain

$$\begin{aligned}\dot{\Psi}_{12}(\lambda_{1j}) &= \frac{M_-}{2e}\prod_{k\neq j}\left(1 - \frac{\lambda_{1j}}{\lambda_{1k}}\right)\dot{\lambda}_{1j},\\ \dot{\Psi}_{21}(\lambda_{2j}) &= -\frac{M_+ e}{2}\prod_{k\neq j}\left(1 - \frac{\lambda_{2j}}{\lambda_{2k}}\right)\dot{\lambda}_{2j}.\end{aligned} \tag{2.30}$$

Combining (2.28) and (2.30), we obtain

$$\begin{aligned}\frac{\dot{\lambda}_{1j}}{2} &= \frac{1}{\prod_{k\neq j}(1 - \lambda_{1j}/\lambda_{1k})}\frac{\Psi_{11}(\lambda_{1j}) - \Psi_{22}(\lambda_{1j})}{e - e^{-1}},\\ \frac{\dot{\lambda}_{2j}}{2} &= -\frac{1}{\prod_{k\neq j}(1 - \lambda_{2j}/\lambda_{2k})}\frac{\Psi_{11}(\lambda_{2j}) - \Psi_{22}(\lambda_{2j})}{e - e^{-1}}.\end{aligned} \tag{2.31}$$

We note that because $\det \Psi = 1$,

$$(\Psi_{11} - \Psi_{22})^2 = P(\lambda)^2 - 4\Psi_{11}\Psi_{22} = P(\lambda)^2 - 4 \tag{2.32}$$

when $\Psi_{12}\Psi_{21} = 0$. Therefore, Eq. (2.31) can be written as

$$\begin{aligned}\dot{\lambda}_{1j} &= \frac{2}{\prod_{k\neq j}(1 - \lambda_{1j}/\lambda_{1k})}\sqrt{Q(\lambda_{1j})},\\ \dot{\lambda}_{2j} &= -\frac{2}{\prod_{k\neq j}(1 - \lambda_{2j}/\lambda_{2k})}\sqrt{Q(\lambda_{2j})},\end{aligned} \tag{2.33}$$

where

$$Q(\lambda) \equiv \frac{P(\lambda)^2 - 4}{(e - e^{-1})^2}. \tag{2.34}$$

In view of the factor 2 in (2.33) and in (2.38) below, it is convenient to take $s = \lambda/2$ as a new variable. We consider the hyperelliptic curve

$$X = \{(s, z) \colon z^2 = Q(2s)\}. \tag{2.35}$$

This curve has genus $g = d - 1$. Let ξ_0 be a point of X, let $\omega_1, \ldots, \omega_{d-1}$ be a normalized basis of Abelian differentials for X, and let $\Lambda \subset \mathbb{C}^{d-1}$ be the period lattice. The corresponding Abel map $A \colon X^{d-1} \to \mathbb{C}^{d-1}/\Lambda$ takes a point $(\xi_1, \ldots, \xi_{d-1}) \in X^{d-1}$ to the point whose mth coordinate is

$$\sum_{k=1}^{d-1} \int_{\xi_0}^{\xi_k} \omega_m.$$

Theorem 2.1. *The Abel map for curve* (2.35) *linearizes the flow of the roots* $\{\lambda_{1j}\}$ *and* $\{\lambda_{2j}\}$, *i.e., if* $\xi_j(t) \in X$ *respectively projects to* $s_{1j} = \lambda_{1j}(t)/2$ *and* $s_{2j} = \lambda_{2j}(t)/2$, $j = 1, \ldots, d-1$, *then*

$$\frac{d}{dt} A(\xi(t)) = \text{const.} \tag{2.36}$$

Proof. The differentials ω_m are linear combinations of the holomorphic differentials

$$\alpha_m = \frac{s^m \, ds}{\sqrt{Q(2s)}}, \quad m = 0, \ldots, d-2.$$

It therefore suffices to prove the analogue of (2.36) for each of the α_m. According to (2.33), taking the correct choice of the square root gives

$$\frac{d}{dt} \sum_{j=1}^{d-1} \int_{\xi_0}^{\xi_j(t)} \frac{s^m \, ds}{\sqrt{Q(2s)}} = \sum_{j=1}^{d-1} \frac{s_{1j}^m}{\prod_{k \neq j} (1 - s_{1j}/s_{1k})}. \tag{2.37}$$

Let

$$R_m(s) = -s^{m-1} \prod_{k=1}^{d-1} \left(1 - \frac{s}{s_{1k}}\right)^{-1}, \quad m = 0, 1, \ldots, d-2.$$

Because R_m is $O(s^{m-d})$ as $s \to \infty$, the sum of its residues is zero. For $m = 1, \ldots, d-2$, the last sum in the right-hand side in (2.37) is this sum of the residues of R_m and is therefore zero. For $m = 0$, R_m has an additional residue -2 at $s = 0$, and the sum in the right-hand side in (2.37) is 2. The same argument applies to s_{2j}.

