Stability of Bifurcating Solutions by Leray-Schauder Degree

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1. Introduction

Let the equilibrium states of some physical system be given by an equation of the form

$$(L-\mu B)u+F(\mu;u)=0 \tag{1.1}$$

where L and B are linear and F is a non-linear operator defined in some Banach space $\mathscr{B}$. The quantity μ is a real parameter. Assume that $F(\mu, 0)=0$ so that $u=0$ is always an equilibrium.

In many physical situations the null solution is a stable equilibrium for μ less than some critical value μ_0 but becomes unstable when μ is increased beyond μ_0. One is then interested in knowing conditions under which other stable equilibria bifurcate from the trivial solution at criticality. Such questions arise, for example, in convection problems in fluid dynamics; buckling problems in elasticity; criticality problems in nuclear reactor design, *etc.*

For the purposes of this paper the principle of linearized stability is assumed to hold: that is, $\tilde{u}$ is a stable equilibrium if all the eigenvalues of the "derivative" operator

$$L-\mu B+F'_u(\mu;\tilde{u}) \tag{1.2}$$

have positive real parts, and $\tilde{u}$ is unstable if some of the eigenvalues have negative real parts. We assume that $F'_\mu(\mu;0)=0$ so that the stability of the trivial solution is determined by the location of the eigenvalues of $L-\mu B$.

In this paper a topological degree argument is used to prove the following: Suppose that for $0\leqq\mu<\mu_0$ the trivial solution is stable, and that instability occurs as μ crosses μ_0 by a simple eigenvalue of $L-\mu B$ crossing the origin into the left half plane. Then *(under suitable conditions to be discussed later)* there exist, besides the trivial solution, either

(i) two stable solutions of (1.1) for $\mu>\mu_0$ and no solutions for $\mu<\mu_0$,

or

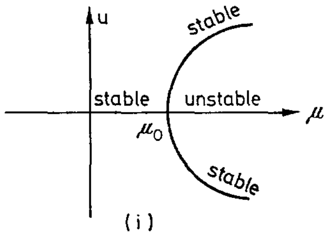
(ii) one stable solution for $\mu>\mu_0$ and one unstable solution for $\mu<\mu_0$,

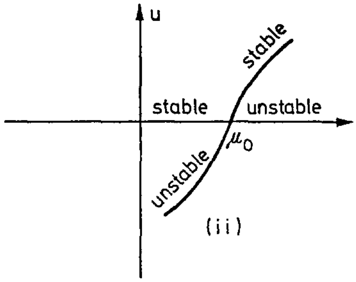
or

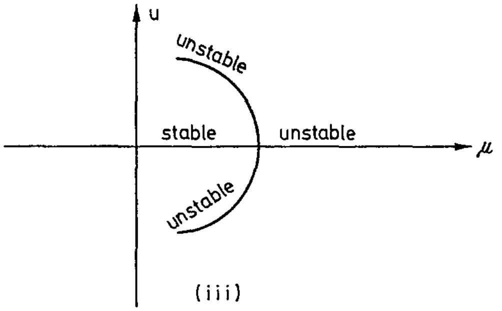
(iii) two unstable solutions for $\mu<\mu_0$ and no solutions for $\mu>\mu_0$.

The non-trivial solutions tend to zero as μ tends to μ_0.

These three possibilities are indicated schematically in the three diagrams below:







In short, solutions which bifurcate above criticality are stable and solutions bifurcating below criticality are unstable.

One might attempt to prove this result by a direct perturbation analysis of the critical eigenvalue. This approach has been carried out in a number of specific cases, for example in [7] for the Taylor problem; but the success of the perturbation approach depends on the non-vanishing of the second coefficient in the perturbation series for the critical eigenvalue. To prove a result of the generality stated here, the general first non-vanishing coefficient in the perturbation series would have to be determined and related to the direction (subcritical or supercritical) of the bifurcation. By contrast, the topological degree argument presented here is relatively simple and requires relatively little specific information about the particular bifurcation problem at hand. There is, in addition, a natural motivation for using degree theory which is explained in § 4.

The hypothesis of linearized stability, usually assumed to be valid in practical investigations of stability, has been shown to hold in a number of important physical cases [11]–[13]. The assumptions on L, B, and F are not overly restrictive, and the theorem is sufficiently general to apply to a variety of physical problems, including the Taylor and Bénard problems of hydrodynamic stability.

