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Acoustic Scattering and the Extended Korteweg–de Vries Hierarchy

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The acoustic scattering operator on the real line is mapped to a Schrödinger operator under the Liouville transformation. The potentials in the image are characterized precisely in terms of their scattering data, and the inverse transformation is obtained as a simple, linear quadrature. An existence theorem for the associated Harry Dym flows is proved, using the scattering method. The scattering problem associated with the Camassa–Holm flows on the real line is solved explicitly for a special case, which is used to reduce a general class of such problems to scattering problems on finite intervals. © 1998 Academic Press

Key Words: acoustic scattering; Harry Dym; Camassa–Holm flows.

1. INTRODUCTION

In this paper we consider the forward and inverse scattering problems on the line for operators

$$L_k = D^2 + k^2\rho^2 - q, \qquad D = d/dx. \tag{1.1}$$

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For $\rho = 1$, (1.1) is the Schrödinger operator and k is the wave number, or momentum. As is well-known, the scattering data of (1.1) evolves linearly as q evolves according to the nonlinear Korteweg–de Vries flows.

For $q = 0$ and $\rho \to 1$ as $x \to \pm\infty$ (1.1) is the *acoustic* scattering problem. In this interpretation, k is the frequency. The associated hierarchy of flows, of which the first is

$$(\rho^2)_t = \frac{1}{2} D^3 \left(\frac{1}{\rho}\right), \tag{1.2}$$

were introduced by Martin Kruskal [14] and attributed to an unpublished paper of Harry Dym.

For $q = 1/4$ and $\rho \to 0$ at infinity, (1.1) was introduced by Camassa and Holm [5], [6], in connection with a nonlinear shallow water model. The Camassa–Holm equation itself,[1] in the normalization determined by (1.1) with $q = 1$, is

$$\begin{aligned}(1 - \tfrac{1}{4}D^2)\, u_t &= \tfrac{3}{2}(u^2)_x - \tfrac{1}{8}(u_x^2)_x - \tfrac{1}{4}(uu_{xx})_x, \\ \rho^2 &= 2(1 - \tfrac{1}{4}D^2)u.\end{aligned} \tag{1.3}$$

For this equation, considered on the line, it is natural to assume that the potential $\rho \to 0$ as $|x| \to \infty$; the scattering problem on the line is therefore singular. The case of periodic potentials has been resolved by Constantin and McKean [7]. Camassa and Holm [5], and Camassa, Holm, and Hyman [6] have obtained blow-up results for certain initial data and have constructed two-soliton solutions by direct methods—that is, methods not based on the inverse scattering technique. Constantin and Escher [8] have proven global existence theorems for (1.3) for a large class of initial data.

The scattering problem for (1.1) when $\rho \equiv 1$ is by now well studied and completely understood; the inverse problem can be solved by the integral equation method of Gel'fand and Levitan [10]. In the acoustic problem, $\rho \to 1$ as $|x| \to \infty$ and $q = 0$, it is well-known that the classical Liouville transformation takes L_k to the Schrödinger operator [4]. The inverse transformation from the Schrödinger to the acoustic problem requires the inversion of a differential equation. We show in this paper that this step can be reduced to a simple linear quadrature. This greatly simplifies the inversion problem.

[1] The equation itself had been obtained previously by Fuchsteiner [9] by the method of recursion operators; but that method does not give the isospectral operator.

It is the acoustic problem, rather than the Schrödinger equation, that is relevant to physical applications in elasticity and reflection seismology [1], [2], [3], [4], [16].[2]

Although the scattering problem for (1.1) when $\rho \to 0$ at infinity and $q \equiv 1$ is *a priori* singular, a particular change of variables reduces it to the Dirichlet problem

$$\left(\frac{d^2}{d\xi^2} + k^2 \rho_1^2(\xi)\right) \psi(\xi, k) = 0, \qquad -1 \leqslant \xi \leqslant 1; \qquad \psi(\pm 1, k) = 0.$$

This is a density problem on a finite interval [12], [13]. A second Liouville transformation converts this to a Schrödinger problem on a finite interval. Thus the inversion problem reduces to solving an inverse Sturm–Liouville problem on a finite interval [11], [15].

