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Multipeakons and the Classical Moment Problem

Richard Beals[1]

Yale University, New Haven, Connecticut 06520

David H. Sattinger[2]

Utah State University, Logan, Utah 84322

and

Jacek Szmigielski[3]

University of Saskatchewan, Saskatoon, Saskatchewan, Canada

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Classical results of Stieltjes are used to obtain explicit formulas for the peakon–antipeakon solutions of the Camassa–Holm equation. The closed form solution is expressed in terms of the orthogonal polynomials of the related classical moment problem. It is shown that collisions occur only in peakon–antipeakon pairs, and the details of the collisions are analyzed using results from the moment problem. A sharp result on the steepening of the slope at the time of collision is given. Asymptotic formulas are given, and the scattering shifts are calculated explicitly. © 2000 Academic Press

Key Words: continued fractions; multipeakons; scattering; collisions.

1. INTRODUCTION

The Korteweg–deVries equation is a simple mathematical model for gravity waves in water, but it fails to model such fundamental physical phenomena as the extreme wave of Stokes [23]. The failure of weakly nonlinear dispersive equations, such as the Korteweg–deVries equation, to model the observed breakdown of regularity in nature, is a prime motivation in the search for alternative models for nonlinear dispersive waves [20, 24].

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In 1976 Green and Naghdi [13] derived a system of water wave equations to model fluid flows in thin domains, such as internal waves in coastal regions. The Green–Naghdi equations have a Hamiltonian structure, and in 1993 Camassa and Holm [7] used scaling and an asymptotic expansion to obtain, in the one dimensional case, an approximate Hamiltonian which is formally integrable by the method of inverse scattering. The strongly nonlinear equation they obtained,

$$u_t - \tfrac{1}{4} u_{xxt} + \tfrac{3}{2} (u^2)_x - \tfrac{1}{8} (u_x^2)_x - \tfrac{1}{4} (uu_{xx})_x = 0, \tag{1.1}$$

supports solutions, dubbed "peakons," that are continuous but only piecewise analytic. Equation (1.1) had originally been obtained by B. Fuchssteiner [12] by the method of recursion operators in 1981. He showed the equation was Hamiltonian, but gave no physical interpretation, nor an isospectral operator; and the equation attracted no special attention until its rediscovery by Camassa and Holm.

Motivated by the form of traveling wave solutions of (1.1), Camassa and Holm proposed solutions of the form (in the normalization used in this paper)

$$u(x, t) = \tfrac{1}{2} \sum_{j=1}^{n} m_j(t) \exp(-2\,|x - x_j(t)|) \tag{1.2}$$

to represent n interacting traveling waves. They substituted this *Ansatz* into (1.1) and obtained a Hamiltonian system of equations for m_j, x_j; the Hamiltonian is obtained directly by substituting (1.2) into the formal Hamiltonian for (1.1). This system describes geodesic flow on a manifold with metric tensor $g^{ij} = \exp(-2\,|x_j - x_j|)$. A solution (1.2) can contain both peaks $m_j > 0$ and antipeaks $m_j < 0$; at large positive or negative time it is asymptotic to a superposition of single travelling waves: peakons and antipeakons.

In [8] the Hamiltonian system for two peakons is integrated, and explicit expressions for the relative position $x_1 - x_2$ and relative momentum $p_1 - p_2$ are obtained. A qualitative analysis of the interaction of peakons, and of the collision of a perfectly antisymmetric peakon-antipeakon pair, together with a number of numerical studies are carried out, and a formula for the phase shifts of the interaction of two solitons was obtained (see Fig. 1).

The strongly nonlinear dispersive wave equation (1.1) possesses a remarkable mathematical structure and is the subject of a steadily growing literature. In [6] we used the theory of continued fractions and formulas of Stieltjes [22] (based on prior results of Frobenius [11]) to give

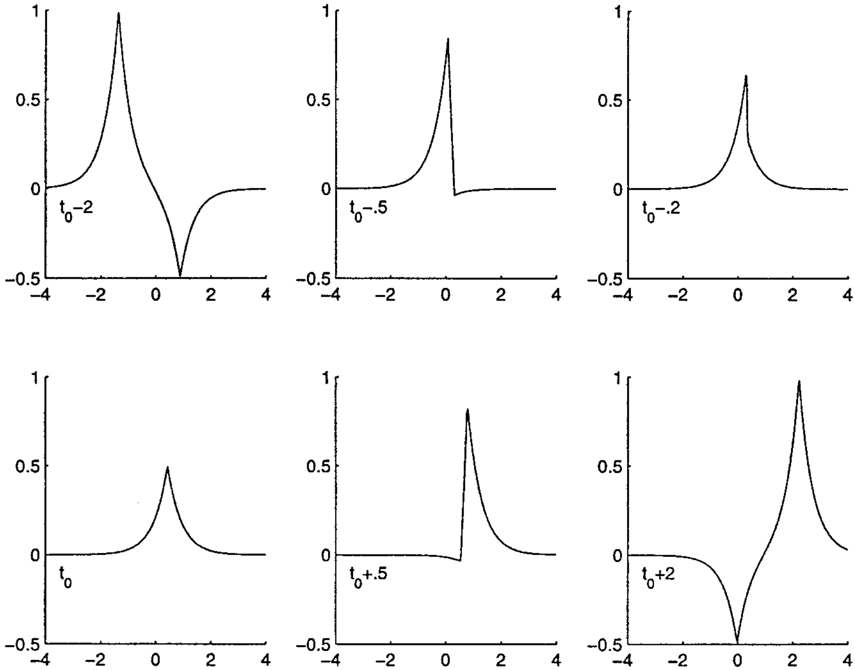

FIG. 1. A peakon moving from left to right collides at time $t_0 = 0.2310$ with an antipeakon moving right to left, computed from the explicit formulas using Matlab. Here $\lambda_- = -1$, $\lambda_+ = 2$, $a_\pm(0) = 0.5$. The slope becomes infinite at the instant of collision. Sharp results on the steepening of the slope are given in Section 7. The vertical scale is exaggerated; the peakons are not as sharp as they appear here.

algebraic formulas for the multipeakon solutions (all $m_j > 0$). These formulas led to explicit expressions for the asymptotic behavior of the positions $\{x_j\}$ and momenta $\{m_j\}$ as $t \to \pm\infty$, and thus to explicit calculation of the scattering shifts undergone by the peaks as a result of their interaction.

In this paper we give a brief but self-contained discussion of the spectral problem associated with (1.1), detailed proofs of the results announced in [6], and an extension of those results to the case of peakon/antipeakon interactions. In the remainder of this section we outline the paper, place it in the context of related work, and describe the conclusions concerning the solutions (1.2).

In Section 2 we obtain the multipeakon/antipeakon solutions as a restriction of (1.1) to a singular subclass. This necessitates a reinterpretation of the Lax pair in the sense of distributions. In Section 3 we describe the Liouville transformation that transforms the spectral problem on the line to a formal density problem for a finite string, first investigated by

M. G. Krein. In Section 4 we describe the Weyl function, which encodes the spectral data, and its expansion in continued fractions. In Section 5 we describe the formulas by which the continued fraction expansion is recovered from the Laurent series for the Weyl function. In Section 6 we use the explicit formulas to obtain the asymptotic behavior of the multipeakon solutions and the scattering shifts of the peakons and antipeakons that result from their interactions.

A closed form of the multipeakon/antipeakon solution is given (Theorem 7.3) in terms of the orthogonal polynomials associated with the moment problem.

Stieltjes' original theory was restricted to positive weights, but the formulas are algebraic and extend to weights of mixed sign, so long as certain determinants do not vanish. In the present context, these determinants are functions of time. Collisions of peakons and antipeakons occur precisely when one or more of these determinants vanish. At such a point some of the weights m_j become infinite, but the solution itself remains bounded throughout the collision. Moreover, there can be no triple collisions: at a given time, collisions occur only in distinct peakon-antipeakon pairs: a peakon moving to the right encounters and passes an antipeakon moving to the left. We prove these facts in Section 7, using the explicit solutions together with the oscillation properties of the orthogonal polynomials of the associated moment problem.

A primary reason for the interest in (1.1) is that it is an integrable model for the breakdown of regularity, a phenomenon not modeled by the Korteweg–deVries equation. A number of qualitative results of a general nature on the steepening of the slope at the instant of breakdown have been obtained by analytic methods [8–10, 17]. Using the exact results obtained from the Stieltjes formulas, we give a sharp result (Theorem 7.2) concerning the steepening of the slope as the instant of collision of a peakon and antipeakon is approached.