The requirement that μ_0 be simple cannot be relaxed in general; as the examples in § 5 show, the main theorem, Theorem 4.2, is not true in the case of bifurcation at a double eigenvalue. On the other hand the critical parameter turns out to be simple in a large number of physical problems or can be made to be simple by a judicious choice of $\mathscr{B}$. (For example, by looking for bifurcating solutions of a distinct periodicity, [10], [18], [19]. See [7] for a discussion of the relationship between the stability of bifurcating solutions of a distinct wave number and the simplicity of μ_0.) Examples illustrating each of the three alternative possibilities at criticality are given in § 5.

There is an extensive literature concerning the bifurcation of solutions of non-linear equations at critical parameter values. We mention, in addition to the references cited above, [5], [9], [14], [16]–[21]. In some cases the stability of the bifurcating solutions has been established by perturbation methods [7], [14], [20]. In § 5 we give a brief account of two problems in hydrodynamic stability to which our results apply.

2. Functional Analytic Preliminaries

Typically L might be a second order elliptic differential operator and B and F lower order operators, but it is desirable to state and prove the results in more general terms. In this section we discuss the general assumptions which must be made in the problem.

We assume L, B, and F are defined on dense subsets of a Banach space $\mathscr{B}$ and that the domain of L is contained in that of B and F. Let L have a compact inverse K and assume that KB and KF have completely continuous extensions to all of $\mathscr{B}$. Denote these by T and N, respectively. Furthermore, let (1.1) be equivalent to

$$(I-\mu T)u+N(\mu; u)=0 \tag{2.1}$$

in the sense that any solution of (2.1) belongs to $\mathscr{D}(L)$ (domain of L) and satisfies (1.1). In the case of partial differential equations this amounts to assuming that solutions of the weak form (2.1) are regular and satisfy equation (1.1) in strong form.

We assume that N has a continuous Fréchet derivative, denoted by N'_u, in a neighborhood of the origin in $\mathscr{B}$. Thus N'_u is a bounded linear transformation on $\mathscr{B}$ such that for $\tilde{u}$, v in $\mathscr{B}$, [15]

$$N(\mu; \tilde{u}+v)=N(\mu; \tilde{u})+N'_u(\mu; \tilde{u})v+R(\mu; \tilde{u}; v)v \tag{2.2}$$

where $\|R(\mu; \tilde{u}; v)\| \to 0$ as $\|v\| \to 0$. (Note: We use $\| \ \|$ to denote the norm of an element of $\mathscr{B}$ as well as that of a linear transformation on $\mathscr{B}$.) From the complete continuity of N it follows that N'_u is completely continuous [8]. We assume $N'_u(\mu; \tilde{u})$ depends continuously on μ and $\tilde{u}$ in the uniform operator topology and that $N'_u(\mu; 0)=0$. Finally, let N be differentiable with respect to μ and let $\|N'_u(\mu; \tilde{u})\|=o(\|\tilde{u}\|)$; it follows that N is uniformly continuous in u.

For any $\tilde{u}$, v in $\mathscr{D}(F)$ we assume $N'_u(\mu; \tilde{u})v$ and $R(\mu; \tilde{u}; v)v$ lie in $\mathscr{D}(L)$. Then F'_u is defined to be the linear operator $LN'_u(\mu; \tilde{u})$ whose domain is $\mathscr{D}(F)$.

In many physical applications we deal with real solutions. In that case the operators (1.1) and (1.2) are "real" in the sense that $\overline{L\varphi}=L\bar{\varphi}$, where the bar denotes complex conjugation. It follows that the eigenvalues of (1.2) occur in complex conjugate pairs. This fact, which we take to be the case, is very important to our main argument; for with the perturbation assumption below it implies that a simple, isolated, real eigenvalue of (1.2) must remain on the real axis for small variations in μ.

At criticality $L-\mu_0 B$ is assumed to have a simple eigenvalue at the origin and the rest of its spectrum to the right of some line $\operatorname{Re}\lambda=\sigma>0$. We assume that for small variations in μ the eigenvalues of (1.2) vary continuously with μ and, except for the eigenvalue near the origin, remain to the right of $\operatorname{Re}\lambda=\frac{\sigma}{2}$.