2. ACOUSTIC SCATTERING

The equation

$$(D^2 + k^2 \rho^2)\, \psi = 0, \; -\infty < x < \infty, \tag{2.1}$$

arises in scattering problems for the wave equation

$$u_{tt} - c^2(x)\, u_{xx} = 0,$$

where the sound speed c is $1/\rho$. Therefore it is natural to assume ρ to be bounded away from 0 and to have a finite limit as $|x| \to \infty$. We call (2.1) the acoustic scattering problem. We assume throughout that ρ is real and positive, while $\rho - 1$ belongs to $\mathscr{S}$, the Schwartz class of rapidly decaying functions.

We begin by constructing the wave functions of (2.1), normalized at $\pm\infty$, by the WKB method. We write the wave functions in the form

$$\varphi_+(x, k) = \ell_+(x, k)\, e^{-ikS(x)}, \qquad \psi_+(x, k) = m_+(x, k)\, e^{ikS(x)},$$

where ℓ_+ and m_+ are normalized to be 1 at $-\infty$ and $+\infty$ respectively. Substituting this form of the wave functions into equation (2.1), we find that m_+ satisfies the differential equation

$$m'' + 2ikS'm' + [\, ikS'' + k^2(\rho^2 - S'^2)]\, m = 0.$$

[2] We thank Fadil Santosa, of the University of Minnesota, for his helpful information regarding the applied literature in acoustic scattering problems.

The term involving k^2 is eliminated by requiring $S(x)$ to satisfy the *eiconal equation*

$$(S')^2 = \rho^2.$$

We take

$$S'(x) = \rho(x), \qquad S(x) = x + \int_{-\infty}^{x} [\rho(y) - 1]\, dy.$$

Then

$$S(x) = \begin{cases} x + o(1) & x \to -\infty \\ x + \gamma + o(1) & x \to \infty \end{cases}$$

where

$$\gamma = \int_{-\infty}^{\infty} [\rho(y) - 1]\, dy.$$

We now have

$$m'' + 2ik\rho m' + ik\rho' m = 0, \qquad m \to 1 \quad \text{as} \quad x \to \infty. \tag{2.2}$$

The solution to this differential equation is constructed by converting it to the Volterra integral equation

$$\begin{aligned} m(x, k) &= 1 + \int_{x}^{\infty} \frac{e^{2ik(y-x)} - 1}{2ik} (2ik(1-\rho)\, m' - ik\rho' m)\, dy \\ &= 1 + \int_{x}^{\infty} G(x, y, k)\, m(y, k)\, dy, \end{aligned} \tag{2.3}$$

where

$$G(x, y, k) = 2ike^{2ik(y-x)}(\rho(y) - 1) + \tfrac{1}{2}(e^{2ik(y-x)} - 1)\, \rho'(y).$$

This integral equation can be solved for any k in the upper half plane $\operatorname{Im} k \geqslant 0$ by the method of successive approximations, since $\rho - 1$ and ρ' belong to $\mathscr{L}^1(\mathbb{R})$. The solution is analytic with respect to k; we denote it by $m_+(x, k)$. We denote the Schwarz reflection of m_+ to the lower half plane by $m_-(x, k) = \overline{m_+(x, \bar{k})}$, and extend ψ_+ to the lower half plane accordingly:

$$\psi_-(x, k) = \bar{\psi}_+(x, \bar{k}) = m_-(x, k)\, e^{-ikS(x)}.$$

Similarly the function ℓ_- defined by

$$\ell_-(x,k) = 1 - \int_{-\infty}^{x} G(x,y,k)\,\ell_-(y,k)\,dy, \tag{2.4}$$

is analytic with respect to k in the lower half plane and has a Schwarz reflection $\ell_+(x,k)$ analytic in the upper half plane. We denote the corresponding wave functions by

$$\varphi_\pm(x,k) = \ell_\pm(x,k)\,e^{\mp ikS(x)}.$$

The boundary values on the real line satisfy the asymptotic conditions

$$\lim_{x\to-\infty} e^{\pm ikS(x)}\varphi_\pm = \lim_{x\to\infty} e^{\mp ikS(x)}\psi_\pm = 1. \tag{2.5}$$

Note that

$$\ell_+(x,k) = \ell_-(x,-k), \qquad m_-(x,k) = m_+(x,-k), \qquad k\in\mathbb{R}. \tag{2.6}$$