In [18, 19], J. Moser applied the theory of continued fractions to finite Jacobi (*a.k.a.* Toda) flows. He obtained an explicit solution for 2×2 and 3×3 Jacobi matrices, though he did not use the full results of Stieltjes. He nevertheless deduced qualitative properties of the solution to the isospectral deformations of finite Jacobi matrices, and calculated the scattering shifts. The relationship of Moser's work, as well as the classical formulas of Frobenius and Stieltjes, to the Riemann theta functions is discussed in an unpublished note of H. P. McKean [16].

Alber *et al.* [2, 3] obtained a number of special solutions of the Camassa–Holm equation, including peakons, n-solitons, and quasiperiodic solutions, by inverting the transformation to action-angle variables. They attacked the Hamiltonian systems directly, without the use of inverse scattering theory. They assert that the general inversion problem can be solved in terms of Riemann theta functions; but explicit results, such as the phase shift undergone by interacting solitons, are obtained only in the case $n=2$.

Moser points out that the moment problem was the forerunner of modern spectral theory. Boris Levitan once remarked to one of the present authors that it was Gelfand's interest in a continuum analogue of the classical moment problem that led to their celebrated solution of the inverse spectral problem. B. Simon in [21] appears to close the circle, and returns to the original view of inverse spectral theory as a continuous analogue of the moment problem.

The theorem of Stieltjes and the method of continued fractions was first applied to an inverse problem for ordinary differential equations by M. G. Krein [14, 15]. The method was extended to a special class of fourth order equations by V. Barcilon [4], which suggests that there are interesting isospectral deformations of such fourth order equations.

The multipeakon/antipeakon solutions are intimately related to Jacobi matrices, but differ from the Toda flows in a fundamental way. The Toda flow describes the dynamics of a lattice of constant weights joined by restoring forces; the conjugate variables are the relative positions of the masses, and their momenta. In the multipeakon flow, the weights m_j vary in time and may take on both positive and negative values; the conjugate variables are the positions of the weights, and the weights themselves. We plan to investigate the relationship of the Jacobi flows to the moment problem in a future paper.

2. THE CAMASSA–HOLM EQUATION

Equation (1.1) is obtained as the compatibility condition for an overdetermined system [7]; in the preset normalization we take

$$L_0(z)\,f=0, \qquad \frac{\partial f}{\partial t}=A_0(z)\,f \tag{2.1}$$
$$L_0(z)=D^2-zm(x,t)-1, \qquad A_0(z)=\left(\frac{1}{z}-u(x,t)\right)D+\frac{1}{2}\,u_x(x,t),$$

where $x\in\mathbb{R}$, $z\in\mathbb{C}$.

Differentiating the first equation in (2.1) with respect to t and the second twice with respect to x and setting $f_{xxt}=f_{txx}$, we obtain, after some calculations,

$$m_t+(um)_x+mu_x=0, \qquad 2m=4u-u_{xx}. \tag{2.2}$$

We assume here that m and its derivatives vanish at $\pm\infty$. According to the second equation in (2.2) the same is then true of u and u_x.

Conversely, suppose that the function m evolves according to (2.2), with m and its derivatives vanishing at $x=\pm\infty$. This is also the compatibility condition for two modifications of (2.1):

$$L_0(z)\,\varphi_0=0, \qquad \frac{\partial\varphi_0}{\partial t}=\left(A_0(z)-\frac{1}{z}\right)\varphi_0 \tag{2.3}$$

$$L_0(z)\,\psi_0=0, \qquad \frac{\partial\psi_0}{\partial t}=\left(A_0(z)+\frac{1}{z}\right)\psi_0. \tag{2.4}$$

These evolution equations are consistent with the asymptotic conditions

$$\varphi_0(x,z)\sim e^{x}, \qquad x\to-\infty; \tag{2.5}$$

$$\psi_0(x,z)\sim e^{-x}, \qquad x\to+\infty, \tag{2.6}$$

respectively. It follows that there are unique *wave functions* $\varphi_0(x,t,z)$, $\psi_0(x,t,z)$ that satisfy (2.3), (2.5), and (2.4), (2.6), respectively.

The remaining x-asymptotics of φ_0 are necessarily

$$\varphi_0(x,t,z)\sim b(t,z)\,e^{x}+c(t,z)\,e^{-x}, \qquad x\to+\infty. \tag{2.7}$$

It follows from the time evolution of φ_0 that

$$\frac{\partial b}{\partial t}=0, \qquad \frac{\partial c}{\partial t}=-\frac{2c}{z}. \tag{2.8}$$

The eigenfunctions for this spectral problem are, by definition, the functions φ for those values $z=\lambda_\nu$ for which $b(\lambda_\nu)=0$. Therefore (2.8) implies that the spectrum $\{\lambda_\nu\}$ is invariant under the flow. The coupling constants are the values $c_\nu(t)=c(t,\lambda_\nu)$; they are characterized by the relation

$$\varphi_0(x,\lambda_\nu)=c_\nu\psi_0(x,\lambda_\nu). \tag{2.9}$$

The evolution is given by specializing (2.8):

$$\dot{c}_\nu=-\frac{2c_\nu}{\lambda_\nu}. \tag{2.10}$$

Thus far we have tacitly assumed that m is a continuous density. The interesting base for present purposes is when m is taken to be a discrete measure with weights m_j at locations x_j,

$$m = \sum_{j=1}^{n} m_j \, \delta_{x_j}, \qquad x_1 < x_2 < \cdots < x_n. \tag{2.11}$$

The two equations in (2.1) are readily interpreted in the sense of distributions. As we note below, the function u of (2.2) arises in the same way. It can be calculated explicitly from (2.11) and the second equation in (2.2). In the normalization of the spectral operator L_0 we have chosen, we obtain precisely the expression (1.2). We note that u_x has jump discontinuities on the support of the singular measure m, so the meaning of mu_x in the first equation (2.2) is *a priori* ambiguous. We shall show, however, that the derivative D can be extended to piecewise smooth functions in such a way that all the operations extend to the discrete case without change. The operations which led from the overdetermined pair of equations (2.1) to the nonlinear system (2.2) therefore continue to hold in the discrete case. In particular, the evolution of coupling coefficients (2.10) applies to the multi-peakon solutions as well.

Suppose that a function f has the form

$$f(x) = f_j(x), \qquad x_j < x < x_{j+1}, \tag{2.12}$$

where each f_j belongs to $C^\infty(\mathbb{R})$ and we have set $x_0 = -\infty$, $x_{n+1} = \infty$. We normalize at the jumps by taking the average of the limits from left and right:

$$f(x_j) = \frac{f(x_j+) + f(x_j-)}{2} = \langle f(x_j) \rangle.$$

If g is a second such function normalized in the same way then the distribution derivative D satisfies the Leibniz rule: $D(fg) = f\,Dg + g\,Df$. Moreover, $D(fm) = f\,Dm$.

In the presence of smooth t dependence,

$$f(x, t) = f_j(x, t), \qquad x_j(t) < x < x_{j+1}(t), \tag{2.13}$$

f as distribution has t-derivative

$$\dot{f} = \frac{\partial f}{\partial t} - \sum_{j=1}^{n} [f(x_j+) - f(x_j-)] \, \dot{x}_j \, \delta_{x_j}. \tag{2.14}$$

Moreover

$$\frac{dm}{dt} = \sum_{j=1}^{n} (\dot{m}_j\, \delta_{x_j} - m_j \dot{x}_j\, D\, \delta_{x_j}).$$

In addition, we have

$$D(um) = \frac{d}{dx} \sum_{j=1}^{n} u(x_j)\, m_j\, \delta_{x_j} = \sum_{j=1}^{n} u(x_j)\, m_j\, D\, \delta_{x_j};$$

$$u_x m = \sum_{j=1}^{n} \langle u_x(x_j) \rangle\, m_j\, \delta_{x_j}.$$

The first equation of (2.2) is therefore

$$\sum_{j=1}^{n} \dot{m}_j\, \delta_{x_j} - m_j \dot{x}_j\, D\, \delta_{x_j} + \langle u_x(x_j) \rangle\, m_j\, \delta_{x_j} + u(x_j)\, m_j\, D\, \delta_{x_j} = 0.$$

Setting the coefficients of the independent distributions δ_{x_j} and $D\, \delta_{x_j}$ equal to zero, we obtain the Hamiltonian system

$$\dot{x}_j = \frac{\partial H}{\partial m_j} = u(x_j), \qquad \dot{m}_j = -\frac{\partial H}{\partial x_j} = -\langle u_x(x_j) \rangle m_j, \tag{2.15}$$

where

$$H(x, m) = \int_{-\infty}^{\infty} (u^2 + \tfrac{1}{4} u_x^2)\, dx = \tfrac{1}{4} \sum_{j, k=1}^{n} m_j m_k e^{-2\,|x_j - x_k|}$$
$$= \tfrac{1}{2} \sum_{j=1}^{n} u(x_j)\, m_j = \tfrac{1}{2} \int_{-\infty}^{\infty} u(x)\, dm(x). \tag{2.16}$$

3. A LIOUVILLE TRANSFORMATION

The flow of the amplitudes m_j and the locations x_j of the peaks can be computed explicitly, by virtue of the inverse scattering approach. As we shall see, there are n eigenvalues λ_j and coupling constants c_j. The eigenvalues are fixed under the evolution and the coupling constants evolve according to (2.10). In order to use this information to recover the time dependence of m_j and x_j we find it convenient first to transform the spectral problem.

