Peakons, Strings, and the Finite Toda Lattice

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In Memory of Jürgen Moser

Abstract

As is well-known, the Toda lattice flow may be realized as an isospectral flow of a Jacobi matrix. A bijective map from a discrete string problem with positive weights to Jacobi matrices allows the pure peakon flow of the Camassa-Holm equation to be realized as an isospectral Jacobi flow as well. This gives a unified picture of the Toda, Jacobi, and multipeakon flows, and leads to explicit solutions of the Jacobi flows via Stieltjes' determination of the continued fraction expansion of a Stieltjes transform. A simple modification produces a bijection from generalized strings, with positive and negative weights, to singular Jacobi matrices, and thus brings peakon/antipeakon flows into the same picture. © 2001 John Wiley & Sons, Inc.

1 Introduction

The Toda lattice [15, 16] is one of several integrable systems whose discovery was prompted by the Fermi-Pasta-Ulam experiment. Flaschka [5] found a nonlinear transformation of the infinite Toda lattice flow to an isospectral flow of an infinite tridiagonal matrix—a Jacobi matrix—and found a Lax pair for the flow. He used a discrete version of Korteweg–de Vries (KdV) inverse scattering theory to linearize the flow. The periodic Toda lattice was solved by Date and Tanaka [3] and by Kac and van Moerbeke [6] in terms of hyperelliptic integrals associated with an algebraic curve; Dubrovin, Matveev, and Novikov [4] expressed the solution in terms of the theta function of the curve. The case of a finite tied lattice ($x_0 = -\infty,\ x_{n+1} = \infty$) was investigated by Moser [10], who pointed out the connection of the isospectral problem with Stieltjes' work on continued fractions and the spectral theory of Jacobi matrices. Moser calculated the full solution for two-particle flows and established, by an indirect argument, the asymptotic phase shifts for the positions of n particles for any n. The connections between these

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viewpoints was explored in an unpublished paper by McKean [9]. See also the recent work of Nakamura [12].

In [1, 2] we investigated another finite-dimensional integrable system, the multipeakon solutions of the Camassa-Holm equation, by transforming the isospectral problem to a discrete string problem that (in the pure peakon or pure antipeakon case) is of the type investigated by Kreĭn [7, 8]. The string problem has a Weyl function whose continued fraction expansion gives the string data. The partial fraction decomposition gives the scattering data: vibration frequencies and the coupling constants; equivalently, it expresses the Weyl function as the Stieltjes transform of a discrete measure. A result of Stieltjes [14] gives the continued fraction expansion in terms of moments of the measure and thus provides an explicit solution of the multipeakon flow.

Isospectral flows of Jacobi matrices are discussed in Section 2. These include both the (transformed) finite Toda flow and a flow considered by Kac and van Moerbeke [6] in connection with a discretization of the Korteweg–de Vries equation. We introduce a Weyl function whose residues flow linearly under each of these Jacobi flows; it is a multiple of Moser's spectral function.

In Section 3 we construct a bijective map between discrete strings and negative semidefinite singular Jacobi matrices. The respective Weyl functions coincide under this mapping. The transformed Camassa-Holm flow is taken to an isospectral Jacobi flow. Both the map and its inverse are given explicitly by algebraic functions.

We summarize in Section 4 the results from [1, 2] that show how the result of Stieltjes [14] yields an explicit integration for each such flow. In Section 5 we use the formulas from Section 4 and from [2] to compute the asymptotics for the Jacobi flows. This allows a simple and direct computation of the phase shifts for the finite Toda flow.

In Section 6 the bijection between strings and Jacobi matrices is extended to generalized strings (having both positive and negative masses) in such a way that a string flow that corresponds to the peakon/antipeakon flow also maps to an isospectral flow of the Jacobi matrix. The Jacobi matrix varies smoothly under the flows; collision of a peakon and an antipeakon is manifested as a vanishing of a principal upper minor of the Jacobi matrix.

In short, the discrete string problem provides a common model in which the pure multipeakon and peakon/antipeakon flows, the finite Toda flow, and the isospectral Jacobi flows are all explicitly solvable in terms of Stieltjes' solution of the inverse problem for continued fraction expansions.

A mapping of the flow of n (rather than $n-1$) peakons into an isospectral flow of an $n \times n$ Jacobi matrix was constructed in [13]. The starting point is the r-matrix. The map from peakon variables to Jacobi matrix is given explicitly, but there does not seem to be a simple expression for the inverse mapping.

2 Jacobi Flows

A finite Jacobi matrix J is a tridiagonal, symmetric, real matrix whose off-diagonal elements are positive:

$$J = \begin{bmatrix} b_1 & a_1 & 0 & \cdots & \cdots & 0 \\ a_1 & b_2 & a_2 & \ddots & & \vdots \\ 0 & a_2 & b_3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & a_{n-1} \\ 0 & \cdots & \cdots & 0 & a_{n-1} & b_n \end{bmatrix}, \quad a_j > 0 . \tag{2.1}$$

The eigenvalues of J are simple, because the equation $Jv = \lambda v$ allows determination of v_{j+1} from v_j.