Moreover $G(x,y,0)=0$ so

$$\ell_\pm(x,0) = m_\pm(x,0) \equiv 1. \tag{2.7}$$

The *scattering data* for the acoustic equation is defined just as it is for the Schrödinger equation. For real non-zero k each pair of wave functions $\psi_\pm$, $\varphi_\pm$ is (generically) independent and thus constitutes a fundamental set of solutions. Therefore

$$\varphi_+(x,k) = a(k)\,\psi_-(x,k) + b(k)\,\psi_+(x,k) \tag{2.8}$$

for some functions a and b. For real k we use (2.6), the definitions of the wave functions, and the limits, to see that (2.8) becomes

$$\ell_+(x,k) \sim a(k) + b(k)\,e^{2ikS(x)} \sim a(k) + b(k)\,e^{2ik(x+\gamma)} \qquad \text{as} \quad x\to\infty.$$

In view of (2.4), (2.6) and the form of G, we obtain for real k that

$$\begin{aligned} a(k) &= 1 + \tfrac{1}{2}\int_{-\infty}^{\infty} \rho'(y)\,\ell_+(y,k)\,dy; \\ b(k) &= \int_{-\infty}^{\infty} e^{-2ik(y+\gamma)}\{2ik(\rho(y)-1) - \tfrac{1}{2}\rho'(y)\}\,\ell_+(y,k)\,dy. \end{aligned} \tag{2.9}$$

It follows from (2.9) and (2.7) that

$$a(0) = 1, \qquad b(0) = 0. \tag{2.10}$$

The formula for $a(k)$ in (2.9) extends to the upper half plane. We take the Wronskian of both sides of (2.8) with $\psi_\pm$ and obtain

$$a(k) = \frac{W(\varphi_+, \psi_+)}{W(\psi_+, \varphi_-)} = \frac{W(\varphi_+, \psi_+)}{2ik}, \qquad 0 \neq k \in \mathbb{R}. \tag{2.11}$$

In fact, taking asymptotics as $x \to \infty$,

$$W(\psi_-, \psi_+) = W(e^{-ikS}, e^{ikS}) = 2ikS' \sim 2ik = 2ik\rho \sim 2ik.$$

The expression (2.11) for $a(k)$ also extends to the upper half plane.

THEOREM 2.1. *The reduced wave functions m_+, ℓ_+ are analytic in the upper half plane, and m_-, ℓ_- are analytic in the lower half plane.*
The function $a(k)$ is analytic in the upper half plane. Moreover $a(0) = 1$ and a has no zeros.

Proof. We have proved everything but the last statement. As in the case the Schrödinger equation, the acoustic scattering data for real k satisfy

$$a(k) = \overline{a(-k)}, \qquad b(k) = \overline{b(-k)}; \qquad |a(k)|^2 - |b(k)|^2 = 1. \tag{2.12}$$

Therefore a has no real zeros. According to (2.11) a zero at k in the upper half place would correspond precisely to a *bound state*: an L^2 wave function. This would be an eigenfunction for the operator $L = \rho^{-2}D^2$ with eigenvalue $-k^2$. However L is selfadjoint and negative in $L^2(\mathbb{R}, \rho^2\,dx)$, so it cannot have such an eigenvalue. ▮

We define the reflection coefficient r by

$$r(k) = \frac{b(k)}{a(k)}, \qquad \operatorname{Im} k = 0.$$

In the absence of bound states, r constitutes the complete scattering data for the problem. In fact from (2.12) we have

$$|a|^2 = \frac{1}{1 - |r|^2}.$$

Since a is analytic in $\operatorname{Im} k \neq 0$, tends to 1 as $k \to \infty$, and has no zeros, arg a can be recovered from $\log |a|$ on the real axis by the Hilbert transform:

$$\arg a(k) = \frac{1}{\pi}\,\text{P.V.} \int_{-\infty}^{\infty} \frac{\log |a(t)|}{k - t}\,dt.$$

Then $\log a$ is obtained for $\operatorname{Im} k \neq 0$ by the Cauchy integral representation.