In [5] we obtained a Liouville transformation which transforms the spectral problem $L_0(z)\, \varphi = 0$ to a "density" problem for a finite string, in which the density can take on both positive and negative values. The density problem with positive sign has been studied extensively by Krein [14, 15]. In the research announcement [6] we used the classical results on continued fractions to obtain solutions of (1.1) in the case m is a discrete measure, with weights of one sign. The argument carries over to weights of arbitrary sign.

For completeness, we repeat the details here. The first step is to transform (a multiple of) the operator $D^2 - 1$ that occurs in (2.1) into the operator D^2 on the interval $(-1, 1)$; then the wave functions for discrete m become piecewise linear. The appropriate coordinate transformation is

$$y = \tanh x, \qquad \frac{dy}{dx} = \rho(x) = \frac{1}{\cosh^2 x} = 1 - y^2. \tag{3.1}$$

The operator

$$\frac{1}{\rho^2} L_0(z) = \cosh^4 x \left(\frac{d^2}{dx^2} - zm - 1 \right) \tag{3.2}$$

is selfadjoint in $L^2(\mathbb{R}, \rho^2\, dx)$. The operator

$$\begin{aligned} &U: L^2((-1, 1), dy) \to L^2(\mathbb{R}, \rho^2\, dx), \\ &[Uf](x) = \rho(x)^{-1/2} f(\tanh x) = (1 - y^2)^{1/2} f(y) \end{aligned} \tag{3.3}$$

is unitary and carries (3.2) to

$$L(z) = \frac{d^2}{dy^2} - zg(y), \qquad g(y) = m(\tanh^{-1} y)(1 - y^2)^2 = \frac{m(x)}{\rho(x)^2}, \tag{3.4}$$

with Dirichlet boundary conditions. As noted above, the resulting spectral problem

$$\frac{d^2 v}{dy^2}(y) = zg(y)\, v(y), \qquad -1 < y < 1; \qquad v(-1) = 0 = v(1), \tag{3.5}$$

is familiar when g is positive: it determines the natural vibration frequencies a string with mass density g.

The Liouville transformation takes the wave functions φ_0, ψ_0 to the corresponding functions for the problem (3.5):

$$\begin{aligned} &\varphi'' - zg\varphi = 0; \qquad \varphi(-1, z) = 0, \qquad \varphi'(-1, 0) = 1; \\ &\psi'' - zg\psi = 0; \qquad \psi(1, z) = 0, \qquad \psi'(1, z) = -1; \end{aligned} \tag{3.6}$$

here the primes denote derivatives with respect to $y \in [-1, 1]$. The eigenvalues $\{\lambda_\nu\}$ and coupling constants $\{c_\nu\}$ are the same as for the original problem.

In particular, the evolution of the coupling coefficients is preserved under the Liouville transformation.

The discrete measure (2.11) is transformed into the discrete measure g on the interval $(-1, 1)$ given by

$$g = \sum_{j=1}^{n} g_j \delta_{y_j}, \qquad -1 = y_0 < y_1 < \cdots < y_n < y_{n+1} = 1. \tag{3.7}$$

The terms in (2.11) are related to the terms here by [6]

$$m_j = g_j(1 - y_j^2), \qquad x_j = \frac{1}{2} \log\left(\frac{1 + y_j}{1 - y_j}\right). \tag{3.8}$$

We remark that the Liouville transformation is a canonical transformation from the equations on the line to a Hamiltonian system on the interval $(-1, 1)$, but we shall not need that here. The multipeakon solution is obtained by solving the inverse spectral problem for the discrete string, and then pulling the solution back to the real line, using (3.8).

In the next two sections we treat the direct and inverse spectral problems for the discrete string.

4. THE WEYL FUNCTION

Equation (3.5) implies that φ is continous and piecewise linear, with jump discontinuities in the derivative at the points $\{y_j\}$. Then both sides of (3.5) are interpreted in the sense of distributions. We denote left and right derivatives with respect to y by $D_\pm$,

$$D_+ \varphi(y_j) = \frac{\varphi(y_{j+1}) - \varphi(y_j)}{l_j}, \qquad D_- \varphi(y_j) = \frac{\varphi(y_j) - \varphi(y_{j-1})}{l_{j-1}},$$

where $l_j = y_{j+1} - y_j$, $0 \leqslant j \leqslant n$. Then (3.5) becomes

$$D_+ \varphi(y_j) - D_- \varphi(y_j) = z g_j \varphi(y_j), \qquad 1 \leqslant j \leqslant n. \tag{4.1}$$

We note that the spectral problem may be written in the matrix form

$$Jq = zGq, \tag{4.2}$$

where $q=(q_1, ..., q_n)^t$, $q_j(t)=\varphi(y_j(t), t)$,

$$J=\begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & a_2 & 0 & \\ 0 & a_2 & b_3 & & \\ \vdots & & & \ddots & a_{n-1} \\ 0 & & & a_{n-1} & b_n \end{pmatrix}, \qquad G=\begin{pmatrix} g_1 & 0 & \cdots & 0 \\ 0 & g_2 & \cdots & 0 \\ \vdots & & & 0 \\ 0 & \cdots & 0 & g_n \end{pmatrix},$$

and $a_j=1/l_j$, $b_j=-(a_{j-1}+a_j)$. The spectral problem for the multipeakon solutions thus differs from that for the Jacobi flows, which simply takes the form $Jq=zq$.

The wave functions $\varphi(\cdot, z)$ may be constructed recursively. We fix z and set

$$q_j=\varphi(y_j, z), \qquad p_j=D_-\varphi(y_j),$$

so that

$$q_j-q_{j-1}=p_j l_{j-1}, \qquad p_j-p_{j-1}=z g_{j-1} q_{j-1}. \tag{4.3}$$

We begin with $q_0=0$ and $p_1=1$. Then q_j and p_j are polynomials of degree $j-1$ in z. In particular, $\varphi(1, z)=q_{n+1}$ and $D_-\varphi(1, z)=p_{n+1}$ are polynomials in z of degree n.

THEOREM 4.1. *The set of roots of $\varphi(1, z)$ is the spectrum of the problem* (3.5), *equivalently* (4.2). *The roots are real, simple, and non-zero. The number of positive* (*resp. negative*) *roots of $\varphi(1, z)$ equals the number of negative* (*resp. positive*) *terms among the g_j.*

Proof. The eigenvalues are given by the zeroes of $\varphi(1, z)$. Any other non-zero solution of (3.5) which vanishes at -1 is a scalar multiple of $\varphi(y, z)$. Therefore there is at most one eigenfunction for each value of z, and the geometric multiplicity is one. Moreover, 0 cannot be an eigenvalue, since $\varphi(y, 0)=1+y$. The eigenvalues are real since J and G are real, symmetric matrices and J is negative definite.

Consider (4.2) with $G=I$. This is a pure eigenvalue problem for the symmetric tridiagonal matrix J, so it has real spectrum. It corresponds to our spectral problem with all $g_j=1$; so if $z\geqslant 0$, then the slope of $\varphi(\cdot, z)$ is non-decreasing from left to right and necessarily $\varphi(1, z)>0$. Consequently J is negative definite and has the form $-B^2$, where B is positive definite. Then (4.2) is equivalent to $B^{-1}GB^{-1}w=-z^{-1}w$, $w=Bq$. Since $B^{-1}GB^{-1}$ is symmetric, the spectrum is real and the algebraic multiplicity of each eigenvalue is its geometric multiplicity, which we have shown to be 1.

To obtain the conclusion about the signs of the roots, we set $B(s) = (1-s)\, I + sB$, $0 \leqslant s \leqslant 1$. Each $B(s)$ is positive definite, so by the previous argument the eigenvalues of $B(s)^{-1}\, GB(s)^{-1}$ are simple and non-zero. Therefore the number of positive roots is independent of s. Comparing $s=1$ and $s=0$, we obtain the result. Q.E.D.