An isospectral flow of J has the Lax form

$$\dot{J} = [J, B], \quad B + B^t = 0 . \tag{2.2}$$

Suppose Φ is a real-valued function on the spectrum $\{\lambda_j\}$. Let $B = B(\Phi)$ denote the skew-symmetric matrix with the property that $B - \Phi(J)$ is lower triangular:

$$B(\Phi)_{jk} = \begin{cases} \operatorname{sgn}(k-j)\Phi(J)_{jk}, & j \neq k, \\ 0, & j = k . \end{cases} \tag{2.3}$$

For such a choice, equation (2.2) preserves the Jacobi form. In fact, $[J, B]$ is symmetric and

$$[J, B] = [J, B - \Phi(J)],$$

so the strictly upper triangular part of the commutator has nonzero entries only on the the first superdiagonal, and they come from the diagonal part of the lower triangular matrix $B - \Phi(J)$. Thus

$$\dot{a}_j = \left\{\Phi(J)_{jj} - \Phi(J)_{j+1,j+1}\right\} a_j ,$$

and it follows that positivity of the off-diagonal elements is preserved.

The finite *Toda flow* [15], as transformed by Flaschka [5], leads to the flow on Jacobi matrices generated by $\Phi(\lambda) \equiv \lambda$. The flow generated by $\Phi(\lambda) = \lambda^k$, $k = 2, \dots, n-1$, is compatible with the additional constraint

$$b_j = 0, \quad 1 \leq j \leq n . \tag{2.4}$$

The case $\Phi(\lambda) = \lambda^2$ with condition (2.4) is the flow considered by Kac and van Moerbeke [6] in their discretization of the Korteweg–de Vries equation. The family of flows (2.2) and (2.3) will be referred to here as the *Jacobi flows*.

It will be convenient to make two normalizations. First, for any real constant λ_0, the study of the flows of $J - \lambda_0 I$ is the same as the study of the flows of J. Therefore we may assume without loss of generality that J is negative semi-definite and singular, i.e., that the spectrum is nonpositive and contains 0. These spectral properties are, of course, preserved under isospectral flows. Second, if functions Φ and Ψ differ by a constant, then $\Phi(J)$ and $\Psi(J)$ differ by a multiple of the identity, so $B(\Phi) = B(\Psi)$. Therefore we may assume that $\Phi(0) = 0$.

DEFINITIONS A *normalized* Jacobi matrix is one that is negative semidefinite and singular. A *J-normalized function* is a real-valued function Φ defined on the spectrum of J such that $\Phi(0) = 0$. Suppose that J is normalized. The *normalized null vector* $v = v(J)$ is the unique vector $v = (v_1, \dots, v_n)^{\mathsf{T}}$ such that

$$Jv = 0\,, \quad \|v\| = 1\,, \quad v_n > 0\,. \tag{2.5}$$

The associated *Weyl function* is

$$W(\lambda) = \frac{1}{v_n^2}(\lambda I - J)^{-1}_{nn}\,. \tag{2.6}$$

The Weyl function has a partial fraction decomposition

$$W(\lambda) = \sum_{j=1}^{n} \frac{c_j}{\lambda - \lambda_j}\,, \quad c_j = \frac{1}{v_n^2}(E_j)_{nn}\,, \tag{2.7}$$

where E_j is the spectral projection for the eigenvalue λ_j.

THEOREM 2.1 *Suppose that J is a normalized Jacobi matrix and Φ is a J-normalized function. Then under the flow* (2.2) *and* (2.3)*, the residues of the Weyl function evolve linearly*:

$$\dot{c}_j = 2\Phi(\lambda_j)c_j\,. \tag{2.8}$$

PROOF: Let $B = B(\Phi)$ and let $v(t)$ be the normalized null vector of $J(t)$. We differentiate $Jv = 0$ and $(v, v) = 1$, where $(\cdot, \cdot)$ denotes the standard inner product in $\mathbb{R}^n$, and recall that B is skew-symmetric, to obtain

$$0 = J(\dot{v} + Bv)\,, \quad 0 = (\dot{v}, v) = (\dot{v} + Bv, v)\,.$$

Therefore $\dot{v} = -Bv$. By (2.3),

$$\begin{aligned} \dot{v}_n = -(Bv)_n &= -\sum_{j=1}^{n-1} B_{nj}v_j = \sum_{j=1}^{n-1} \Phi(J)_{nj}v_j \\ &= \sum_{j=1}^{n} \Phi(J)_{nj}v_j - \Phi(J)_{nn}v_n = [\Phi(J)v]_n - \Phi(J)_{nn}v_n \\ &= -\Phi(J)_{nn}v_n\,. \end{aligned} \tag{2.9}$$

The resolvent $(\lambda I - J)^{-1}$ also satisfies (2.2). By the functional calculus, it follows that every function of J evolves in the same way. In particular,

$$\frac{d}{dt}E_j = [E_j, B]\,.$$

It follows that the evolution of $r_j = (E_j)_{nn}$ is given by

$$\begin{aligned}(2.10)\quad \dot{r}_j = [E_j, B]_{nn} &= 2(E_j B)_{nn} \\ &= 2[E_j \Phi(J)]_{nn} - 2\Phi(J)_{nn} r_j = 2\Phi(\lambda_j) r_j - 2\Phi(J)_{nn} r_j \,.\end{aligned}$$

It follows from (2.9) and (2.10) that $c_j = r_j / v_n^2$ evolves according to (2.8). □

The matrix entries $r_j = (E_j)_{nn}$ are proportional to the c_j and sum to $I_{nn} = 1$. It follows from Theorem 2.1 that

$$(2.11)\qquad r_j(t) = [E_j(t)]_{nn} = \frac{r_j(0) e^{2\Phi(\lambda_j)t}}{\sum_k r_k(0) e^{2\Phi(\lambda_k)t}} \,.$$

This was proved by Moser [10, 11] for the cases $\Phi(\lambda) = \lambda$ and $\Phi(\lambda) = \lambda^2$, corresponding to the Toda flow and the Kac–van Moerbeke flow.