3. THE LIOUVILLE TRANSFORMATION

The acoustic equation may be transformed to the Schrödinger equation by the well-known Liouville transformation. By our assumptions, $S(x)$ is a monotone increasing function on the line, hence we may define a change of variables by $\xi = S(x)$. The variable ξ corresponds physically to the time of travel. By the chain rule,

$$\frac{d}{dx} = \frac{d\xi}{dx}\frac{d}{d\xi} = \rho\,\frac{d}{d\xi}, \qquad \frac{d^2}{dx^2} = \rho\,\frac{d}{d\xi}\,\rho\,\frac{d}{d\xi}.$$

To keep track of the relevant variables we define $\rho_s(\xi)$ by

$$\rho_s(\xi) = \rho(x), \qquad \text{at} \quad \xi = S(x).$$

The mapping $f \to f \circ S^{-1}$ is a unitary map from $L^2(\mathbb{R}, \rho^2\,dx)$ to $L^2(\mathbb{R}, \rho_s\,d\xi)$. It carries the negative selfadjoint operator $\rho^{-2}D^2$ to the operator

$$D_\xi^2 + \frac{D_\xi \rho_s}{\rho_s}\,D_\xi, \qquad D_\xi = \frac{d}{d\xi}.$$

To complete the Liouville transformation we use the unitary map $f \to f\sqrt{\rho_s}$ from $L^2(\mathbb{R}, \rho_s\,d\xi)$ to $L^2(\mathbb{R}, d\xi)$. The corresponding gauge transformation (conjugation by $\rho_s^{-1/2}$) takes the preceding operator to the Schrödinger operator

$$D_\xi^2 - q(\xi), \tag{3.1}$$

where

$$q = \frac{1}{2}\frac{D_\xi^2 \rho_s}{\rho_s} - \frac{1}{4}\left(\frac{D_\xi \rho_s}{\rho_s}\right)^2. \tag{3.2}$$

Since $D_\xi = \rho^{-1}\,D_x = \rho^{-1}D$, we can also express the potential as

$$q(S(x)) = \frac{1}{2}\frac{D\rho}{\rho^3} - \frac{3}{4}\left(\frac{D\rho}{\rho^2}\right)^2 = \frac{1}{2\rho^2}\,\{\xi, x\} \tag{3.3}$$

where $\{\xi, x\}$ denotes the Schwarzian derivative

$$\{\xi, x\} = \frac{D^3\xi}{D\xi} - \frac{3}{2}\left(\frac{D^2\xi}{D\xi}\right)^2.$$

We have seen that the transformation from the Schrödinger potential q to the associated acoustic potential ρ is given by the equation (3.2), together with a change of variables. We show below that this transformation may be computed by a linear operation on scattering data and a simple quadrature.

THEOREM 3.1. *The normalized wave functions* $\varphi_{s,\pm}, \psi_{s,\pm}$ *for the operator* (3.1), (3.2) *are related to the wave functions for the acoustic operator by*

$$\varphi_{s,\pm}(\xi) = \sqrt{\rho(\xi)}\, \varphi_{\pm}(S^{-1}\xi); \qquad \psi_{s,\pm} = \sqrt{\rho(\xi)}\, \psi_{\pm}(S^{-1}\xi). \tag{3.4}$$

The scattering data (a, b) *of the Schrödinger problem is precisely that obtained for the acoustic problem* (2.1).

Proof. The relation in (3.4) simply implements the two unitary transformations, change of variables and change of gauge. Therefore functions $\varphi_{s,\pm}, \psi_{s,\pm}$ are wave functions for the operator (3.1). Moreover they have the correct asymptotics and analyticity properties. ∎

THEOREM 3.2. *The image of the acoustic problem under the Liouville transformation consists of all Schrödinger operators with Schwartz class potentials and no bound states, such that* $a(0) = 1$.

The inversion of the Liouville transformation is given by

$$\rho_s(\xi) = \psi_s(\xi, 0)^2, \qquad x = \xi + \int_{-\infty}^{\xi} \left(\frac{1}{\rho_s(\xi')} - 1 \right) d\xi'. \tag{3.5}$$