We denote the roots of $\varphi(1, z)$ by $\{\lambda_j\}$. Note that $\varphi(y, 0) = 1 + y$, so $\varphi(1, 0) = 2$ and

$$\varphi(1, z) = 2 \prod_{j=1}^{n} \left(1 - \frac{z}{\lambda_j}\right). \tag{4.4}$$

The *Weyl function* associated with (4.1) is

$$w(z) = \frac{D_-\varphi(1, z)}{\varphi(1, z)} = \frac{p_{n+1}}{q_{n+1}}. \tag{4.5}$$

It will be more convenient to work with the modified Weyl function $w(z)/z$; in particular we use its partial fractions decomposition

$$\frac{w(z)}{z} = \frac{1}{2z} + \sum_{j=1}^{n} \frac{a_j}{z - \lambda_j} = \sum_{j=0}^{n} \frac{a_j}{z - \lambda_j}, \tag{4.6}$$

where we have set

$$a_0 = \tfrac{1}{2}, \qquad \lambda_0 = 0.$$

The *scattering data* for the spectral problem (4.1) consists of the spectrum $\{\lambda_j\}$ and the coupling constants $\{c_j\}$. Recall that the coupling constants relate the eigenfunctions φ to their counterparts ψ normalized at $y=1$ by $\psi(1, z) = 0$, $D_-\psi(1, z) = -1$. In fact

$$\varphi(y, \lambda_j) = c_j \psi(y, \lambda_j). \tag{4.7}$$

The Wronskian of two solutions of (4.1) is constant in the intervals $y_j < y < y_{j+1}$ and continuous across the y_j. Evaluating the Wronskian $W(\varphi, \psi)$ at $y = \pm 1$ establishes that $\varphi(1, z) = \psi(-1, z)$. Differentiating (4.7) with respect to y and setting, $y=1$, we obtain

$$c_j = -\varphi'(1, \lambda_j).$$

The residues $a_j, j \geqslant 1$, in (4.6) are determined by the scattering data. Combining (4.8) and (4.4), we obtain

$$a_j = \frac{1}{\lambda_j} \frac{D_-\varphi(1, \lambda_j)}{\varphi_z(1, \lambda_j)} = \frac{c_j}{2} \prod_{k \neq j} (1 - \lambda_j/\lambda_k)^{-1}. \tag{4.9}$$

Thus under the nonlinear evolution given by (1.1),

$$\dot{a}_j = -\frac{2a_j}{\lambda_j}, \qquad a_j(t) = a_j(0)\, e^{-2t/\lambda_j}. \tag{4.10}$$

THEOREM 4.2. *The residues a_j in the partial fractions decomposition* (4.6) *of $w(z)/z$ are positive and satisfy*

$$\lambda_j^2 \varphi_z^2(1, \lambda_j)\, a_j = \int_{-1}^{1} \varphi'^2(y, \lambda_j)\, dy. \tag{4.11}$$

Proof. Differentiation with respect to z commutes with the distribution derivative D, so

$$D^2\varphi = z\varphi g, \qquad D^2\varphi_z = z\varphi_z g + \varphi g.$$

We multiply the first of these equations by φ_z and the second by φ, subtract, and integrate with respect to y. When $z = \lambda_j$ we obtain

$$\varphi'(1, \lambda_j)\, \varphi_z(1, \lambda_j) = -\int_{-1}^{1} \varphi^2(y, \lambda_j)\, dg(y), \tag{4.12}$$

where dg is the measure defined by the distribution g defined by (3.7). Integrating the equation $\varphi\, D^2\varphi = \lambda_j \varphi^2 g$ with respect to y, we obtain

$$\int_{-1}^{1} \varphi'^2(y, \lambda_j)\, dy = -\lambda_j \int_{-1}^{1} \varphi^2(y, \lambda_j)\, dg(y). \tag{4.13}$$

Combining (4.9), (4.12), and (4.13), we obtain (4.11). Q.E.D.

5. A THEOREM OF STIELTJES

For the inverse spectral problem, we assume that the eigenvalues $\{\lambda_j\}$ and the residues $\{a_j\}$ are known, and seek to determine the constants $\{g_j\}$ and the points $\{y_j\}$; in place of the latter we may look for the subinterval lengths $l_j = y_{j+1} - y_j$. A first step in the process is to determine the Laurent expansion at infinity of the modified Weyl function. This is easily obtained from (4.6) by expanding

$$\frac{a_j}{z - \lambda_j} = \sum_{k=0}^{\infty} \frac{a_j \lambda_j^k}{z^{k+1}}.$$

The result is the following.

LEMMA 5.1. *The modified Weyl function has the Laurent expansion*

$$\frac{w(z)}{z} = \sum_{k=0}^{\infty} \frac{(-1)^k A_k}{z^{k+1}}, \tag{5.1}$$

where

$$A_k = \sum_{j=0}^{n} (-\lambda_j)^k a_j. \tag{5.2}$$

The Weyl function itself can also be written as a continued fraction [14].

LEMMA 5.2. *The Weyl function is*

$$w(z) = \cfrac{1}{l_n + \cfrac{1}{zg_n + \cfrac{1}{l_{n-1} + \cdots + \cfrac{1}{zg_1 + 1/l_0}}}}. \tag{5.3}$$

Proof. We use (4.3) inductively,

$$p_1 = 1, \qquad q_1 = l_0, \qquad q_2 = q_1 + l_1 p_2, \qquad p_2 = p_1 + zg_1 l_0;$$

$$\frac{p_2}{q_2} = \frac{p_2}{l_0 + p_2 l_1} = \cdots = \frac{1}{l_1 + 1/(zg_1 + (1/l_0))}.$$

Assuming (5.3) for $\{y_j\}_{j<n}$ and $\{g_j\}_{j<n}$, we adjoint $y_n > y_{n-1}$ and g_n. Then $p_{n+1} = p_n + zg_n q_n$, $q_{n+1} = q_n + l_n p_{n+1}$; and

$$\frac{p_{n+1}}{q_{n+1}} = \frac{p_{n+1}}{q_n + l_n p_{n+1}} = \frac{1}{l_n + q_n/p_{n+1}} = \frac{1}{l_n + q_n/(p_n + zg_n q_n)}$$
$$= \frac{1}{l_n + 1/(zg_n + (p_n/q_n))}. \tag{5.4}$$

Hence (5.3) follows by induction. Q.E.D.

Dividing by z gives the continued fraction decomposition of the modified Weyl function:

$$\frac{w(z)}{z} = \cfrac{1}{l_n z + \cfrac{1}{g_n + \cfrac{1}{l_{n-1} + \cdots + \cfrac{1}{g_1 + 1/l_0 z}}}}. \tag{5.5}$$

A classical result of Stieltjes [22] recovers the coefficients of the continued fraction (5.3) from the Laurent expansion of $w(z)/z$ at infinity.

A key role is played by the Hankel matrix whose entries are the quantities A_k defined by (5.2). In the original theory of Stieltjes, the A_k are all positive. This corresponds to the pure peakon case, in which all the weights g_j are positive and the eigenvalues λ_j are negative. In the peakon-antipeakon case the weights are of both signs, and the A_k are now of both signs; nevertheless—and this will be an important point in the analysis—they are still the moments of a positive measure. In fact (5.2) play be rewritten as

$$A_k = \int_{-\infty}^{\infty} \lambda^k \, d\mu(\lambda), \qquad \mu = \sum_{j=0}^{n} a_j \, \delta_{-\lambda_j}, \tag{5.6}$$

where again we take $\lambda_0 = 0$ and $a_0 = 1/2$.

In view of (5.2) and (4.7), we see that the A_j are rational functions of the scattering data. The following result provides explicit formulas.

THEOREM 5.3 (Stieltjes). *The Laurent series* (5.1) *can be uniquely developed in a continued fraction*

$$\cfrac{1}{b_1 z + \cfrac{1}{b_2 + + \cfrac{1}{b_3 z + \cdots}}}.$$

where

$$b_{2k} = \frac{(\Delta_k^0)^2}{\Delta_k^1 \, \Delta_{k-1}^1}, \qquad b_{2k+1} = \frac{(\Delta_k^1)^2}{\Delta_k^0 \, \Delta_{k+1}^2}. \tag{5.7}$$

Moreover

$$b_1 + b_3 + \cdots + b_{2k+1} = \frac{\Delta_k^2}{\Delta_{k+1}^0}. \tag{5.8}$$

Here the Δ_k^1 are certain $k \times k$ minors of the infinite Hankel matrix

$$H = \begin{pmatrix} A_0 & A_1 & A_2 & A_3 & \cdots \\ A_1 & A_2 & A_3 & A_4 & \cdots \\ A_2 & A_3 & A_4 & A_5 & \cdots \\ A_3 & A_4 & A_5 & A_6 & \cdots \\ & & \vdots & & \end{pmatrix}.$$

Δ^l_k *is the determinant of the* $k \times k$ *submatrix of* H *whose* (i, j) *entry is* $A_{l+i+j-2}$. *By convention,* $\Delta^0_l = 1$.