The following result is important for the connection between Jacobi matrices and string problems established in the next section:

LEMMA 2.2 *Let v be the normalized null vector of the normalized Jacobi matrix J. Then each entry v_j is positive.*

PROOF: Set $D_0 = 1$ and let D_k, $1 \le k \le n$, be the $k \times k$ upper principal minor of J. Expanding this minor along the last row and then the last column, we find

$$(2.12)\qquad D_k = b_k D_{k-1} - a_{k-1}^2 D_{k-2} \,, \quad k = 1, \dots, n \,,$$

where we take $a_0 = 0$. We claim that

$$(2.13)\qquad (-1)^k D_k > 0 \,, \quad k = 0, \dots, n-1 \,.$$

By assumption, $D_n = 0$. Since J is negative semidefinite, $(-1)^k D_k \ge 0$. Suppose that $k-1$ is the first index for which $D_k = 0$. If $k-1 < n$, then (2.13) implies that D_{k-2} and D_k have opposite signs, a contradiction.

We define a vector w by

$$(2.14)\qquad w_1 = 1 \,, \quad w_k = \frac{(-1)^{k-1} D_{k-1}}{a_1 a_2 \cdots a_{k-1}} \,, \quad k = 2, \dots, n \,.$$

It follows from (2.12), (2.14), and $D_n = 0$ that

$$(2.15)\qquad a_k w_{k+1} = -b_k w_k - a_{k-1} w_{k-1} \,, \quad k = 1, \dots, n-1 \,,$$

where again we set $a_0 = 0$. The equations (2.15) are equivalent to $Jw = 0$. Now (2.13) shows that each w_j is positive. Thus the normalized null vector $v = \|w\|^{-2} w$ has positive entries. □

3 Discrete Strings

By a *discrete string* we mean a collection of masses $m_1, \dots, m_{n-1}$ located at points $y_1 < \cdots < y_{n-1}$, which we take to lie in the interval (0, 1). Thus the string is a discrete measure

$$m = \sum_{j=1}^{n-1} m_j \delta_{y_j}, \quad m_j > 0. \tag{3.1}$$

The associated *string problem* is the vibration problem: Determine the nontrivial solutions $\{u, \lambda\}$ of

$$D^2 u = \lambda m u, \quad u(0) = u(1) = 0. \tag{3.2}$$

The precise interpretation of (3.2) is that u is a continuous piecewise linear function whose derivative has jumps only at the $\{y_j\}$. We set

$$y_0 = 0, \quad y_n = 1; \qquad l_j = y_j - y_{j-1}, \quad j = 1, \dots, n;$$
$$q_j = u(y_j), \quad j = 0, \dots, n.$$

Note that

$$l_1 + \cdots + l_n = 1. \tag{3.3}$$

Then the string problem (3.2) is the system

$$q_j - q_{j-1} = l_j p_j, \qquad j = 1, \dots, n, \tag{3.4}$$

$$p_{j+1} - p_j = \lambda m_j q_j, \quad j = 1, \dots, n-1, \tag{3.5}$$

$$q_0 = q_n = 0. \tag{3.6}$$

Note that for $\lambda = 0$, the full system (3.4) and (3.5) has only the trivial solution.

THEOREM 3.1 *The problem* (3.4), (3.5), *and* (3.6) *is equivalent to a matrix problem for the vector of slopes* $p = (p_1, \dots, p_n)^{\mathsf{T}}$:

$$\lambda L p = M p, \quad \lambda \neq 0, \tag{3.7}$$

where $L = \operatorname{diag}\{l_1, \dots, l_n\}$ *and* M *is the Jacobi matrix with*

$$M_{jj} = -\frac{1}{m_{j-1}} - \frac{1}{m_j}, \quad M_{j,j+1} = \frac{1}{m_j}. \tag{3.8}$$

(*We use here the convention* $1/m_0 = 1/m_n = 0$.)

PROOF: The values q_j can be determined from the p_j by setting $q_0 = 0$ or $q_n = 0$ and using (3.4) recursively. Note in particular that

$$q_1 = p_1 l_1, \quad q_{n-1} = -p_n l_n. \tag{3.9}$$

To determine the slopes p_j, we solve equations (3.5) for the q_j and insert the results into (3.4) to obtain the system

$$\frac{p_{j+1}}{m_j} + \frac{p_{j-1}}{m_{j-1}} = \left\{ \frac{1}{m_j} + \frac{1}{m_{j-1}} + \lambda l_j \right\} p_j, \quad j = 1, \dots, n. \tag{3.10}$$

(To obtain the cases $j = 1$ and $j = n$, we use (3.9), as well as the convention that $1/m_0 = 1/m_n = 0$.) The system (3.10) is precisely (3.7). □

Note that for $\lambda = 0$, (3.7) has the solution $p = (1, 1, \dots, 1)^{\mathsf{T}}$.