Proof. Theorems 2.1 and 3.1 imply that operators in the range of the Liouville transformation have no bound states and have $a(0) = 1$. The transformation equation (3.2) shows that the potential is of Schwartz class. The acoustic potential ρ may be recovered from the Schrödinger potential q as follows. By (3.4) the wave functions at $k = 0$ are related by

$$\psi_s(\xi, 0) = \sqrt{\rho_s(\xi)}\, \psi(\xi, 0).$$

On the other hand we have observed that the normalized acoustic wave functions at $k = 0$ are identically 1. It follows that

$$\psi_s(\xi, 0) = \sqrt{\rho_s(\xi)}.$$

Therefore to reconstruct the function $x = x(\xi)$ from the Schrödinger potential, we can compute $\psi_s(\cdot, 0)$ and use

$$\frac{dx}{d\xi} = \frac{1}{\rho_s(\xi)} = \frac{1}{\psi_s(\xi, 0)^2}.$$

Suppose, conversely, that q is a Schwartz class Schrödinger potential with no bound states, such that $a(0)=1$. Then $\psi_s(\cdot, 0)=\varphi_s(\cdot, 0)$ is real and asymptotically 1 in each direction. We prove that $\psi_s(\xi, 0=m^s(\xi, 0)$ has no zeros when there are no bound states. First, note that $m^s_+(\cdot, i\omega)$ is real and converges uniformly to $m^s_+(\cdot, 0)$ as ω decreases to 0. If the latter function had any zeros they would necessarily be simple, since m^s is a solution of a second order differential equation, and therefore $m^s_+(\cdot, i\omega)$ would have zeros for small $\omega > 0$. This function converges uniformly to 1 as $\varepsilon \to +\infty$ and the zeros remain simple, so there would be a value $\omega > 0$ for which m^s_+ has limit 0 as $x \to \infty$. But the corresponding ψ^s_+ would be a bound state. Therefore $\psi_s(\cdot, 0)$ is positive.[3] Because of this we can use (3.5) to construct a change of variables and a potential ρ. The assumption that q belongs to $\mathscr{S}$ implies that $\psi_s(\cdot)-1$ belongs to $\mathscr{S}$. Therefore $\rho - 1$ belongs to $\mathscr{S}$. Clearly $D^2 - q$ is the Liouville transform of the acoustic operator associated to ρ. ∎

We have shown that the density ρ in the time of travel coordinate ξ is obtained immediately from the Schrödinger wave function at $k=0$. The latter can be obtained directly from the Gel'fand Levitan kernel $K(x, y)$, since the wave functions are given by

$$\psi(\xi, k) = e^{ik\xi} + \int_\xi^\infty K(\xi, y)\, e^{iky}\, dy.$$

4. THE HARRY DYM FLOWS

The Harry Dym flows are related to the operator $L = \rho^{-2} D^2$ exactly as the KdV flows are to $D^2_\xi - q$. They are derived from commutator conditions $[L, A]=0$ where A is the Liouville transformation of an operator that determines one of the KdV flows. The computation of A is rather complicated, and seems to shed no light on the issue. It is therefore more convenient to work with $L_k = \rho^2(L + k^2)$ instead. Then the commutator condition must be modified to

$$[L_k, A] = BL_k \tag{4.1}$$

[3] By the same argument one can prove that the number of zeros of $m^s(\xi, 0)$ is precisely equal to the number of bound states. If there are n bound states, then by oscillation theory $m^s(\xi, i\omega_n)$ has n zeros. The number of zeros of m^s is constant as ω decreases from ω_n to zero; so $m^s(\xi, 0)$ also has n zeros.

where L_k, A, B act on functions $\psi = \psi(x, t, k)$, and L_k and A are given by:

$$L_k = D^2 + k^2\rho^2, \qquad A = -\frac{\partial}{\partial t} + \sum_{j=0}^{n} f_j D^j.$$

The coefficients f_j are taken to be polynomials in k^2. The significance of (4.1) is that $[L, A]$ vanishes on the wave functions, i.e. solutions of $L_k\psi = 0$. The conditions relating the coefficients of L_k and A may be obtained from cross-differentiation of the pair of equations $L_k\psi = 0$, $A\psi = 0$.