Equations (5.7) and (5.8) appear in [22, Eqs. (7) and (11)], respectively. By comparing the continued fraction in (5.5) with that in Theorem 5.3, we obtain

$$l_j = \frac{(\Delta^1_{n-j})^2}{\Delta^0_{n-j}\,\Delta^0_{n-j+1}}, \qquad 0 \leqslant j \leqslant n; \tag{5.9}$$

$$g_j = \frac{(\Delta^0_{n-j+1})^2}{\Delta^1_{n-j+1}\,\Delta^1_{n-j}}, \qquad 1 \leqslant j \leqslant n; \tag{5.10}$$

while from (5.8) and (5.9) we obtain

$$y_j = 1 - (l_n + l_{n-1} + \cdots + l_j) = 1 - \frac{\Delta^2_{n-j}}{\Delta^0_{n-j+1}}. \tag{5.11}$$

These results allow us to characterize the range of the forward spectral map in terms of conditions on the A_k.

LEMMA 5.4. *For even* l, *the minors* Δ^l_k *are strictly positive,* $0 \leqslant k \leqslant n$, *as is* Δ^0_{n+1}. *For odd* l, *and* $0 \leqslant k \leqslant n$, *if all the weights* g_j *have the same sign, then* Δ^l_k *is non-zero and has the opposite sign, while* $\Delta^l_{n+1} = 0$.

Proof. The quadratic form associated with the $k \times k$ submatrix with upper left hand element A_l is

$$\sum_{i,\,j=0}^{k-1} A_{l+i+j}\xi_i\xi_j = \int_{-\infty}^{\infty} \sum_{i,\,j=0}^{k-1} \xi_i\xi_j\lambda^{i+j+l}\,d\mu(\lambda) = \int_{-\infty}^{\infty} \lambda^l\xi^2(\lambda)\,d\mu(\lambda),$$

where

$$\xi(\lambda) = \sum_{j=0}^{k-1} \xi_j\lambda^j.$$

Thus the form is positive definite when l is even and $k \leqslant n$: a non-zero polynomial of degree $k-1$ cannot vanish at the n non-zero points $-\lambda_j$ in the support of $d\mu$. (When $l=0$ the point $\lambda=0$ must also be considered.) If all these points $-\lambda_j$ are positive (respectively negative) then the form is also positive (respectively negative) for odd l. The associated determinant is Δ^l_k.

Finally, for odd l and $k > n$, there is a nonzero polynomial $\sum c_j\lambda_j$ of degree $k-1$ that vanishes at each $\lambda_1, ..., \lambda_n$, and it follows that the vector $(c_0, ..., c_{k-1})^t$ is in the null space of the matrix associated with Δ^l_k. Q.E.D.

In the pure peakon or antipeakon case, i.e., when all the weights m_j have the same sign, Theorem 4.1 says that all eigenvalues λ_j have the same sign. It follows immediately from Lemma 5.4 that the minors Δ_j^1 cannot vanish; hence $l_j > 0$ for all time, and there are no exact collisions: the distances $x_{k+1} - x_k$ are always strictly positive.

THEOREM 5.5. *The real non-zero constants* $\{\lambda_j, c_j\}$, $j = 1, ..., n$ *are scattering data for* (3.5) *if and only if the* λ_j *are distinct, the* a_j *given by* (4.9) *are positive, and the moment matrix constructed from the associated measure has the property that the determinants* Δ_j^1, $0 \leqslant j \leqslant n$, *do not vanish.*

Proof. Suppose that the $\{\lambda_j\}$, $\{c_j\}$ are scattering data. We showed in Section 4 that the eigenvalues are distinct and the a_j are positive. It follows from Lemma 5.4 that $\Delta_j^0 > 0$, $0 \leqslant j \leqslant n+1$. Since the l_j are positive, it follows from (5.9) that $\Delta_j^1 \neq 0$, $0 \leqslant j \leqslant n$.

Conversely, starting from distinct $\{\lambda_j\}$ and $\{c_j\}$ such that $\{\alpha_j\}$ are positive and $\Delta_j^1 \neq 0$, (5.9) and (5.10) can be used to define $\{l_j\}$ and $\{g_j\}$. The Laurent series (5.1) corresponds to the continued fraction (5.3). The proof of Lemma (5.2) is reversible. The associated continued fraction (5.3) terminates, because $\Delta_{n+1}^1 = 0$, and gives the Weyl function for the spectral problem with constants $\{g_j\}$ and $\{l_j\}$. Q.E.D.

To relate the measure m to the scattering data, we begin with some notation and an identity. We denote by $\tilde{A}_k$ and $\tilde{\Delta}_k^1$ the corresponding moments and determinants with respect to the modified measure

$$\tilde{\mu} = \sum_{j=1}^{n} a_j \delta_{-\lambda_j}. \tag{5.12}$$

These coincide with the A_k and Δ_k^l except that $A_0 = 1/2 + \tilde{A}_0$, and consequently

$$\Delta_k^0 = \tilde{\Delta}_k^0 + \tfrac{1}{2} \Delta_{k-1}^2, \qquad k \geqslant 1. \tag{5.13}$$

It follows from (5.11) and (5.13) that

$$1 - y_j = \frac{\Delta_{n-j}^2}{\Delta_{n-j+1}^0}, \qquad 1 + y_j = \frac{2\tilde{\Delta}_{n-j+1}^0}{\Delta_{n-j+1}^0}. \tag{5.14}$$

Combining (5.14) with (5.10) and (3.8) we obtain the following.

THEOREM 5.6. *The weights* m_j *and positions* x_j *associated with the distribution* m *in* (2.11) *are given by*

$$x_j = \frac{1}{2} \log\left(\frac{2\tilde{\Delta}_{n-j+1}^0}{\Delta_{n-j}^2}\right); \qquad m_j = \frac{2\tilde{\Delta}_{n-j+1}^0 \Delta_{n-j}^2}{\Delta_{n-j+1}^1 \Delta_{n-j}^1}. \tag{5.15}$$

We illustrate with the cases $n=1, 2, 3$. For $n=1$ we have

$$x_1=\frac{1}{2}\log\left(\frac{2\tilde{\Delta}_1^0}{\Delta_0^2}\right)=\frac{1}{2}\log 2a_1;$$

$$m_1=\frac{2\tilde{\Delta}_1^0\,\Delta_0^1}{\Delta_1^1\,\Delta_0^1}=\frac{2a_1}{-\lambda_1 a_1}=-\frac{2}{\lambda_1}.$$

For $n=2$ the result is

$$x_1=\frac{1}{2}\log\frac{2(\lambda_1-\lambda_2)^2\,a_1a_2}{\lambda_1^2a_1+\lambda_2^2a_2},\qquad x_2=\frac{1}{2}\log 2(a_1+a_2);$$

$$m_1=-\frac{2(\lambda_1^2a_1+\lambda_2^2a_2)}{\lambda_1\lambda_2(\lambda_1a_1+\lambda_2a_2)},\qquad m_2=-\frac{2(a_1+a_2)}{\lambda_1a_1+\lambda_2a_2}.$$

Finally, for $n=3$ we have

$$x_1=\frac{1}{2}\log\frac{2(\lambda_1-\lambda_2)^2\,(\lambda_2-\lambda_3)^2\,(\lambda_3-\lambda_1)^2\,a_1a_2a_3}{\sum_{j<k}\lambda_j^2\lambda_k^2(\lambda_j-\lambda_k)^2\,a_ja_k},$$

$$x_2=\frac{1}{2}\log\left(\frac{2\sum_{j<k}(\lambda_j-\lambda_k)^2\,a_ja_k}{\sum\lambda_j^2a_j}\right),\qquad x_3=\frac{1}{2}\log\left(2\sum a_j\right);$$

$$m_1=-\frac{2\sum_{j<k}\lambda_j^2\lambda_k^2(\lambda_j-\lambda_k)^2\,a_ja_k}{\lambda_1\lambda_2\lambda_3\sum_{j<k}\lambda_j\lambda_k(\lambda_j-\lambda_k)^2\,a_ja_k},$$

$$m_2=-\frac{2\sum\lambda_j^2a_j\sum_{j<k}(\lambda_j-\lambda_k)^2\,a_ja_k}{\sum\lambda_ja_j\sum_{j<k}\lambda_j\lambda_k(\lambda_j-\lambda_k)^2\,a_ja_k},\qquad m_3=-\frac{2\sum a_j}{\sum\lambda_ja_j}.$$

Moser [19] applied the theory of continued fractions for Jacobi matrices and obtained explicit solutions similar to these for the isospectral flow of 3×3 Jacobi matrices in a special case.

6. ASYMPTOTICS

The function u in (1.2) is a superposition of single peakons ($m_k>0$) and antipeakons ($m_k<0$) with peaks and troughs at the x_k; however both the m_k and the x_k vary in time. We show now that u is asymptotic at large times to superpositions of non-interacting peakons and antipeakons with constant heights/depths: At large negative time peakons are found to the far left, decreasing in height from left to right, travelling to the right at speeds proportional to their heights, with antipeakons to the far right, increasing in depth from left to right, travelling to the left at speeds proportional to their depths. At large positive times antipeakons are to the left, with decreasing depths, and peakons to the right, with increasing heights. The difference, relative to a pure superposition of such travelling solutions, is that each peak or trough has undergone a phase shift. The total (asymptotic) phase shift for each peak or trough is precisely the sum of the phase shifts that would occur in interactions with each of the other terms separately (in a calculation with $n=2$); see (6.8).