COROLLARY 3.2 *The problem* (3.7) *is equivalent to the Jacobi spectral problem*

$$\lambda u = Ju\,, \qquad \lambda \neq 0\,, \tag{3.11}$$

$$u = L^{1/2} p\,, \quad J = L^{-1/2} M L^{-1/2}\,. \tag{3.12}$$

In the notation of the preceding section,

$$a_j = J_{j,j+1} = \frac{1}{m_j\sqrt{l_j l_{j+1}}}\,, \qquad j = 1, \dots, n-1\,, \tag{3.13}$$

$$b_j = J_{jj} = -\frac{1}{l_j}\left\{\frac{1}{m_j} + \frac{1}{m_{j-1}}\right\}, \quad j = 1, \dots, n\,, \tag{3.14}$$

continuing the convention that $1/m_0 = 1/m_n = 0$.

Remarks. The system of difference equations (3.4)–(3.6) can be transformed to a spectral problem for an $(n-1) \times (n-1)$ Jacobi matrix using the position vector q [2]. However, a dimension count shows that such a map cannot be injective. A string with $n-1$ masses has parameters $l_1, \dots, l_n$ and $m_1, \dots, m_{n-1}$. The only constraint, apart from signs, is that the lengths sum to a fixed total length, so there are $2n-2$ parameters. However, an $(n-1) \times (n-1)$ Jacobi matrix is characterized by $2n-3$ parameters: $a_1, \dots, a_{n-2}$, and $b_1, \dots, b_{n-1}$.

On the other hand, using the vector of slopes p, the same problem is transformed into an $n \times n$ Jacobi matrix with zero determinant. The manifold of $n \times n$ Jacobi matrices has dimension $2n-1$; the condition that the determinant vanish reduces the dimension to $2n-2$. Thus the dimension count is correct. We show now that this mapping is explicitly invertible.

THEOREM 3.3 *The map* $(L, M) \to J = L^{-1/2} M L^{-1/2}$ *is a bijection between discrete string problems and normalized Jacobi matrices.*

PROOF: The $n-1$ vibration frequencies of the string problem are negative; see, for example, [1, 2]. The remaining eigenvalue of J is $\lambda = 0$, with eigenvector

$$v = L^{1/2}(1, \dots, 1)^{\mathsf{T}} = (\sqrt{l_1}, \dots, \sqrt{l_n})^{\mathsf{T}}\,. \tag{3.15}$$

By (3.3), this is a unit vector. Thus $J \le 0$ and J is singular, with normalized null vector v.

Conversely, suppose that J is a normalized Jacobi matrix with normalized null vector v. By Lemma 2.5 the entries v_j are positive, so we may define l_j by

$$\sqrt{l_j} = v_j\,, \quad j = 1, \dots, n\,. \tag{3.16}$$

Since $\|v\| = 1$, the l_j sum to 1. Equations (3.13) define positive $m_1, \dots, m_{n-1}$. Since v is annihilated by J,

$$a_{j-1}\sqrt{l_{j-1}} + b_j\sqrt{l_j} + a_j\sqrt{l_{j+1}} = 0\,, \quad j = 1, \dots, n\,, \tag{3.17}$$

with $a_0 = a_n = 0$. Equations (3.13) and (3.17) imply (3.14), so the string problem with data $\{l_j\}$ and $\{m_j\}$ maps into the given Jacobi matrix J. □

Remark. In the proof of Lemma 2.5 we constructed the normalized null vector v explicitly as an algebraic function of the entries of J. Combining this with (3.16) and (3.13), we obtain an explicit algebraic inversion of the algebraic map (3.13) and (3.14) from strings (L, M) to normalized Jacobi matrices J.

In [1, 2] we introduced a Weyl function associated to the string problem (3.2), defined as follows: Let $q_0 = 0$, $p_1 = 1$, and use (3.4) and (3.5) to determine $q_j = q_j(\lambda)$ and $p_j = p_j(\lambda)$ recursively. Then the associated Weyl function is

$$W(\lambda) = \frac{p_n(\lambda)}{\lambda q_n(\lambda)} \,. \tag{3.18}$$

THEOREM 3.4 *The Weyl function* (3.18) *of the string problem coincides with the Weyl function* (2.6) *of the associated Jacobi matrix.*

PROOF: The preceding construction of p_j and q_j is arranged so that the first $n-1$ entries of $(\lambda L - M)p$ vanish. To find the last entry, we note that (3.4) with $j = n$ and (3.5) with $j = n-1$ imply

$$\lambda p_n l_n + \frac{p_n}{m_{n-1}} - \frac{p_{n-1}}{m_{n-1}} = \lambda q_n \,. \tag{3.19}$$

Therefore

$$(\lambda L - M)p = \lambda q_n e_n, \quad e_n = (0, 0, \ldots, 0, 1)^{\mathsf{T}},$$

and so

$$\frac{p_n}{\lambda q_n} = (\lambda L - M)^{-1}_{nn} = \frac{1}{l_n}(\lambda I - J)^{-1}_{nn} \,. \tag{3.20}$$