The momentum $M = \sum m_j$ is constant; this follows either by direct computation using the Hamilton equations for Hamiltonian (2.13) or because the linear term of the invariant polynomial $P(\lambda)$ has the coefficient $(e - e^{-1})M$. We now compute the flow of the terms $M_\pm$. Expressions (2.29) imply that

$$\frac{\dot{M}_-}{M_-} = \lim_{\lambda \to 0} \frac{\dot{\Psi}_{12}}{\Psi_{12}}, \qquad \frac{\dot{M}_+}{M_+} = \lim_{\lambda \to 0} \frac{\dot{\Psi}_{21}}{\Psi_{21}}. \tag{2.38}$$

To evaluate the limits, we turn to (2.27). We note that (2.17) and (2.19) imply that

$$\Psi_{11} - \Psi_{22} = e - e^{-1} + (e + e^{-1})\frac{M}{2}\lambda + O(\lambda^2).$$

This and (2.29) imply that

$$\frac{M_- e^{-2}}{1 - e^{-2}} \frac{\Psi_{11} - \Psi_{22}}{\Psi_{12}} = \frac{2}{\lambda}\left\{\frac{1 + (e + e^{-1})(e - e^{-1})^{-1}M\lambda/2 + \cdots}{1 - \lambda\sum \lambda_{ij}{}^{-1} + \cdots}\right\},$$
$$\frac{M_+}{1 - e^{-2}} \frac{\Psi_{11} - \Psi_{22}}{\Psi_{21}} = -\frac{2}{\lambda}\left\{\frac{1 + (e + e^{-1})(e - e^{-1})^{-1}M\lambda/2 + \cdots}{1 - \lambda\sum \lambda_{2j}{}^{-1} + \cdots}\right\}.$$

Combining these equations with (2.27), we find that (2.38) becomes

$$\begin{aligned} \frac{\dot{M}_-}{M_-} &= \frac{e + e^{-1}}{e - e^{-1}}M + 2\sum_{j=1}^{d-1}\frac{1}{\lambda_{1j}}, \\ \frac{\dot{M}_+}{M_+} &= -\frac{e + e^{-1}}{e - e^{-1}}M - 2\sum_{j=1}^{d-1}\frac{1}{\lambda_{2j}}. \end{aligned} \tag{2.39}$$

3. Periodic two-particle systems

In this section, we consider the periodic Camassa–Holm equation in the case where $d = 2$. The Hamiltonian is

$$\begin{aligned} H(x_1, x_2, m_2, m_2) &= \frac{1}{2}m_1^2 G(0) + m_1 m_2 G(x_2 - x_1) + \frac{1}{2}m_2{}^2 G(0) = \\ &= \frac{1}{2}M^2 G(0) + m_1 m_2[G(x_1 - x_2) - G(0)]. \end{aligned} \tag{3.1}$$

Because H and M are constants of motion, it follows that the product $m_1 m_2[G(x_1 - x_2) - G(0)]$ is constant. Therefore, m_j do not change sign except at a singularity, where $x_2 - x_1$ is necessarily 0 or 1, $m_1 m_2$ blows up, and both m_j change sign. Moreover, if the m_j have the same sign at some instant, then $|m_j| \leq |M|$, there is therefore no blow-up, and the distance $x_2 - x_1$ is bounded away from 0 and 1. The case where $m_1, m_2 > 0$ is the peakon/peakon case, the case where $m_1, m_2 < 0$ is the antipeakon/antipeakon case, and the case where $m_1 m_2 < 0$ is the peakon/antipeakon case. The basic result is the same in each of these three cases, although there is an important difference between the first two and the third.