The key to the analysis of the long-term asymptotics is the evaluation of the determinants Δ_k^l and $\tilde{\Delta}_k^0$.

THEOREM 6.1. *The determinants Δ_k^l of the moment matrix for the measure $d\mu$ in* (5.6) *are given by*

$$\Delta_k^l = \sum_{J \subset \{0, 1, ..., n\},\ |J| = k} a^J (-\lambda)^{lJ} \pi_J, \tag{6.1}$$

where

$$a^J = \prod_{j \in J} a_j, \qquad \lambda^J = \prod_{j \in J} \lambda_j, \qquad \pi_J = \prod_{j, m \in J,\ j < m} (\lambda_j - \lambda_m)^2.$$

The determinants $\tilde{\Delta}_k^0$ are

$$\tilde{\Delta}_k^0 = \sum_{J \subset \{1, ..., n\},\ |J| = k} a^J \pi_J. \tag{6.2}$$

The proof of this theorem is given at the end of this section. We begin by using it to deduce the asymptotics. We number the eigenvalues λ_j so that

$$\lambda_{m+1} < \cdots < \lambda_n < \lambda_0 = 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m. \tag{6.3}$$

An examination of the formulas (6.1) and (5.13) in the light of (4.10) and the assumption (6.3) yields the following asymptotics of the determinants.

LEMMA 6.2. *As $t \to -\infty$, if $l > 0$ and $k \leqslant n$, then*

$$\Delta_k^l \sim \left(\prod_{j=1}^{k} a_j(0)(-\lambda_j)^l \right) \left(\prod_{1 \leqslant j < r \leqslant k} (\lambda_j - \lambda_r)^2 \right) \exp\left(-\sum_{j=1}^{k} \frac{2t}{\lambda_j} \right).$$

If $k \leqslant m$ then

$$\Delta_k^0 \sim \left(\prod_{j=1}^{k} a_j(0) \right) \left(\prod_{1 \leqslant j < r \leqslant k} (\lambda_j - \lambda_r)^2 \right) \exp\left(-\sum_{j=1}^{k} \frac{2t}{\lambda_j} \right).$$

If $m<k\leqslant n+1$, *then*

$$\Delta_k^0 \sim \frac{1}{2}\left(\prod_{j=1}^{k-1} \lambda_j^2 a_j(0)\right)\left(\prod_{1\leqslant j<r<k} (\lambda_j-\lambda_r)^2\right) \exp\left(-\sum_{j=1}^{k-1} \frac{2t}{\lambda_j}\right).$$

In all cases, for $k\leqslant n$

$$\tilde{\Delta}_k^0 \sim \left(\prod_{j=1}^{k} a_j(0)\right)\left(\prod_{1\leqslant j<r\leqslant k} (\lambda_j-\lambda_r)^2\right) \exp\left(-\sum_{j=1}^{k} \frac{2t}{\lambda_j}\right).$$

Proof. We discuss only the first case; the remaining are treated similarly. The sum in (6.1) is taken over all subsets of integers J of size k. The exponential term that corresponds to any such subset is

$$\exp\left(-\sum_{j\in J} \frac{2t}{\lambda_j}\right).$$

Since $-t\to\infty$, the dominant term is that for which $J=\{1, ..., k\}$. Hence

$$\begin{aligned}\Delta_k^l(t) &\sim \prod_{j=1}^{k} a_j(t)(-\lambda_j)^l \prod_{s<m\leqslant k} (\lambda_s-\lambda_m)^2\\ &= \prod_{j=1}^{k} a_j(0)(-\lambda_j)^l \prod_{s<m\leqslant k} (\lambda_s-\lambda_m)^2 \exp\left(-\sum_{j=1}^{k} \frac{2t}{\lambda_j}\right),\end{aligned}$$

and the first result follows.

LEMMA 6.3. *As* $t\to+\infty$, *if* $l>0$ *then for* $k\leqslant n$

$$\Delta_k^l \sim \left(\prod_{j>n-k} a_j(0)(-\lambda_j)^l\right)\left(\prod_{n-k<j<r} (\lambda_j-\lambda_r)^2\right) \exp\left(-\sum_{j>n-k} \frac{2t}{\lambda_j}\right).$$

If $k\leqslant n-m$ *then*

$$\Delta_k^0 \sim \left(\prod_{j>n-k}^{n} a_j(0)\right)\left(\prod_{n-k<j<r}^{n} (\lambda_j-\lambda_r)^2\right) \exp\left(-\sum_{j>n-k}^{n} \frac{2t}{\lambda_j}\right).$$

If $n-m<k\leqslant n+1$, *then*

$$\Delta_k^0 \sim \frac{1}{2}\left(\prod_{j>n-k+1}^{n} \lambda_j^2 a_j(0)\right)\left(\prod_{n-k+1<j<r}^{n} (\lambda_j-\lambda_r)^2\right) \exp\left(-\sum_{j>n-k+1}^{n} \frac{2t}{\lambda_j}\right).$$

In all cases, for $k \leqslant n$

$$\tilde{\Delta}_k^0 \sim \Bigl(\prod_{j>n-k}^{n} a_j(0)\Bigr)\Bigl(\prod_{n-k<j<r}^{n} (\lambda_j-\lambda_r)^2\Bigr) \exp\Bigl(-\sum_{j>n-k}^{n} \frac{2t}{\lambda_j}\Bigr).$$

Combining these results with (5.6), we obtain the asymptotics of the problem on the line.

THEOREM 6.4. *As* $t \to -\infty$,

$$x_{n-j+1}(t) \sim -\frac{t}{\lambda_j} + \frac{1}{2}\log\Bigl[2a_j(0)\prod_{k=1}^{j-1}\Bigl(1-\frac{\lambda_j}{\lambda_k}\Bigr)^2\Bigr]; \tag{6.4}$$

$$m_j(t) \sim -\frac{2}{\lambda_j}. \tag{6.5}$$

As $t \to +\infty$,

$$x_j(t) \sim -\frac{t}{\lambda_j} + \frac{1}{2}\log\Bigl[2a_j(0)\prod_{k=j+1}^{n}\Bigl(1-\frac{\lambda_j}{\lambda_k}\Bigr)^2\Bigr]; \tag{6.6}$$

$$m_{n-j+1}(t) \sim -\frac{2}{\lambda_j}. \tag{6.7}$$

It follows that the solution $u(x, t)$ is asymptotically a sum of free peakons ($\lambda_j < 0$) and antipeakons ($\lambda_j > 0$). The term with asymptotic velocity $-2/\lambda_j$ undergoes a phase shift due to the interactions in the amount

$$\sum_{k=j+1}^{n} \log\Bigl|1-\frac{\lambda_j}{\lambda_k}\Bigr| - \sum_{k=1}^{j-1} \log\Bigl|1-\frac{\lambda_j}{\lambda_k}\Bigr|. \tag{6.8}$$

Proof of Theorem 6.1. The proof is based on the following lemma.

LEMMA 6.5. *Let* A_k, $k=0, 1, 2, ...,$ *be the moments of the discrete measure* $\sum_{j=0}^{n} b_j \delta_{\lambda_j}$,

$$A_k = \sum_{j=0}^{n} b_j \lambda_j^k,$$

where the weights b_j are non-zero, but need not be positive. Then the determinant

$$\Delta_{n+1}^0 = \begin{vmatrix} A_0 & A_1 & \cdots & A_n \\ A_1 & A_2 & \cdots & A_{n+1} \\ \vdots & & & \\ A_n & A_{n+1} & \cdots & A_{2n} \end{vmatrix} \tag{6.9}$$

is given by

$$\Delta_{n+1}^0 = \prod_{j=0}^{n} b_j \prod_{k>j}^{n} (\lambda_k - \lambda_j)^2. \tag{6.10}$$

Proof. Δ_{n+1}^0 is a polynomial in the b_j and λ_j, so it is enough to verify the identity (6.10) when all the b_j are positive. Under this assumption, Δ_{n+1}^0 is a moment matrix for a positive measure with weights at the λ_j. By Lemma 5.4, Δ_{n+1}^0 is positive if and only the λ_j are distinct. Therefore each root $\lambda_j - \lambda_k$ is double, and Δ_{n+1}^0 is divisible by $\prod_{j<k} (\lambda_j - \lambda_k)^2$. Comparing the total degree in the λ_j, we find

$$\Delta_{n+1}^0 = c_{n+1}(b_0, ..., b_n) \prod_{k>j} (\lambda_k - \lambda_j)^2. \tag{6.11}$$

The coefficient is a polynomial of total degree $n+1$ in the b_j and vanishes if any $b_j = 0$, hence has the form $c_{n+1} \prod b_j$. The constant c_{n+1} may be determined inductively: we take $\lambda_n = 0$, so only the moment A_0 has a b_n term and Δ_{n+1}^0 is the product of b_n and the $n \times n$ minor Δ_n^2. The latter is Δ_n^0 for the measure

$$\sum_{j=0}^{n-1} \lambda_j^2 b_j \, \delta_{\lambda_j}$$

and it follows that $c_{n+1} = c_n = \cdots = c_1 = 1$. Q.E.D.