The left side of (3.20) is (3.18). We established above that $l_n = v_n^2$, where v is the normalized solution of $Jv = 0$. Therefore the right side of (3.20) is (2.6). □

4 The Inverse Problem

The inverse problem is the problem of recovering discrete string data or a normalized Jacobi matrix from the associated Weyl function, i.e., from the eigenvalues $\{\lambda_j\}$ and residues $\{c_j\}$. By the previous section, it is enough to treat the string problem. This is done in [2], and we simply summarize the results here. The details differ notationally from [2] in three respects: In [2] the interval had length 2 rather than 1, the discrete measure m was supported on n sites rather than $n-1$, and the subinterval lengths were indexed $l_0, l_1, \ldots, l_n$ rather than $l_1, l_2, \ldots, l_n$.

THEOREM 4.1 *The Weyl function W for the discrete string* (3.2) *has a continued fraction expansion*

$$W(\lambda) = \cfrac{1}{\lambda l_n + \cfrac{1}{m_{n-1} + \cfrac{1}{\lambda l_{n-1} + \cdots + \cfrac{1}{m_1 + \cfrac{1}{\lambda l_1}}}}}. \tag{4.1}$$

Formulas of Stieltjes [2] give the solution of the inverse problem in closed form. The formulas are expressed in terms of the moments of a discrete measure ν determined by the eigenvalues and residues:

$$\nu = \sum_{j=1}^{n} c_j \delta_{-\lambda_j}\,. \tag{4.2}$$

The moments are

$$A_k = \int_{-\infty}^{\infty} \lambda^k d\nu(\lambda) = \sum_{j=1}^{n} (-\lambda_j)^k c_j\,. \tag{4.3}$$

The associated moment matrix is the $n \times n$ Hankel matrix

$$H = \begin{bmatrix} A_0 & A_1 & A_2 & \dots & A_{n-1} \\ A_1 & A_2 & A_3 & \dots & A_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n-1} & A_n & A_{n+1} & \dots & A_{2n-2} \end{bmatrix}. \tag{4.4}$$

Finally, Δ_k^0 denotes the determinant of the $k \times k$ submatrix of H that consists of the first k rows and columns, $k \le n$; Δ_k^1 denotes the determinant of the $k \times k$ submatrix consisting of the first k rows and columns 2 through $k+1$, $k \le n-1$. By convention $\Delta_0^0 = \Delta_0^1 = 1$. Because the λ_j are all negative and the measure ν is positive, these submatrices are positive definite and the determinants are positive. Below, we shall use the convention that $\Delta_{-1}^1 = \Delta_n^1 = 0$.

THEOREM 4.2 (Stieltjes) *The coefficients of the continued fraction expansion* (4.1) *can be expressed in terms of the minors of the moment matrix H of the measure ν*:

$$l_k = \frac{\{\Delta_{n-k}^1\}^2}{\Delta_{n-k}^0 \Delta_{n-k+1}^0}\,, \quad k = 1, \dots, n\,, \tag{4.5}$$

$$m_k = \frac{\{\Delta_{n-k}^0\}^2}{\Delta_{n-k-1}^1 \Delta_{n-k}^1}\,, \quad k = 1, \dots, n-1\,. \tag{4.6}$$

Combining the previous result with (3.13) and (3.14), we obtain the inversion formulas for a normalized Jacobi matrix.

THEOREM 4.3 *Let $J \le 0$ be a normalized Jacobi matrix with eigenvalues $\{\lambda_j\}$, and let $\{c_j\}$ be the residues of the Weyl function. Then*

$$a_k = \frac{\{\Delta^0_{n-k-1}\Delta^0_{n-k+1}\}^{1/2}}{\Delta^0_{n-k}}, \qquad k = 1, \dots, n-1, \tag{4.7}$$

$$-b_k = \frac{\Delta^0_{n-k+1}}{\Delta^1_{n-k}} \cdot \frac{\Delta^1_{n-k-1}}{\Delta^0_{n-k}} + \frac{\Delta^0_{n-k}}{\Delta^1_{n-k}} \cdot \frac{\Delta^1_{n-k+1}}{\Delta^0_{n-k+1}}, \quad k = 1, \dots, n. \tag{4.8}$$

5 Asymptotics of Jacobi Flows

The multipeakon flows are transformed to an isospectral flow of a discrete string [1] and therefore to an isospectral flow of the associated Jacobi matrix. This Jacobi flow belongs to the hierarchy discussed in Section 2: The generator is

$$B(\Phi), \quad \Phi(\lambda) = \begin{cases} 0, & \lambda = 0, \\ -\frac{1}{\lambda}, & \lambda \neq 0. \end{cases} \tag{5.1}$$

We showed in [2] how to compute the asymptotics of the associated string flow from Theorem 4.2 and (5.1). The equations (3.13) and (3.14) then give the asymptotics of the associated Jacobi flow. The asymptotics of the general Jacobi flow can be obtained in the same way. To simplify matters somewhat, we restrict to the case when all values $\Phi(\lambda_j)$ are distinct and have the same sign. Up to reversing the time direction, we may assume the values are nonnegative.