Let $y_j = e^{2x_j}$. Then

$$M = m_1 + m_2, \qquad M_+ = m_1 y_1 + m_2 y_2, \qquad M_- = \frac{m_1}{y_1} + \frac{m_2}{y_2}. \tag{3.2}$$

The scattering matrix Φ is given by

$$\Phi(\lambda) = I + \frac{\lambda}{2}\begin{bmatrix} M & M_- \\ -M_+ & -M \end{bmatrix} + \frac{\lambda^2 m_1 m_2}{4}\begin{bmatrix} 1 - \dfrac{y_1}{y_2} & \dfrac{1}{y_1} - \dfrac{1}{y_2} \\ y_1 - y_2 & 1 - \dfrac{y_2}{y_1} \end{bmatrix}. \tag{3.3}$$

We let N_1 and N_2 denote the diagonal coefficients of $\lambda^2/4$,

$$N_1 = m_1 m_2\left(1 - \frac{y_1}{y_2}\right), \qquad N_2 = m_1 m_2\left(1 - \frac{y_2}{y_1}\right). \tag{3.4}$$

The invariant polynomial $\mathrm{Tr}\,\Psi$ is then given by

$$P(\lambda) = e + e^{-1} + M(e - e^{-1})\frac{\lambda}{2} + (N_1 e + N_2 e^{-1})\frac{\lambda^2}{4}. \tag{3.5}$$

Let Q be the polynomial given by (2.34). In this case, it is a quartic, and the curve X is elliptic. A direct calculation shows that the roots of Q are real and distinct. (An exception occurs when $x_2 - x_1 = 1/2$ and $m_1 = m_2$, but this is simply the case where $d = 1$ and the period is $1/2$.)

Theorem 3.1. *Let $d = 2$. Representatives $x_1(t)$ and $x_2(t)$ can be chosen such that $0 < x_2(t) - x_1(t) < 1$ and*

$$m(x,t) = \sum_{n=-\infty}^{+\infty} \big\{ m_1(t)\,\delta(x - x_1(t) - n) + m_2(t)\delta(x - x_2(t) - n) \big\},$$

where $m_1(t)$, $m_2(t)$, $x_2(t) - x_1(t)$, and $x_1(t) + x_2(t) - Kt$ are periodic with the period $2\omega_1$ and K is a constant. Each of these functions can be written in terms of the Weierstrass elliptic functions with the periods $2\omega_1$ and $2\omega_2$ given by (3.16).

The proof consists of two parts: an analysis of the evolution of the roots of Ψ_{12} and Ψ_{21} and of $M_\pm$, and the recovery of the m_j and x_j from this data. We begin with the recovery problem.

Let N denote the coefficient of $\lambda^2/4$ in the invariant polynomial $\mathrm{Tr}\,\Psi$. Then

$$eN_1 + e^{-1}N_2 = N, \qquad N_1 + N_2 = M^2 - M_+M_-. \tag{3.6}$$

These equations allow expressing N_j in terms of the invariants N and M and the product M_+M_-. To solve the inverse problem, we need at least one more expression whose evolution can be solved explicitly. We note that Ψ_{12} and Ψ_{21} each have one nonzero root, respectively denoted by λ_1 and λ_2. It follows from (3.3) that

$$\begin{aligned} \lambda_1 &= -\frac{2M_- y_1 y_2}{m_1 m_2 (y_2 - y_1)} = -2\frac{m_1 y_2 + m_2 y_1}{m_1 m_2 (y_2 - y_1)}, \\ \lambda_2 &= -\frac{2M_+}{m_1 m_2 (y_2 - y_1)} = -2\frac{m_1 y_1 + m_2 y_2}{m_1 m_2 (y_2 - y_1)}. \end{aligned} \tag{3.7}$$

Lemma 3.1. *The data m_j and $y_j = e^{2x_j}$, $j = 1, 2$, can be expressed in terms of the invariants M and N and the quantities M_+, M_-, λ_1, and λ_2:*

$$y_1 = -\frac{\lambda_1}{2}\frac{N_1}{M_-}, \qquad y_2 = -\frac{2}{\lambda_2}\frac{M_+}{N_1}, \tag{3.8}$$

$$m_1 = \frac{M - M_- y_2}{1 + N_2 N_1^{-1}}, \qquad m_2 = \frac{M - M_- y_1}{1 + N_1 N_2^{-1}}. \tag{3.9}$$

Proof. Equation (3.8) follows from (3.7). It follows from (3.4) that

$$\frac{y_1}{y_2} = -\frac{N_1}{N_2}. \tag{3.10}$$

Combining this with (3.2), we obtain (3.9). Equations (3.7) imply that

$$y_1 y_2 = \frac{\lambda_1 M_+}{\lambda_2 M_-}. \tag{3.11}$$

The lemma is proved.