Any operator product of the form CL may be added to A without affecting (4.1). Therefore even powers of D may be eliminated in favor of even powers of k, and it is enough to take $n = 1$: $A = -\partial_t + fD + g$. Then

$$\begin{aligned}[L, A] &= 2f_x D^2 + (f_{xx} + 2g_x)\, D + [k^2(\rho^2)_t - k^2 f(\rho^2)_x] + g_{xx} \\ &= (f_{xx} + 2g_0)\, D + [k^2(\rho^2)_t - k^2(\rho^2)_x + g_{xx} - 2k^2\rho^2 f_x] + 2f_1 DL.\end{aligned}$$

The coefficient of D must vanish, so we take

$$g = -\tfrac{1}{2} f_x.$$

Then the conditions for (4.1) become

$$\begin{aligned}k^2(\rho^2)_t &= \tfrac{1}{2} f_{xxx} + k^2[(\rho^2)_x f + 2\rho^2 f_x] \\ &= \tfrac{1}{2} f_{xxx} + 2\rho(\rho f)_x.\end{aligned}$$

Setting

$$f = \sum_{j=1}^{n} F_j k^{2j},$$

substituting this expession into the previous identity, and comparing coefficients of powers of k, we find that F_n is a constant multiple of $1/\rho$, $F_0 = 0$, and the remaining coefficients can be determined from the recursion relation

$$4\rho D(\rho F_{j-1}) = -D^3 F_j.$$

The flow of ρ is

$$(\rho^2)_t = \tfrac{1}{2}\, D^3 F_1.$$

For $n = 1$, we take $F_1 = 1/\rho$ and obtain (1.2).

THEOREM 4.1. *Under the Harry Dym flow, the scattering data $a(k, t)$ and $b(k, t)$ evolve according to*

$$\dot{a} = 0, \qquad \dot{b} = 2ik^3 b. \tag{4.2}$$

Proof. The commutator relation (4.1) implies that the kernel of L is invariant under A; that is,

$$L\varphi(x, t, k) = 0 \Rightarrow LA\varphi(x, t, k) = 0.$$

The wave function

$$\varphi_+(x, t, k) = \ell_+(x, t, k)\, e^{-ikS(x)} \sim e^{-ikx}, \qquad x \to -\infty.$$

On the other hand, for the Dym equation itself,

$$A = -\partial_t + \frac{k^2}{\rho} D - \frac{1}{2} D\left(\frac{1}{\rho}\right) \sim -\partial_t + k^2 D, \qquad x \to \pm\infty.$$

Therefore,

$$A\varphi_+(x, t, k) \sim (-\partial_t + k^2 D)\, e^{-ikx} = -ik^3 e^{-ikx}, \qquad x \to -\infty.$$

Since the wave functions are uniquely determined by their asymptotic behavior as $x \to -\infty$, we conclude that

$$A\varphi_+ = -ik^3 \varphi_+.$$

Similarly, we find that

$$A\psi_\pm = \pm ik^3 \psi_\pm, \qquad A\varphi_- = ik^3 \varphi_-.$$

Therefore, applying A to (2.8), we obtain

$$\begin{aligned} A\varphi_+ = -ik^3\varphi_+ &= -ik^3(a\psi_- + b\psi_+) \\ &= \dot{a}\psi_- - \dot{b}\psi_+ + aA\psi_- + bA\psi_+ \\ &= \dot{a}\psi_- - \dot{b}\psi_+ - ik^3 a\psi_- + ik^3 b\psi_+; \end{aligned}$$

and equations (4.2) follow immediately from the independence of ψ_+ and ψ_-. ∎

5. THE CAMASSA–HOLM SPECTRAL PROBLEM

We consider the operators

$$L = D^2 + k^2\rho^2 - 1, \tag{5.1}$$

where $D = d/dx$, introduced by Camassa and Holm [5]. We are interested in a singular case of (5.1): ρ is positive but $\rho \to 0$ rapidly at infinity. As a consequence, the natural normalization for the wave functions is independent of k:

$$\lim_{x\to-\infty} e^{-x}\varphi(x) = 1 = \lim_{x\to+\infty} e^{x}\psi(x). \tag{5.2}$$

The Camassa–Holm equation (1.3) implies that the evolution of ρ^2 is given by

$$(\rho^2)_t = u(\rho^2)_x + 2u_x\rho^2.$$

Under our assumption that ρ is strictly positive, we have the equivalent form,

$$\rho_t = (u\rho)_x, \qquad (4 - D^2)\, u = 2\rho^2. \tag{5.3}$$

Let us assume that

$$\int_{-\infty}^{\infty} e^{2|y|}\rho(y)^2\, dy < \infty. \tag{5.4}$$

Then, from the second equation in (5.3),

$$u(x) = \tfrac{1}{2}\int_{-\infty}^{\infty} e^{-2|x-y|}\rho(y)^2\, dy.$$

It follows that

$$\lim_{x\to\pm\infty} e^{2|x|}u(x) = \int_{-\infty}^{\infty} e^{\pm 2y}\rho(y)^2\, dy. \tag{5.5}$$

Therefore the evolution (5.3) is consistent with the assumption (5.4), and with the stronger assumption $\rho(x) = O(e^{-2|x|})$ as $|x| \to \infty$.