THEOREM 5.1 *Suppose that $\Phi(0) = 0$ and that Φ takes distinct positive values on the negative eigenvalues of the normalized Jacobi matrix $J = J(0)$. Suppose that the eigenvalues are numbered so that*

$$0 = \mu_1 < \mu_2 < \cdots < \mu_n, \quad \mu_j = \Phi(\lambda_j).$$

Let c_j be the residue of the Weyl function of $J(0)$ at λ_j. Then the asymptotics of the flow (2.2) generated by $B = B(\Phi)$ as $t \to -\infty$ are

$$a_{n-k}(t) = e^{t\{\mu_{k+1}-\mu_k\}} \left\{ \frac{\sqrt{c_{k+1}}}{\sqrt{c_k}} \frac{\prod_{l<k+1} |\lambda_l - \lambda_{k+1}|}{\prod_{l<k} |\lambda_l - \lambda_k|} + O(e^{\delta t}) \right\}, \tag{5.2}$$

$$b_{n-k+1}(t) = \lambda_k + O(e^{\delta t}). \tag{5.3}$$

The asymptotics as $t \to +\infty$ are

$$a_k(t) = e^{t\{\mu_k-\mu_{k+1}\}} \left\{ \frac{\sqrt{c_k}}{\sqrt{c_{k+1}}} \frac{\prod_{l>k} |\lambda_l - \lambda_k|}{\prod_{l>k+1} |\lambda_l - \lambda_{k+1}|} + O(e^{-\delta t}) \right\}, \tag{5.4}$$

$$b_k(t) = \lambda_k + O(e^{-\delta t}). \tag{5.5}$$

Here $\delta = \inf\{\mu_{k+1} - \mu_k\} > 0$.

PROOF: By Theorem 2.1, the residues at time t are

$$c_j(t) = e^{2\mu_j t} c_j. \tag{5.6}$$

It follows from (5.6) and from results in [2, section 6], that as $t \to -\infty$,

$$\frac{\Delta_{k+1}^0}{\Delta_k^0} = e^{2t\mu_{k+1}}\left\{c_{k+1}\prod_{l<k+1}(\lambda_l - \lambda_{k+1})^2 + O(e^{\delta t})\right\}, \tag{5.7}$$

$$\frac{\Delta_k^1}{\Delta_{k+1}^0} = \frac{(-1)^k}{c_1\lambda_2\lambda_3\cdots\lambda_{k+1}} + O(e^{\delta t}). \tag{5.8}$$

The behavior (5.2) of a_{n-k} follows from (4.7) and (5.7). To obtain (5.3) we note that (5.7) and (5.8) imply that the second summand on the right in (4.8) b_{n-k+1} is

$$O\left(e^{2t\{\mu_{k+2}-\mu_{k+1}\}}\right) = O(e^{\delta t}), \quad t \to -\infty,$$

while (5.8) implies that the first summand converges to $-\lambda_k$.

Similarly, it follows from (5.6) and from results in [2, section 6] that as $t \to +\infty$,

$$\frac{\Delta_{n-k+1}^0}{\Delta_{n-k}^0} = e^{2t\mu_k}\left\{c_k\prod_{l>k}(\lambda_l - \lambda_k)^2 + O(e^{-\delta t})\right\}, \tag{5.9}$$

$$\frac{\Delta_{n-k}^1}{\Delta_{n-k}^0} = (-1)^{n-k}\lambda_n\lambda_{n-1}\cdots\lambda_{k+1} + O(e^{-\delta t}). \tag{5.10}$$

The behavior (5.4) of a_k follows from (4.7) and (5.9). To obtain (5.5) we note that (5.9) and (5.10) imply that the first summand on the right in (4.8) is

$$O(e^{2t\{\mu_k-\mu_{k+1}\}}) = O(e^{-\delta t}), \quad t \to +\infty,$$

while (5.10) implies that the second summand converges to $-\lambda_k$. □

The following is an immediate consequence:

COROLLARY 5.2 *Under the assumptions of Theorem* 5.1*,*

$$\lim_{t\to+\infty} 4a_k(t)a_{n-k}(-t)e^{2t(\mu_{k+1}-\mu_k)} = \frac{\prod_{l>k}|2\lambda_l - 2\lambda_k|\prod_{l<k+1}|2\lambda_l - 2\lambda_{k+1}|}{\prod_{l<k}|2\lambda_l - 2\lambda_k|\prod_{l>k+1}|2\lambda_l - 2\lambda_{k+1}|}. \tag{5.11}$$

As shown by Moser [10], (5.11) determines the scattering shifts for the finite Toda lattice. The Toda flow is mapped into the Jacobi flow corresponding to $\Phi(\lambda) = -\lambda$ by the Flaschka transformation

$$a_k = \frac{1}{2}e^{(x_k-x_{k+1})/2}, \quad \dot{x}_k = -2b_k. \tag{5.12}$$

The Hamiltonian for the finite Toda lattice is

$$H = -\operatorname{tr} J^2 = -\sum_{j=1}^n b_j^2 - 2\sum_{j=1}^{n-1} a_j^2, \tag{5.13}$$

and the Toda flow is given by the second-order equation

$$\ddot{x}_j + 4\left(a_j^2 - a_{j-1}^2\right) = 0.$$

The equations of motion for the points x_k are obtained from the corresponding Jacobi flow using (5.12). These determine $x_k(t)$ only up to a common constant, reflecting the translational invariance of the Toda lattice. After a Galilean transformation of the positions, which amounts to a shift of J by a multiple of the identity, we may assume that J is normalized. The asymptotics (5.3) and (5.5) imply that

$$x_k(t) \sim -2\lambda_k t + \beta_k^+ , \quad x_{n-k+1}(-t) \sim 2\lambda_k t + \beta_k^- , \quad t \to +\infty . \tag{5.14}$$