Note that any larger-moment matrix for the measure $d\mu$ supported at $n+1$ sites vanishes, e.g., by introducing additional sites with weights $b_{n+k} = 0$.

The formula (6.1) for $l = 0$, $k = n+1$ coincides with (6.10), under the replacement $\lambda_j \mapsto -\lambda_j$. The remaining formulas may be derived from this one. First, suppose that $l = 0$ and $k < n$. We express the determinant Δ_k^0 as a sum of terms indexed by the J's; each term is itself the determinant of the $k \times k$ matrix associated with the measure (5.6). This proves (6.1) when $l = 0$.

When $l > 0$, Δ_k^l coincides with a determinant of the form Δ_k^0 taken with respect to the measure

$$\sum_{j=0}^{n} (-\lambda_j)^l \, a_j \, \delta_{-\lambda_j}.$$

Therefore the remaining formulas of (6.1) are consequences of the formulas for $l = 0$. Q.E.D.

7. COLLISIONS AND CLOSE ENCOUNTERS

The phase shift formula (6.8) suggests that interactions occur in pairs. This is borne out by an analysis of close encounters ($x_{k+1}(t_0) - x_k(t_0)$ positive but locally minimal) and collisions ($x_{k+1}(t_0) = x_k(t_0)$). When only peakons or only antipeakons are present, collisions cannot occur: the determinants Δ_j^1 do not vanish; see Lemma 5.4. The equation $\dot{x}_j = u(x_j)$ in (2.15) shows that an overtaking peakon must be higher, and that after the event, the higher peakon must be to the right; whereas an overtaking trough must be lower, and moving to the left (see Fig. 2).

When both peaks and troughs are present, collisions occur. At a collision, some $\Delta_{n-k}^1 = 0$ and the solution of the inverse problem for (3.5) breaks down: the terms m_k and m_{k+1} blow up. We have $\Delta_0^1 = 1$, so m_n cannot become singular unless m_{n-1} does also, and (6.1) implies that $\Delta_n^1 > 0$, so m_1 cannot become singular unless m_2 does also.

We show by a direct analysis of the inverse problem that u has a well-behaved continuation throughout the collision. (This can also be seen from the fact that the Hamiltonian (2.16) is conserved under the flow, so that the $W^{1,2}$ norm of u, and hence the L^∞ norm, remain bounded.)

The qualitative facts about collisions between peakons and antipeakons are summarized in the following theorem.

THEOREM 7.1. *Suppose that some of the determinants Δ_{n-k}^1 vanish at $t = t_0$. As t approaches t_0, collisions occur within distinct peakon-antipeakon pairs. For $t < t_0$ each such pair consists of a peakon with peak at x_k and an antipeakon with trough at x_{k+1}; for $t_0 < t$ the trough is at x_k and the peak is at x_{k+1}.*

As $t \to t_0 \pm$ the function $u(x, t)$ converges uniformly (with respect to x) to a function that has the same form (but with the x_j no longer distinct).

These aspects of the peakon-antipeakon collision are illustrated in the figure in the introduction.

Although $u(x, t)$ is well-behaved in the supremum norm at collisions, this is not the case for the derivative.

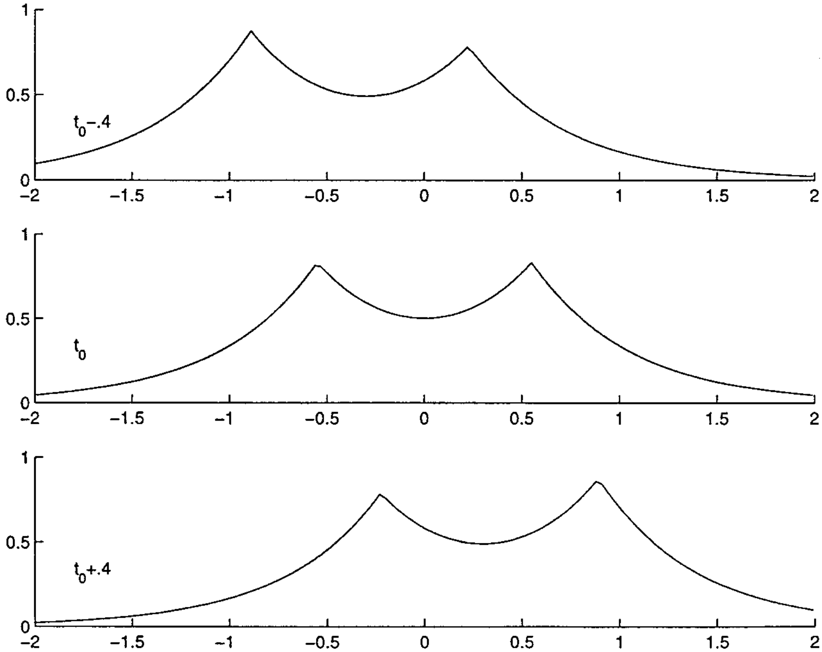

FIG. 2. A close encounter, computed from the exact formulas; $\lambda_1 = -1$, $\lambda_2 = -2$, $a_1(0) = 1$; $a_2(0) = 0.5$, $t_0 = 0$. After the encounter, for $t > t_0$, the peakon on the right picks up speed and moves away.

THEOREM 7.2. *Suppose that $\Delta_{n-k}(t_0) = 0$. Then on the interval (x_k, x_{k+1}),*

$$u_x(x, t) = \frac{\alpha}{t - t_0} + O(1) \qquad \text{as} \qquad t \to t_0, \qquad \alpha > 0. \tag{7.1}$$

The proofs of Theorems 7.1 and 7.2 use some facts from the classical moment problem; proofs may be found in the monograph by Akhiezer, [1], Chapter 1. Given the positive measure $d\mu$ of (5.6), one obtains a sequence of $n+1$ polynomials $\{P_j(\lambda), 0 \leqslant j \leqslant n\}$, such that P_j has degree j and the P_j are orthonormal in $L^2(\mathbb{R}, d\mu)$. Explicitly

$$P_j(\lambda) = \frac{1}{(\Delta_{j+1}^0 \Delta_j^0)^{1/2}} \begin{vmatrix} A_0 & A_1 & \cdots & A_j \\ A_1 & A_2 & \cdots & A_{j+1} \\ \vdots & & & \\ A_{j-1} & A_j & \cdots & A_{2j-1} \\ 1 & \lambda & \cdots & \lambda^j \end{vmatrix}, \qquad 0 \leqslant j \leqslant n. \tag{7.2}$$

In particular,

$$P_j(0) = (-1)^j \frac{\Delta_j^1}{(\Delta_{j+1}^0 \Delta_j^0)^{1/2}}, \qquad 0 \leqslant j \leqslant n, \tag{7.3}$$

hence, by (5.9), a collision between the kth and $(k+1)$st places occurs precisely when the constant term in the polynomial $P_{n-k}(\lambda)$ vanishes.

The $P_j(\lambda)$ satisfy a second order recursion relation

$$\lambda P_j(\lambda) = b_j P_{j+1}(\lambda) + d_j P_j(\lambda) + b_{j-1} P_{j-1}(\lambda), \qquad 1 \leqslant j \leqslant n-1, \tag{7.4}$$

where

$$b_j = \frac{(\Delta_j^0 \Delta_{j+2}^0)^{1/2}}{\Delta_{j+1}^0} > 0, \qquad 0 \leqslant j \leqslant n-1.$$

The recursion relation implies a well-known formula of Darboux–Christoffel,

$$b_j \frac{P_{j+1}(\lambda) P_j(\lambda') - P_j(\lambda) P_{j+1}(\lambda')}{\lambda - \lambda'} = \sum_{i=0}^{j} P_i(\lambda) P_i(\lambda'), \qquad 0 \leqslant j \leqslant n-1.$$

We shall use the limiting form

$$b_j(P'_{j+1} P_j - P_{j+1} P'_j) = \sum_{i=0}^{j} P_i^2, \qquad 0 \leqslant j \leqslant n-1, \tag{7.5}$$

where primes denote differentiation with respect to λ.