We next analyze the evolution of λ_j and $M_\pm$. As we remarked above, calculation shows that invariant polynomial (3.5) has four distinct real roots $r_1 < r_2 < r_3 < r_4$. These roots are negative in the peakon/peakon case and positive in the antipeakon/antipeakon case. In the peakon/antipeakon case, two roots are negative and two are positive. The quartic Q is positive in the interval (r_2, r_3), and inspection shows that in each case, λ_1 and λ_2 lie in the closure of this interval.

When $d = 2$, each of Φ_{12} and Φ_{21} has only one nonzero root, respectively denoted by λ_1 and λ_2. The product in Eqs. (2.31) is vacuous, and the equations simplify to

$$\dot{\lambda}_1 = 2\sqrt{Q(\lambda_1)}, \qquad \dot{\lambda}_2 = -2\sqrt{Q(\lambda_2)} \tag{3.12}$$

with an appropriate choice of the square root. It follows that λ_j are periodic as functions of real t and that there is an instant in each period at which $\lambda_1 = \lambda_2$. After a translation in x and t, we can assume that

$$\lambda_1(0) = \lambda_2(0), \qquad x_1(0) + x_2(0) = 0. \tag{3.13}$$

By (3.7),

$$\lambda_1 - \lambda_2 = \frac{2}{m_1} - \frac{2}{m_2}.$$

Equality occurs only when $m_1 = m_2$ or m_1 and m_2 are singular. The former cannot occur when $m_1 m_2 < 0$, and the latter cannot occur if $m_1 m_2 > 0$. Therefore,

$$\begin{aligned} &m_1(0) = m_2(0) \quad \text{if } m_1 m_2 > 0, \\ &m_1, m_2 \text{ are singular} \quad \text{at } t = 0 \quad \text{if } m_1 m_2 < 0. \end{aligned} \tag{3.14}$$

Equations (2.39) show that we also want to work with the reciprocals $\mu_j = 2/\lambda_j$. Equations (3.12) imply

$$\dot{\mu}_1 = -\sqrt{\widetilde{Q}(\mu_1)}, \qquad \dot{\mu}_2 = \sqrt{\widetilde{Q}(\mu_2)}, \tag{3.15}$$

where $\widetilde{Q}$ is the quartic

$$\widetilde{Q}(\mu) = \mu^4 Q\left(\frac{2}{\mu}\right).$$

Using the Hamilton equations once again, it can be shown that near $t = 0$, the appropriate choice in each of these equations is the positive square root.

Let $s_1 < s_2 < s_3 < s_4$ be the roots of $\widetilde{Q}$. Then $\widetilde{Q}$ is positive on the interval (s_2, s_3) and negative on the intervals (s_1, s_2) and (s_3, s_4). The elliptic curve $\widetilde{X} = \{(z, \mu)\colon\ z^2 = \widetilde{Q}(\mu)\}$ has the periods

$$2\omega_1 = 2\int_{s_2}^{s_3} \frac{ds}{\sqrt{\widetilde{Q}(s)}}, \qquad 2\omega_2 = 2i\int_{s_1}^{s_2} \frac{ds}{\sqrt{-\widetilde{Q}(s)}}. \tag{3.16}$$

(A change of the integration variables shows that these are also the periods of elliptic curve (2.35).) Choosing a point $p_0 \in \mathbb{C}$, we integrate (3.15) to obtain

$$-t - a = \int_{p_0}^{\mu_1(t)} \frac{ds}{\sqrt{\widetilde{Q}(s)}}, \qquad t - a = \int_{p_0}^{\mu_2(t)} \frac{ds}{\sqrt{\widetilde{Q}(s)}}, \tag{3.17}$$

where

$$a=\int_{\mu_1(0)}^{p_0}\frac{ds}{\sqrt{\widetilde{Q}(s)}}. \tag{3.18}$$

We can recover μ_j in terms of the Weierstrass function $\wp$ with the periods $2\omega_j$ by taking p_0 in (3.17) to be a root of $\widetilde{Q}$, for example, $p_0 = s_2$. In fact, with this choice, (3.17) can be inverted to

$$\mu_1(t)=s_2+\frac{c}{\wp(t+a)-\wp(b)}, \tag{3.19}$$

$$\mu_2(t)=s_2+\frac{c}{\wp(t-a)-\wp(b)}, \tag{3.20}$$

where $\wp(b)=\widetilde{Q}''(s_2)/24$ and $c=\widetilde{Q}'(s_2)/4$ (p. 439 in [12]). (In (3.19), we use the fact that $\wp$ is even.)