The phase shifts $\beta_k^+ - \beta_k^-$ satisfy

$$\beta_k^+ - \beta_k^- = \sum_{l>k} \log(2\lambda_l - 2\lambda_k)^2 - \sum_{l<k} \log(2\lambda_l - 2\lambda_k)^2 . \tag{5.15}$$

The derivation of (5.15) from (5.11) is the same as that given by Moser. First, tr J is constant, so $x_1 + \cdots + x_n$ flows linearly and the sum of the phase shifts is zero. Thus to prove (5.15) it is enough to prove

$$\begin{aligned}[\beta_k^+ - \beta_k^-] - [\beta_{k+1}^+ - \beta_{k+1}^-] &= \sum_{l>k} \log(2\lambda_l - 2\lambda_k)^2 + \sum_{l<k+1} \log(2\lambda_l - 2\lambda_{k+1})^2 \\ &\quad - \sum_{l<k} \log(2\lambda_l - 2\lambda_k)^2 - \sum_{l>k+1} \log(2\lambda_l - 2\lambda_{k+1})^2 .\end{aligned} \tag{5.16}$$

But (5.16) follows from (5.11), (5.12), and (5.14).

Remark. The above result may be extended in the following way. The Flaschka transformation connects to each flow in the Jacobi hierarchy a particle system that is asymptotically free: As $t \to \pm\infty$ the velocity of each particle tends to a constant characteristic velocity. We call these flows the generalized Toda flows. The generalized Toda flows associated to powers $\Phi_p(\lambda) = \lambda^p$ were discussed briefly by Moser [10].

Consider the flow in the Jacobi hierarchy associated with the function $\Phi(\lambda)$, where we assume that Φ is a continuous function on the real line that vanishes at $\lambda = 0$. The Lax equations $\dot{J} = [J, B]$ of this Jacobi flow are given in coordinate form by

$$\dot{a}_j = a_j(\Phi(J)_{jj} - \Phi(J)_{j+1,j+1}) , \tag{5.17}$$

$$\dot{b}_j = 2[a_{j-1}\Phi(J)_{j-1,j} - a_j\Phi(J)_{j,j+1}] . \tag{5.18}$$

Taking

$$a_j = \frac{1}{2} e^{(x_j - x_{j+1})/2} ,$$

the first equation is

$$\dot{x}_j - \dot{x}_{j+1} = 2\Phi(J)_{jj} - 2\Phi(J)_{j+1,j+1} . \tag{5.19}$$

Since $\operatorname{tr}\Phi(J)$ is an integral of the motion, we may fix

$$\sum_{j=1}^{n} \dot{x}_j = 2\operatorname{tr}\Phi(J)\,. \tag{5.20}$$

The $\dot{x}_j$ are then determined uniquely by (5.19) and (5.20), and

$$\dot{x}_j = 2\Phi(J)_{jj}\,. \tag{5.21}$$

Equations (5.21) and (5.18) are Hamilton's equations for conjugate variables x_j and b_j and Hamiltonian

$$H(b_1,\dots,b_n,x_1,\dots,x_n) = 2\operatorname{tr}\Psi(J)\,, \quad \Psi' = \Phi\,.$$

(Here we must drop the restriction that J be singular.) In fact,

$$\frac{\partial H}{\partial b_j} = 2\operatorname{tr}\frac{\partial}{\partial b_j}\Psi(J) = 2\operatorname{tr}\left[\Phi(J)\frac{\partial J}{\partial b_j}\right] = 2\operatorname{tr}[\Phi(J)e_{jj}] = 2\Phi(J)_{jj}\,.$$

Moreover,

$$\frac{\partial H}{\partial x_j} = 2\operatorname{tr}\left[\Phi(J)\sum_{k=1}^{n}\frac{\partial J}{\partial a_k}\frac{\partial a_k}{\partial x_j}\right].$$

But

$$\frac{\partial J}{\partial a_k} = e_{k,k+1} + e_{k+1,k}\,,$$

while

$$\frac{\partial a_j}{\partial x_j} = \frac{1}{2}a_j\,, \quad \frac{\partial a_{j-1}}{\partial x_j} = -\frac{1}{2}a_{j-1}\,.$$

Therefore

$$\begin{aligned}\frac{\partial H}{\partial x_j} &= 2\operatorname{tr}\Phi(J)\left[\frac{\partial J}{\partial a_{j-1}}\left(-\frac{1}{2}a_{j-1}\right) + \frac{\partial J}{\partial a_j}\left(\frac{1}{2}a_j\right)\right] \\ &= \operatorname{tr}\Phi(J)[(e_{j+1,j}+e_{j,j+1})a_j - (e_{j-1,j}+e_{j,j-1})a_{j-1}] \\ &= 2[a_j\Phi(J)_{j,j+1} - a_{j-1}\Phi(J)_{j-1,j}]\,.\end{aligned}$$