These assumptions are valid, for example, if the following two conditions hold:

(1) There is a real number γ and an angle θ, $0<\theta<\frac{\pi}{2}$, such that the spectrum of

$$L-\mu B+F_u'(\mu,\tilde{u})+\gamma \tag{2.3}$$

lies in the interior of the wedge $\mathscr{W}=\{\lambda\colon |\arg\lambda|\leqq\theta\}$ for μ in a neighborhood of μ_0.

(2) The inverse of operator (2.3) is given by

$$(L-\mu B+F_u'+\gamma)^{-1}=(I-\mu T+N_u'+\gamma K)^{-1}K. \tag{2.4}$$

Both of these conditions are fulfilled when L is an elliptic operator with symmetric principal part acting on functions defined on a bounded domain with smooth boundary. (See [1], especially Chapters 9, 15.) The second condition is again in the nature of a regularity assumption as above.

Lemma 2.1. *Suppose conditions* (1) *and* (2) *above hold and that* $\tilde{u}(\mu)$ *varies continuously with* μ *and tends to zero as* μ *tends to* μ_0. *Then for* μ *sufficiently near* μ_0 *the spectrum of the derivative operator* (1.2) *is discrete; the eigenvalues vary continuously with* μ; *and except for the critical eigenvalue near the origin, they remain to the right of* $\operatorname{Re}\lambda=\frac{\sigma}{2}$.

Proof. The spectrum of (2.3) is the same as that of (1.2) shifted to the right by γ, so it suffices to prove the corresponding statements for (2.3). By assumption (2) we see that (2.3) has a compact inverse, hence that its spectrum consists of a discrete set of eigenvalues whose reciprocals are the eigenvalues of (2.4). For $\mu=\mu_0$ $(L-\mu_0 B+\gamma)^{-1}$ has a simple eigenvalue at γ^{-1}, and the remaining eigenvalues lie (in the complex plane) in the intersection of $\mathscr{W}$ with the interior of the circle with center $[2(\gamma+\sigma)]^{-1}$ and radius $[2(\gamma+\sigma)]^{-1}$. (N.B. The line $\operatorname{Re}\lambda=\gamma+\sigma$ goes into this circle under the transformation $z=\lambda^{-1}$.)

As μ varies the operator $(I-\mu T+N_u'+\gamma K)^{-1}$ varies continuously with μ. Hence the isolated eigenvalue near γ^{-1} varies continuously. Given any $\varepsilon>0$, the remaining eigenvalues lie in an ε neighborhood of the spectrum of $(L-\mu_0 B+\gamma)^{-1}$ for μ sufficiently close to μ_0. (See [3], Chapt. 7, § 6.) By assumption (1) these eigenvalues must also remain in $\mathscr{W}$. Therefore, for μ sufficiently close to μ_0 they lie in the intersection of $\mathscr{W}$ with the circle of radius $(2\gamma+\sigma)^{-1}$

whose center lies at $(2\gamma+\sigma)^{-1}$. Under the reciprocal transformation the interior of this circle is mapped into the region $\operatorname{Re}\lambda \geqq (\gamma+\sigma/2)$. The conclusion now follows immediately.

3. Bifurcation at a Simple Critical Value

In this section the Lyapounov-Schmidt method is used to discuss the nature of the bifurcation at criticality. The remarks here provide a convenient reference for our principal assumption about the bifurcation point and for the proof of Lemma 4.1, which is important in the development of the main theorem. The methods of this section are standard in bifurcation theory and need not be reproduced in full.

Operating on (1.1) by K and letting $\tau=\mu-\mu_0$, we get

$$(I-\mu_0 T)u-\tau T u+N(\mu; u)=0 \tag{3.1}$$

where $T=KB$.

Let φ_0 satisfy $(I-\mu_0 T)\varphi_0=0$. Denote the dual space of $\mathscr{B}$ by $\mathscr{B}^*$ and elements of $\mathscr{B}^*$ by φ^*, ψ^*, ... We use the notation $\varphi^*(\psi)=(\psi, \varphi^*)$. By the Fredholm alternative there is a vector φ_0^* such that $(I-\mu_0 T^*)\varphi_0^*=0$. Since μ_0 is simple φ_0^* can be normalized so that $(\varphi_0, \varphi_0^*)=1$.