We here recall that $\wp$ is related to the Weierstrass σ and ζ functions by

$$\wp=-\zeta', \qquad \zeta=\frac{\sigma'}{\sigma}. \tag{3.21}$$

The function ζ has only a simple pole with residue 1 at each point of the period lattice $\Lambda=\{2m\omega_1+2n\omega_2\}$; it has no other poles. Therefore, $\wp$ has double poles on Λ and no other poles. Because $\wp$ is doubly periodic, the two differences $\delta_j=\zeta(z+2\omega_j)-\zeta(z)$, $j=1,2$, are constant.

There are two poles for each of the expressions (3.19) and (3.20) in each period parallelogram. Because $\wp$ is even, the poles occur at $t=\pm a\pm b$ modulo Λ. In view of (3.19) and (3.20), we have

$$b=\int_{s_2}^{\infty}\frac{ds}{\sqrt{\widetilde{Q}(s)}}. \tag{3.22}$$

Taking the integral along the real axis, we find that $b-\bar{b}=\pm 2\omega_2$. It follows that $\wp(s\pm b)$ is real for real s:

$$\overline{\wp(s+b)}=\wp(s+\bar{b})=\wp(s+b\mp 2\omega_2)=\wp(s+b). \tag{3.23}$$

We compute the residues of (3.19) and (3.20) as follows. We note that $\widetilde{Q}$ has the leading coefficient 1 and Eqs. (3.17) and (3.22) therefore imply that

$$t-a-b=-\int_{\mu_2(t)}^{\infty}\frac{ds}{\sqrt{\widetilde{Q}(s)}}\sim-\frac{1}{\mu_2(t)} \tag{3.24}$$

as $\mu_2(t)\to+\infty$. It follows that μ_2 has the residue -1 at $t=a+b$; hence, $c/\wp'(b)=1$. The residue at $t=a-b$ is $c/\wp'(-b)=-c/\wp'(b)=1$. Therefore, μ_1 has the residue 1 at $-t=a+b$ and the residue -1 at $-t=a-b$. The μ_j are doubly periodic. Therefore, up to additive constants, they are equal to the corresponding linear combinations $\mp\big[\zeta(\pm t-a-b)-\zeta(\pm t-a+b)\big]$.

The additive constants can be evaluated by taking $t=\pm a$. Because ζ is odd, the results can be written as

$$\begin{aligned}\mu_1(t)&=s_2+\zeta(t+a+b)-\zeta(t+a-b)-2\zeta(b),\\ \mu_2(t)&=s_2+\zeta(t-a+b)-\zeta(t-a-b)-2\zeta(b).\end{aligned} \tag{3.25}$$

These functions have the period $2\omega_1$ as functions of real t.

Because ζ is the logarithmic derivative of the Weierstrass function σ, we can integrate (2.39) using the periodicity of the μ_j,

$$\begin{aligned} M_- &= M_-(0)e^{-\kappa t}\,\frac{\sigma(t+a+b)\sigma(a-b)}{\sigma(t+a-b)\sigma(a+b)}, \\ M_+ &= M_+(0)e^{\kappa t}\,\frac{\sigma(t-a-b)\sigma(-a+b)}{\sigma(t-a+b)\sigma(-a-b)}, \end{aligned} \tag{3.26}$$

where $\kappa = 2\zeta(b) - s_2 - M\coth 1$.

Proof of Theorem 3.1. It follows from (3.26) that M_-M_+ has the real period $2\omega_1$. Equations (3.6) and (3.10) then imply that N_1, N_2, and $x_2 - x_1 = \log(y_2/y_1)/2$ have the real period $2\omega_1$. Equations (3.7) imply that

$$\frac{\lambda_1}{\lambda_2} = \frac{y_2/y_1 + m_2/m_1}{1 + (m_2/m_1)(y_2/y_1)}.$$

It follows that m_2/m_1 has the real period $2\omega_1$. Because $m_1 + m_2 = M$ is constant, it follows that m_1 and m_2 each have the real period $2\omega_1$. Finally, the derivative

$$\dot{x}_1 + \dot{x}_2 = M\{G(0) + G(x_2 - x_1)\}$$

has the real period $2\omega_1$, and therefore

$$x_1 + x_2 - t\int_0^{2\omega_1} M\bigl\{G(0) + G(x_2(s) - x_1(s))\bigr\}\,ds$$

has the real period $2\omega_1$. This completes the proof of Theorem 3.1.