For reasons that will become clear, a natural class of functions ρ to consider are positive C^∞ functions that have asymptotic expansions

$$\rho(x) \sim \sum_{\nu=1}^{\infty} a_\nu^\pm e^{-2\nu x} \qquad \text{as} \quad x \to \pm\infty, \quad a_1^- = a_1^+ > 0. \tag{5.6}$$

It follows that u has similar expansions. The leading terms $b_1^{\pm}$ in these expansions are given by (5.5) and the remaining terms can be computed from (5.6) and the second equation in (5.3). These expansions are consistent with the evolution (5.3) and the time dependence of the coefficients can be computed from these equations.

We observe next that for a particular choice of potential ρ the Liouville transformation trivializes the operator $\rho^{-2}(D^2-1)$. According to the calculations in Section 3, the latter transforms to a Schrödinger operator with potential

$$q(S(x)) = \frac{1}{\rho^2}\left[\frac{1}{2}\frac{D^2\rho}{\rho} - \frac{3}{4}\left(\frac{D\rho}{\rho}\right)^2 + 1\right]$$
$$= \frac{1}{\rho^2}\left[-\sqrt{\rho}\, D^2\left(\frac{1}{\sqrt{\rho}}\right) + 1\right].$$

Therefore if we choose

$$\rho_0(x) = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x \tag{5.7}$$

the potential q_0 vanishes. Note that $\rho_0(x)$ has an expansion (5.6). Since $\operatorname{sech}^2 x = D \tanh x$, we can take as transformed variable $\zeta = S_0(x) = \tanh x$. Then the range of the transformation is the interval $-1 < \zeta < 1$.

The positive smooth potentials ρ that have asymptotic expansions (5.6) are precisely the potentials that can be written in the form

$$\rho(x) = g(\tanh x)\, \rho_0(x) = g(\tanh x) \operatorname{sech}^2 x, \tag{5.8}$$

$$g \in C^\infty([-1, 1]), \qquad g > 0, \qquad g(-1) = g(1). \tag{5.9}$$

The Liouville transformation generated by $\rho_0(x)$ brings $\rho_0^{-2}(D^2 + k^2\rho^2 - 1)$ to the form

$$\left(\frac{d}{d\zeta}\right)^2 + k^2 g(\zeta)^2. \tag{5.10}$$

This is the density problem for a string on the finite interval. [12], [13].

The Liouville transformation from (5.1) with $d\zeta/dx = \rho$ and the Liouville transformation from (5.10) with $d\xi/d\zeta = g(\zeta)$ arrive at the same point: a Schrödinger operator on a finite interval. The length of the interval is $2M$ where

$$M = \tfrac{1}{2}\int_{-\infty}^{\infty} \rho(x)\, dx = \tfrac{1}{2}\int_{1}^{1} g(\zeta)\, d\zeta, \tag{5.11}$$

and we normalize the interval to be $(-M, M)$. The expression for the potential $q(\xi)$ in terms of g is given by (3.3):

$$q(\xi(\zeta)) = \frac{1}{2g^2}\{\xi, \zeta\}. \tag{5.12}$$

6. CAMASSA–HOLM SCATTERING DATA

In this section we describe the scattering data for (5.1) and find its evolution under the Camassa–Holm flows.

We have seen that the Liouville transformation is a unitary equivalence between the negative operator $\rho^{-2}(D^2-1)$ and the Schrödinger problem on a finite interval. Under this transformation the normalized wave functions (5.2) are multiplied by ρ, which vanishes at ∞. On the finite interval, the asymptotic conditions become Dirichlet conditions: vanishing at an endpoint. Thus it is natural to take as scattering data for (5.1) the Dirichlet spectrum and the associated coupling coefficients. Note that the wave functions for (5.10) at $k=0$ have the form $a\xi_0+b$; the corresponding wave functions for the Schrödinger operator do not satisfy Dirichlet conditions. Thus the Dirichlet eigenvalues are strictly negative.