Let P_0 be the projection onto φ_0 which commutes with T, namely, $P_0 u=(u, \varphi_0^*)\varphi_0$; and let $Q_0=I-P_0$. Operating on (3.1) with P_0 and Q_0 we get the equivalent system of equations

$$(I-\mu_0 T)Q_0 u-\tau T Q_0 u+Q_0 N(\mu; u)=0, \tag{3.2}$$

$$f(\alpha, \tau)\overset{\text{def}}{=}-\frac{\alpha\tau}{\mu_0}+(N(\mu; u), \varphi_0^*)=0, \tag{3.3}$$

where $\alpha=(u, \varphi_0^*)$. By the Riesz theory of compact operators there is a bounded operator S such that $S(I-\mu_0 T)Q_0=Q_0$. Operating on (3.2) with S and using $u=\alpha\,\varphi_0+Q_0 u$, we get

$$u-\tau S T Q_0 u+S Q_0 N(\mu; u)=\alpha\varphi_0. \tag{3.4}$$

Equation (3.3) with u given by (3.4) is called the bifurcation equation. Solutions of (3.4) may be constructed by successive approximations for small α and τ:

Lemma 3.1. *Let N be continuously Fréchet differentiable in u in a neighborhood of the origin in $\mathscr{B}$ and let $N_u'(\mu; 0)=0$. Then* (3.4) *is uniquely solvable for u for small α and τ, and the solution $u=u(\alpha, \tau)$ is continuously differentiable in α and τ. The derivatives $\frac{\partial u}{\partial\alpha}$ and $\frac{\partial u}{\partial\tau}$ satisfy the equations*

$$\frac{\partial u}{\partial\alpha}-\tau S T Q_0\frac{\partial u}{\partial\alpha}+S Q_0 N_u'(\mu; u)\frac{\partial u}{\partial\alpha}=\varphi_0, \tag{3.5}$$

$$\frac{\partial u}{\partial\tau}-\tau S T Q_0\frac{\partial u}{\partial\tau}+S Q_0 N_u'(\mu; u)\frac{\partial u}{\partial\tau}=+S T Q_0 u-S Q_0 N_\mu'(\mu; u) \tag{3.6}$$

where u is the solution of (3.4). *The function* $(N(\mu; u), \varphi_0^*)$ *is continuously differentiable in α and τ and its derivatives are obtained by formal differentiation. Furthermore if* $\|N'_\mu(\mu; u)\| = O(\|u\|)$, *then* $\|u\|$ *and* $\left\|\frac{\partial u}{\partial \tau}\right\|$ *are* $O(|\alpha|)$ *as* $\alpha \to 0$. *Finally, if N is analytic in μ and μ, then the solution u given by* (3.4) *is analytic in α and τ.*

Proof. Lemma 3.1 is an immediate consequence of the implicit function theorem in Banach spaces. We need only set $x = (\alpha, \tau)$ (the ordered pair) and

$$H(x, u) = u - \tau S T Q_0 u + S Q_0 N(\mu; u) - \alpha \varphi_0.$$

We easily see that $H(0,0) = 0$ and $H'_u(0,0) = I$. The first part then follows from [22], Theorem 10.2.1. The formulae (3.5) and (3.6) are obtained by differentiating $H(\alpha, \tau, u(\alpha, \tau))$ with respect to α and τ. The estimates for $\|u\|$ and $\|u_\tau\|$ are left to the reader.

The theory of analytic operator valued functions is discussed in [15]. In case N is analytic, the analyticity of u in α and τ follows from its differentiability [22] p. 225. In the analytic case it is of course necessary to assume that N is extended to a complex Banach space containing $\mathscr{B}$ and that it is differentiable with respect to u as a complex variable. This completes the proof of Lemma 3.1.