As noted above, the roots λ_1 and λ_2 are confined to the closed interval $[r_2, r_3]$ whose endpoints are the middle two roots of Q. In the peakon/peakon and antipeakon/antipeakon cases, we have $m_1m_2 > 0$, the roots of Q are respectively all negative or all positive, and the λ_j therefore stay away from 0. Consequently, μ_j and $M_\pm$ are regular and nonvanishing. This confirms the observation at the beginning of this section that in these cases, the solution is regular, and $x_2 - x_1$ is bounded away from 0. By contrast, in the peakon/antipeakon case where $m_1m_2 < 0$, the μ_j have a simple pole at $t = 0$. It follows that the m_j have simple poles at $t = 0$, while $x_2 - x_1$ has a double zero at $t = 0$. The latter offsets the former, and it follows that the solution

$$u(x,t) = m_1(t)G(x - x_1(t)) + m_2(t)G(x - x_2(t))$$

is uniformly continuous in x and t; the argument here is the same as in [7].

4. Calogero–Françoise flows

As noted in the introduction, Calogero and Françoise [1], [3] introduced a family of completely integrable finite-dimensional Hamiltonian systems with the Hamiltonian in a form that generalizes (2.8) and (3.1). With these flows, we associate a spectral problem that is a generalization of (2.1). We fix a complex parameter $\nu \neq 0$ and consider flows of

$$L(\lambda, \nu) = D^2 - \nu^2 - \lambda m(x), \tag{4.1}$$

which is again compatible with generalized Lax evolution (2.2), where $A(\lambda)$ is given by (2.3). Here, (2.2) is equivalent to the conditions

$$m_t = u_x m + (um)_x, \qquad 2m_x = 4\nu^2 u_x - u_{xxx}. \tag{4.2}$$

Integrating the second equation in (2.4) gives evolution (4.2).

We assume that m is a discrete measure of form (2.6), $x_1 < \cdots < x_d$. Any even fundamental solution for $(-D^3 + 4\nu^2 D)m = 2Du$ has the form

$$F(x) = \frac{\beta_+}{2\nu} e^{2\nu|x|} + \frac{\beta_-}{2\nu} e^{-2\nu|x|} + \gamma, \tag{4.3}$$

where β_-, β_+, and γ are constants and

$$\beta_- - \beta_+ = 1. \tag{4.4}$$

Once again, (4.2) for u is equivalent to the Hamilton equations for the Hamiltonian

$$H(x, m) = \frac{1}{2} \sum_{j,k=1}^{d} m_j m_k F(x_j - x_k). \tag{4.5}$$

The general problem can be reduced to the case $\gamma = 0$ as follows. It is easily seen that $M = \sum m_j$ is a constant of motion. Temporarily, we indicate the γ dependence with a subscript. We note that

$$H_\gamma(x, m) = H_0(x, m) + \frac{1}{2} M^2 \gamma.$$

It follows that if $\{x_j(t), m_j(t)\}$ is a solution of the Hamilton equations for H_0, then $\{x_j(t) + \gamma M t, m_j(t)\}$ is a solution for H_γ. From now on, we assume that $\gamma = 0$. We take u as analogous to (2.7):

$$u(x, t) = \sum_{j=1}^{d} m_j(t) F(x - x_j(t)). \tag{4.6}$$

In analogy with the periodic problem considered in the previous sections, we set

$$M_\pm = \sum_{j=1}^{d} e^{\pm 2\nu x_j} m_j. \tag{4.7}$$

Then u given by (4.6) has precise asymptotics:

$$u(x) = \begin{cases} \dfrac{\beta_-}{2\nu} M_- e^{2\nu x} + \dfrac{\beta_+}{2\nu} M_+ e^{-2\nu x}, & x < x_1, \\ \dfrac{\beta_+}{2\nu} M_- e^{2\nu x} + \dfrac{\beta_-}{2\nu} M_+ e^{-2\nu x}, & x > x_d. \end{cases} \tag{4.8}$$

We note that, once again, M is constant under the flow.