THEOREM 7.3. *The multipeakon solutions may be expressed in terms of the orthogonal polynomials $P_j(0, t)$ as*

$$l_j = P_{n-j}(0)^2, \qquad g_j = -\frac{1}{b_{n-j} P_{n-j+1}(0) P_{n-j}(0)} \tag{7.6}$$

$$y_j = 1 + b_{n-j}(P'_{n-j}(0) P_{n-j+1}(0) - P_{n-j}(0) P'_{n-j+1}(0)). \tag{7.7}$$

Proof. The formulas follow immediately from (5.9), (5.10), (7.3), and (7.5).

An immediate consequence of (7.5) and the assumption that $P_0(0) \neq 0$ is that no two consecutive $P_j(0)$ can vanish simultaneously. Therefore no two consecutive Δ_j^1 can vanish, $0 \leqslant j \leqslant n-1$; and collisions can occur only in distinct pairs m_k, m_{k+1}. Moreover, if $P_j(0) = 0$ then $P'_j(0) \neq 0$. From now on we consider the time-dependent case, but we shall mainly suppress the time dependence in the notation, e.g., $P_i = P_i(\lambda, t)$, $P_i(0) = P_i(0, t)$.

LEMMA 7.4. *Suppose that the measure* (5.6) *evolves according to* (4.10), *and let* $\{P_j(\lambda, t)\}$ *be the associated orthogonal polynomials. Suppose that* $P_j(0, t_0) = 0$. *Then the derivative* $\dot{P}_j$ *at* $t = t_0$ *satisfies*

$$\dot{P}_j + 2P_j/\lambda = P_j'(0) \sum_{i<j} P_i(0)\, P_i. \tag{7.8}$$

Proof. Let (,) denote the inner product with respect to $d\mu$. Differentiating $(P_j, P_i) = \delta_{ij}$, with respect to t (and suppressing the time variable) gives

$$0 = (\dot{P}_j, P_i) + (P_j, \dot{P}_i) + \sum_{\lambda_m \neq 0} \frac{2P_j(-\lambda_m)}{-\lambda_m} P_i(-\lambda_m)\, a_m. \tag{7.9}$$

We assume that $P_j(0)$ vanishes at $t = t_0$; then P_j/λ is a polynomial of degree less than j with constant term $P_j'(0)$ and we can add a summand at $\lambda = 0$ in (7.9) to obtain

$$(\dot{P}_j, P_i) + (P_j, \dot{P}_i) + 2(P_j/\lambda, P_i) = P_j'(0)\, P_i(0).$$

Note that $(P_j, \dot{P}_i) = 0$ if $i < j$. It follows that the polynomial $\dot{P}_j + 2P_j/\lambda$ has degree $< j$ and its expansion in the orthonormal basis $\{P_i\}$ is (7.8).Q.E.D.

Note that (7.8) implies

$$\dot{P}_j(0) + 2P_j'(0) = P_j'(0) \sum_{i<j} P_i(0)^2. \tag{7.10}$$

The next lemma implies that the zeros of the $\Delta_j^1(t)$ are simple.

LEMMA 7.5. *Under the assumptions of Lemma* 7.4, $\dot{P}_j(0)\, P_j'(0) < 0$ *at* $t = t_0$.

Proof. Since the polynomial $\dot{P}_j + 2P_j/\lambda$ has degree $< n$, it cannot vanish at all the non-zero points $-\lambda_m$ in the support of the measure. Therefore we have a strict inequality

$$\tfrac{1}{2}(\dot{P}_j(0) + 2P_j'(0))^2 < \|\dot{P}_j + 2\lambda^{-1} P_j\|^2 = \sum_{i<j} P_j'(0)^2\, P_i(0)^2, \tag{7.11}$$

where $\| \ \|^2$ is the norm on $L^2(\mathbb{R}, d\mu)$. The equality in (7.11) follows from (7.8) and the orthonormality of the polynomials P_j.

It follows from (7.10) and (7.11) that

$$-2\dot{P}_j(0)\, P_j'(0) > \dot{P}_j(0)^2. \qquad \text{Q.E.D.}$$

The next lemma allows us to determine the direction of the sign change for interacting peakon/antipeakon pairs.

LEMMA 7.6. *Under the assumptions of Lemma* 7.4, *at* $t = t_0$

$$\frac{d}{dt}(P_{j+1}(0, t)\, P_j(0, t)) > 0, \qquad \frac{d}{dt}(P_j(0, t)\, P_{j-1}(0, t)) < 0, \tag{7.12}$$

for $1 \leqslant j \leqslant n-1$ *and* $1 \leqslant j \leqslant n$, *respectively*.

Proof. By Lemma 7.5 and the assumption that $P_j(0, t_0) = 0$, the derivatives in (7.12) have the same sign as

$$-P_{j+1}(0, t_0)\, P_j'(0, t_0), \qquad -P_j'(0, t_0)\, P_{j-1}(0, t_0),$$

respectively. The conclusion (7.12) follows from (7.5). Q.E.D.

The final lemma gives an algebraic proof that the singularities in successive g_k's, hence in successive m_k's, cancel each other in a collision.

LEMMA 7.7. *Under the assumptions of Lemma* 7.4 *the sum* $g_k + g_{k+1}$ *has a finite limit as* $t \to t_0$.

Proof. Let $j = n-k$. From (7.6) and (7.4) we obtain

$$g_k + g_{k+1} = -\frac{b_j P_{j+1}(0) + b_{j-1} P_{j-1}(0)}{b_j b_{j-1} P_{j+1}(0)\, P_j(0)\, P_{j-1}(0)} = \frac{d_j}{b_j b_{j-1} P_{j+1}(0)\, P_{j-1}(0)}.$$

The denominator does not vanish for t near t_0, and it follows from (7.4) that $d_j = (\lambda P_j, P_j)$, which has a finite limit. Q.E.D.

Proof of Theorem 7.1. As noted above, (7.5) implies that collisions occur only in pairs m_k, m_{k+1}, where Δ^1_{n-k} vanishes. According to Theorem 5.6 and Eq. (7.3), m_k and m_{k+1} have the same signs as $-P_{j+1}(0)\, P_j(0)$ and $-P_j(0)\, P_{j-1}(0)$, respectively, $j = n-k$. It follows from (7.12) that m_k changes from positive to negative and m_{k+1} changes from negative to positive. In other words, a peakon on the left changes places with an antipeakon on the right.

The assertion about uniform convergence of $u(x, t)$ follows from Lemma 7.7. In fact suppose that $\Delta^1_{n-k}(t_0) = 0$, so that m_k and m_{k+1} become singular. At the same time, however, $y_k - y_{k+1} \to 0$, so Lemma 7.7 and (3.8) imply that the sum $m_k + m_{k+1}$ has a finite limit. For such a pair

$$\begin{aligned} &m_k(t)\, e^{-2\,|x - x_k(t)|} + m_{k+1}(t)\, e^{-2\,|x - x_{k+1}(t)|} \\ &\quad = \{m_k(t) + m_{k+1}(t)\}\, e^{-2\,|x - x_k(t)|} + m_{k+1}(t)\{e^{-2\,|x - x_{k+1}(t)|} - e^{-2\,|x - x_k(t)|}\}. \end{aligned}$$

The difference between the exponentials is uniformly $O(|x_k(t)-x_{k+1}(t)|)$, so the sum converges uniformly to

$$m_k(t_0)\, e^{-2\,|x-x_k(t_0)|}+m_{k+1}(t_0)\, e^{-2\,|x-x_{k+1}(t_0)|},$$

where

$$m_k(t_0)=m_{k+1}(t_0)=\tfrac{1}{2}\lim_{t\to t_0}\{m_k(t)+m_{k+1}(t)\},$$

$$x_k(t_0)=\lim_{t\to t_0} x_k(t)=\lim_{t\to t_0} x_{k+1}(t)=x_{k+1}(t_0).$$

The same argument applies to any other pair that become singular at $t=t_0$, so $u(x,t)$ converges uniformly. Q.E.D.

Proof of Theorem 7.2. It follows from the results of this section that if $\Delta^1_{n-k}(t_0)=0$, then

$$x_{k+1}-x_k=O(l_k)=\alpha_0(t-t_0)^2+O((t-t_0)^3);$$

$$m_k=-\frac{\alpha_1}{t-t_0}+O(1);$$

$$m_{k+1}=\frac{\alpha_1}{t-t_0}+O(1),$$

for some positive constants α_j which can be computed explicitly, from these facts and (1.2) we deduce that on the interval in question we may compute the derivative to order $O(1)$ by dropping all but two summands and replacing them by

$$\frac{\alpha_1}{t-t_0}\left(e^{2x-2x_{k+1}}-e^{2x_k-2x}\right)$$

whose derivative on the interval (x_k, x_{k+1}) is $4\alpha_1/(t-t_0)+O(1)$. ∎

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