Thus (5.21) and (5.18) have the form

$$\dot{x}_j = \frac{\partial H}{\partial b_j}\,, \qquad \dot{b}_j = -\frac{\partial H}{\partial x_j}\,.$$

Under the conditions of Theorem 5.1 relating Φ and the spectrum of J, J tends to a diagonal matrix as $t \to \pm\infty$: The asymptotic formulas (5.2) and (5.4) show that the a_j tend to zero. This implies that the b_j tend to the eigenvalues of J, since the flow is isospectral. Consequently, (5.21) implies

$$\frac{\dot{x}_j}{2} \sim \Phi(J)_{jj} \to \Phi(\lambda_j) = \mu_j\,, \quad t \to \infty\,.$$

A similar result obtains as $t \to -\infty$.

The computation above shows that scattering shifts (5.15) for the Toda lattice are in fact the same for all these generalized Toda flows to which Theorem 5.1 applies. Unlike the case for the Toda flow itself, for other generalized Toda flows the b_j are no longer fixed multiples of the velocities $\dot{x}_j$.

Remark. The corresponding asymptotic results [11] for the Kac–van Moerbeke flow $\Phi(\lambda) = \lambda^2$ may also be derived by a direct argument. However, Theorem 5.1 does not apply as stated because the generating function Φ is not injective on the spectrum of J when $\operatorname{diag}(J) = cI$. Therefore the computation of the asymptotics of the minors Δ_k^0 and Δ_k^1 is somewhat different.

6 Peakon/Antipeakon Flows

The bijection described in Section 3 mapped the spectral problem of strings with positive masses into the normalized Jacobi matrices. This mapping may be modified in a simple way to map generalized strings, with positive and negative weights, into singular Jacobi matrices with both positive and negative eigenvalues. This allows us to realize the peakon/antipeakon flows also in the hierarchy of Jacobi flows.

We modify the bijection by defining

$$L^{1/2} = \operatorname{diag}\left\{\sigma_1\sqrt{l_1}, \dots, \sigma_n\sqrt{l_n}\right\}, \tag{6.1}$$

where

$$\sigma_1 = 1\,, \quad \sigma_j = \sigma_{j-1}\operatorname{sgn}(m_{j-1})\,, \quad j = 2, \dots, n\,.$$

Again, we set $J = L^{-1/2}ML^{-1/2}$, so

$$a_j = J_{j,j+1} = \frac{1}{m_j\sigma_j\sigma_{j+1}\sqrt{l_jl_{j+1}}}\,, \quad j = 1, \dots, n-1\,. \tag{6.2}$$

The a_j are positive once again, and the b_j are still given by (3.14).

For example, in the peakon/antipeakon case, we have $m_1 > 0$ and $m_2 < 0$ before the collision, since the antipeakons move to the left while peakons move to the right. (The weight m_j refers to the weight of the mass at the j^{th} site.) Therefore $\sigma_1 = \sigma_2 = -1$ before the collision. After the collision, the antipeakon is on the left and the peakon is on the right, so $m_1 < 0$ and $m_2 > 0$, and $\sigma_2 = \sigma_3 = 1, \sigma_1 = -1$. In both cases the a_j are positive.

THEOREM 6.1 *The map $(L, M) \to J = L^{-1/2}ML^{-1/2}$, defined by the choice* (6.1) *of $L^{1/2}$, is a bijection between generalized strings and singular Jacobi matrices J whose principal upper minors D_j, $j = 1, \dots, n-1$, are nonzero.*

PROOF: The normalized null vector of J is now given by

$$v = \sigma_n L^{1/2}(1, \dots, 1)^{\mathsf{T}} = \sigma_n(\sigma_1\sqrt{l_1}, \dots, \sigma_n\sqrt{l_n})^{\mathsf{T}}\,.$$

None of the v_j vanish, since these are proportional to $\sqrt{l_j}$. However, v is again proportional to the vector w of Lemma 2.5, so the D_j are nonzero.

Conversely, if J is a singular Jacobi matrix for which the upper principal minors $D_1, \dots, D_{n-1}$ do not vanish, then the entries of the normalized null vector v are nonzero, by (2.14). Define $\sqrt{l_j} = |v_j|$, take the absolute value of m_j from (6.2), and set

$$\operatorname{sgn}(m_j) = \operatorname{sgn}\left(\frac{\sigma_j}{\sigma_{j+1}}\right) = \operatorname{sgn}\left(\frac{v_j}{v_{j+1}}\right), \quad j = 1, \dots, n-1\,.$$

Then $J = L`-1/2ML^{-1/2}$. □

Remarks. The entries of J are regular under any of the Jacobi flows, but an upper principal minors D_j may become zero at some time. This corresponds to vanishing of l_{j+1} and thus to the breakdown of the inverse map to generalized strings: m_j and m_{j+1} blow up. For the multipeakon flow, this occurs when a peakon and an antipeakon collide.

The results for the multipeakon flow in [2] imply that as a function of t, the D_j have simple zeros and that D_j and D_{j+1} do not vanish simultaneously: no triple collisions. These results carry over to general Jacobi flows. For example, the fact that the D_j satisfy the second-order difference equation (2.12) implies that two successive D_j cannot vanish.

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