We can analyze the spectral problem

$$D^2\varphi - \nu^2 \varphi = \lambda \varphi m \tag{4.9}$$

in the same way as in the periodic case above. We set $x_0 = -\infty$, $x_{d+1} = +\infty$. On each interval (x_j, x_{j+1}), a solution of (4.9) is a linear combination $a_j e^{\nu x} + b_j e^{-\nu x}$. Equation (2.1) translates to

$$\begin{aligned} a_{j+1} e^{\nu x_j} + b_{j+1} e^{-\nu x_j} &= a_j e^{\nu x_j} + b_j e^{-\nu x_j}, \\ a_{j+1} e^{\nu x_j} - b_{j+1} e^{-\nu x_j} &= a_j e^{\nu x_j} - b_j e^{-\nu x_j} + \lambda m_j (a_j e^{\nu x_j} - b_j e^{-\nu x_j}). \end{aligned} \tag{4.10}$$

The transition is given by

$$\begin{bmatrix} a_{j+1} \\ b_{j+1} \end{bmatrix} = \begin{bmatrix} 1 + \dfrac{\lambda}{2\nu} m_j & \dfrac{\lambda}{2\nu} m_j e^{-2\nu x_j} \\ -\dfrac{\lambda}{2\nu} m_j e^{2\nu x_j} & 1 - \dfrac{\lambda}{2\nu} m_j \end{bmatrix} \begin{bmatrix} a_j \\ b_j \end{bmatrix}. \tag{4.11}$$

With the transition matrix in (4.11) denoted by $T_j(\lambda)$, we see that the scattering matrix for this problem is given by the product

$$\Phi(\lambda) = T_d(\lambda) T_{d-1}(\lambda) \cdots T_1(\lambda) = I + \frac{\lambda}{2\nu} \begin{bmatrix} M & M_- \\ -M_+ & -M \end{bmatrix} + O(\lambda^2). \tag{4.12}$$

Again, the entries $\Phi_{ij}(\lambda)$ are polynomials of degree d in λ, and Φ has determinant 1.

We now consider evolution under the flow. Equation (2.2) implies (2.20). It follows from (4.8) and an argument similar to that in Sec. 2 that the vector representation of $A(\lambda)\varphi$ is

$$A(\lambda)\colon \begin{bmatrix} a \\ b \end{bmatrix} \to A_-(\lambda) \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \dfrac{\nu}{\lambda} & \beta_- M_- \\ -\beta_+ M_+ & -\dfrac{\nu}{\lambda} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}, \quad x < x_1; \tag{4.13}$$

$$A(\lambda)\colon \begin{bmatrix} a \\ b \end{bmatrix} \to A_+(\lambda) \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \dfrac{\nu}{\lambda} & \beta_+ M_- \\ -\beta_- M_+ & -\dfrac{\nu}{\lambda} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}, \quad x > x_d. \tag{4.14}$$

We now take solutions $\varphi_1 = e^{\nu x}$ and $\varphi_2 = e^{-\nu x}$ for $x < x_1$ such that the two vector representations give the identity matrix for $x < x_1$ and the matrix $\Phi(\lambda)$ for $x > x_d$. As before, we deduce that

$$\dot{\Phi}(\lambda) + A_+(\lambda)\Phi(\lambda) = \Phi(\lambda) A_-(\lambda).$$

If $\beta_-\beta_+ = 0$, then the problem simplifies and can be treated in the same way as the corresponding Camassa–Holm problem [7]. We now assume that $\beta_-\beta_+ \neq 0$. We note that

$$A_-(\lambda) = B^{-1} A_+(\lambda) B, \qquad B = \begin{bmatrix} \beta_- & 0 \\ 0 & \beta_+ \end{bmatrix},$$

and setting $\Psi = \Phi B$, we therefore have

$$\dot{\Psi}(\lambda) = \big[\Psi(\lambda), A(\lambda)\big] = \Psi(\lambda) A_+(\lambda) - A_+(\lambda)\Psi(\lambda). \tag{4.15}$$

The polynomial

$$P(\lambda) = \Psi_{11}(\lambda) + \Psi_{22}(\lambda) = \beta_- \Phi_{11}(\lambda) + \beta_+ \Phi_{22}(\lambda) \tag{4.16}$$

is therefore an invariant of motion. The evolution of the off-diagonal entries is

$$\dot{\Psi}_{12} = -\frac{2\nu}{\lambda} \Psi_{12} + \beta_+ M_- (\Psi_{11} - \Psi_{22}), \tag{4.17}$$

$$\dot{\Psi}_{21} = \frac{2\nu}{\lambda} \Psi_{21} + \beta_- M_+ (\Psi_{11} - \Psi_{22}). \tag{4.18}$$

These observations show that the problem can be analyzed in exactly the same way as the periodic Camassa–Holm problem: the Abel map for an associated hyperelliptic curve linearizes the evolution of the roots of Φ_{12} and Φ_{21}.