At the eigenvalues $-k_n^2$ the wave functions φ and ψ are linearly dependent: there is a *coupling coefficient* c_n such that

$$\varphi(x, k_n) = c_n \psi(x, k_n).$$

Thus we take as scattering data for ρ the countable set:

$$\{k_n, c_n\}. \tag{6.1}$$

The Camassa–Holm equation is obtained from the commutator relationship (4.1), with L given by (5.1) and

$$A = -\frac{\partial}{\partial t} + aD - \frac{1}{2}a_x, \qquad a = u(x, t, k) + \frac{1}{k^2}.$$

Note that the interval $(-M, M)$ remains constant, in view of (5.11), and (5.3).

To determine the evolution of the coupling coefficients, we proceed as in the Harry Dym flow, with suitable modifications. The compatibility relation (4.1) implies that the kernel of L is invariant under A; hence $A\varphi(x, t, k)$ is a linear combination of $\varphi(x, t, k)$ and $\psi(x, t, k)$. The exact linear combination is determined by evaluating the asymptotic behavior of

$A\varphi$ as $x \to \pm\infty$. Under our assumptions, u and u_x, tend to zero exponentially as x tends to $-\infty$. Again, the leading asymptotics of φ are independent of t, so we find that the leading asymptotics of $A\varphi$ as $x \to -\infty$ are

$$A\varphi = \left(-\partial_t + aD - \frac{1}{2} a_x \right) \varphi \sim \frac{De^x}{k^2} = \frac{e^x}{k^2},$$

hence

$$A\varphi(x, t, k) = \frac{1}{k^2} \varphi(x, t, k).$$

Similarly,

$$A\psi(x, t, k) = -\frac{1}{k^2} \psi(x, t, k).$$

At a bound state,

$$\varphi(x, t, k_n) = c_n(t)\, \psi(x, t, k_n),$$

hence

$$\begin{aligned} \frac{c_n}{k_n^2} \psi_n &= \frac{1}{k_n^2} \varphi_n = A\varphi_n = A(c_n \psi_n) \\ &= (-\dot{c}_n \psi_n + c_n A\psi_n) = \left(-\dot{c}_n - \frac{c_n}{k_n^2} \right) \psi_n. \end{aligned}$$

Since ψ_n is not identically zero, we have

$$\dot{c}_n(t) = -\frac{2c_n}{k_n^2}, \qquad c_n(t) = e^{-2t/k_n^2} c_n(0). \tag{6.2}$$

We next turn to the characterization of the image of $\rho^{-2}(D^2 - 1)$ under the Liouville transformation to a Schrödinger operator.

THEOREM 6.1. *The necessary and sufficient conditions that the Schrödinger operator $D_\xi^2 - q$ on the interval $(-M, M)$ be in the range of the Liouville transformation associated to* (5.1), *where $\rho = g(\tanh x) \operatorname{sech}^2 x$ and g is a positive smooth function on $[-1, 1]$ with $g(-1) = g(1)$, are that q be smooth on $[-M, M]$ and that the Dirichlet spectrum of $D_\xi^2 - q$ be strictly negative.*

Proof. It is clear from the formula that describes q in terms of g_0 that q must be smooth, and we have noted above that the Dirichlet spectrum must be strictly negative.

Conversely, suppose that the Schrödinger potential q is smooth and the Dirichlet spectrum is strictly negative. By standard oscillation theory the non-zero solutions to

$$(D_\xi^2 - q)\,\varphi_0 = 0, \qquad \varphi_0(-M) = 0, \qquad (D_\xi^2 - q)\,\psi_0 = 0, \qquad \psi_0(M) = 0$$

do not change sign on the interval. Therefore the (unique) solution to the Dirichlet problem

$$(D_\xi^2 - q)\,\psi = 0, \qquad \psi(-M) = \psi(M) = 1$$

is a linear combination of φ_0 and ψ_0 that is strictly positive on the closed interval. The inverse Liouville transformation must be associated to the change of variables $d\zeta/d\xi = \lambda/\psi(\xi)^2$, where λ is determined by the normalization condition

$$\lambda \int_{-M}^{M} \frac{d\xi}{\psi(\xi)^2} = 2. \quad \blacksquare$$

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