For $d = 2$, solutions can be expressed in terms of Weierstrass functions, exactly as in Sec. 3. In this case, the transition matrix Φ is given by (2.17) with $\lambda/2$ replaced with $\lambda/2\nu$, and $y_j = e^{2\nu x_j}$.

The dynamical behavior critically depends on ν and β_- and on the initial conditions. We here discuss the problem for $d = 2$, $m_1 m_2 > 0$, in two special cases:

$$\nu = 1, \qquad \beta_- = -\beta_+ = \frac{1}{2}: \quad F(x) = -\frac{1}{2}\sinh(2|x|), \tag{4.19}$$

$$\nu = i, \qquad \beta_- = -\beta_+ = \frac{1}{2}: \quad F(x) = -\frac{1}{2}\sin(2|x|). \tag{4.20}$$

Theorem 4.1. *Under assumptions* (4.19)*, the CF system with $d = 2$ and $m_1 m_2 > 0$ blows up in finite time in both the positive and negative time directions.*

Proof. Under assumption (4.19), the invariant polynomial is

$$P(\lambda) = M\frac{\lambda}{2} + m_1 m_2 \sinh 2(x_2 - x_1)\frac{\lambda^2}{4}.$$

Here, $-4\beta_-\beta_+ = 1$, and the associated quartic is therefore given by

$$Q(\lambda) = P(\lambda)^2 + 1,$$

which is positive on the real axis. The roots λ_1 of Φ_{12} and λ_2 of Φ_{21} are given by Eqs. (3.7) (with $y_j = e^{2x_j}$), implying that they are negative if the m_j are positive and positive if the m_j are negative. Evolution equations (3.12) imply that each root converges to 0 and to ∞ in finite time in both the positive and negative time directions. Equations (3.7) imply

$$\frac{\lambda_1\lambda_2}{4} = \frac{M^2}{2(m_1m_2)^2(\cosh 2(x_2 - x_1) - 1)} + \frac{1}{m_1m_2} > \frac{1}{m_1m_2} \geq \frac{4}{M^2}.$$

Therefore, as one of λ_j converges to 0, the other converges to $\pm\infty$. On the other hand,

$$\frac{\lambda_1}{\lambda_2} = \frac{m_1 e^{2(x_2 - x_1)} + m_2}{m_2 e^{2(x_2 - x_1)} + m_1}.$$

It follows that when this ratio converges to 0 or to ∞, the separation $x_2 - x_1$ converges to $+\infty$ in finite time in both the positive and negative time directions.

Theorem 4.2. *Under assumptions* (4.20) *with $d = 2$ and $m_1 m_2 > 0$, the functions $m_1(t)$, $m_2(t)$, $x_2(t) - x_1(t)$, and $x_1(t) + x_2(t) - Kt$ for some constant K are periodic, and the period is equal to the real period of the quartic $-\widetilde{Q}(\mu) = \mu^4[P(2/\mu)^2 + 1]$.*

Proof. Under assumptions (4.20),

$$P(\lambda) = -iM\frac{\lambda}{2} - im_1m_2 \sin 2(x_2 - x_1)\frac{\lambda^2}{4}.$$

The coefficients of P are constant and $m_1m_2 \leq M^2/4$. It follows that the factor $\sin 2(x_2 - x_1)$ is bounded away from 0, and the separation $x_2 - x_1$ is hence bounded away from 0. The roots of $P \pm i$ are real and distinct, and the quartic Q therefore has four distinct real roots.

On the other hand, the zeros of $\Phi_{12}\Phi_{21}$ are now complex, and (3.7) (suitably modified for this case by replacing the factor 2 with $2\nu = 2i$) implies that

$$\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2 = \frac{M\cos(x_2 - x_1)}{m_1m_2\sin(x_1 - x_2)}.$$

Because P is an invariant polynomial under the flow, $m_1m_2\sin 2(x_2 - x_1)$ is a constant of motion; because $2\sin 2(x_2 - x_1) = \sin(x_2 - x_1)\cos(x_2 - x_1) \neq 0$, it follows that neither $\sin(x_2 - x_1)$ nor $\cos(x_2 - x_1)$ can vanish. Consequently, neither of the λ_j vanish, and their reciprocals are bounded periodic functions of t. The data x_j and m_j are nonsingular and have the same periodicity properties as in Theorem 3.1.

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