From now on we assume that $N(\mu; \alpha u) = \alpha^2 N_1(\mu; u; \alpha)$, where N_1 is Fréchet differentiable in (μ, u, α). In (3.4) we may set $u = \alpha v$, where v satisfies

$$v - \tau S T Q_0 v + \alpha S Q_0 N_1(\mu; v; \alpha) = \varphi_0.$$

The bifurcation equation then reads $f(\alpha, \tau) = \alpha g(\alpha, \tau) = 0$, where

$$g(\alpha, \tau) = -\frac{\tau}{\mu_0} + \alpha (N_1(\mu; v, \alpha), \varphi_0^*).$$

From our regularity assumptions on N_1 it immediately follows that

$$\left.\frac{\partial g}{\partial \tau}\right|_{\alpha, \tau = 0} = -\frac{1}{\mu_0}.$$

Hence the ordinary implicit function theorem can be applied to g to conclude that there exists a unique, continuously differentiable function $\tau = \tau(\alpha)$ such that $g(\alpha, \tau(\alpha)) \equiv 0$ for small α. The curve $\tau(\alpha)$ corresponds to the non-trivial solution branch bifurcating from the null solution $(\mu_0, 0)$.

It will be important in the sequel to establish that the Fréchet derivative

$$I - \mu T + N'_u(\mu, \tilde{u}) \tag{3.7}$$

is invertible along the bifurcating solution branch. (Here $\tilde{u}(\mu) = u(\alpha, \tau(\alpha))$ is the non-trivial bifurcating solution.)

Definition. *The point* $(\mu_0, 0)$ *is called a regular bifurcation point if the operator in* (3.7) *is invertible for all μ sufficiently close to μ_0 but* $\mu \neq \mu_0$.

Lemma 3.2. *The point* $(\mu_0, 0)$ *is a regular bifurcation point if and only if* $\tau'(\alpha) \neq 0$ *for α small, but* $\alpha \neq 0$. *In particular,* (3.7) *is invertible only if* $\tau'(\alpha) \neq 0$.

The proof of Lemma 3.2 is given below. The following important lemma follows immediately from Lemma 3.2 and gives a general sufficient condition for a bifurcation point to be regular.

Lemma 3.3. *If the non-linear operator N is analytic in μ and u, then a sufficient condition for $(\mu_0, 0)$ to be a regular bifurcation point is that the bifurcation not be vertical—that is, that there does not exist a continuous one parameter family of solutions of* (3.1) *for* $\mu=\mu_0$.

Proof of Lemma 3.3. If N is analytic, then f and hence g are analytic in α and τ. Consequently, the function $\tau(\alpha)$ is analytic, and the zeroes of $\tau'(\alpha)$ cannot cluster at $\alpha=0$ unless $\tau'(\alpha)\equiv 0$. In this case, however, the bifurcation is vertical, and we get a family of solutions of (3.1) for $\tau=0$ and α small but arbitrary.

When the bifurcation is vertical the bifurcating solutions have marginal stability, since in that case (3.7) is not invertible, and hence zero is an eigenvalue of the derivative operator (1.2).

Proof of Lemma 3.2. First note that the bifurcation function defined in (3.3) satisfies

$$\frac{\partial f}{\partial \tau}=\alpha\frac{\partial g}{\partial \tau}\neq 0$$

for small α and τ, $\alpha\neq 0$, since, as we have seen, $g_\tau(0,0)\neq 0$. Now along the curve $\tau(\alpha)$ we have $f_\alpha d\alpha+f_\tau d\tau=0$. Since $\tau'(\alpha)\neq 0$ we conclude that $f_\alpha\neq 0$ for small α and τ; that is, for small τ and $\alpha\neq 0$,

$$-\frac{\tau}{\mu_0}+\left(N_u'(\mu,\tilde{u})\frac{\partial \tilde{u}}{\partial \alpha},\varphi_0^*\right)\neq 0. \tag{3.8}$$

On the other hand, suppose zero is an eigenvalue of $I-\mu T+N_u'(\mu;\tilde{u})$. Then there is a vector φ such that

$$(I-\mu T)\varphi+N_u'(\mu;\tilde{u})\varphi=0. \tag{3.9}$$

Putting $\varphi=\beta\varphi_0+Q_0\varphi$, where Q_0 is the projection defined in § 3, as in the Lyapounov-Schmidt procedure we get

$$-\frac{\tau\beta}{\mu_0}+(N_u'(\mu;\tilde{u})\varphi,\varphi_0^*)=0, \tag{3.10}$$

$$\varphi-\tau STQ_0\varphi+SQ_0N_u'(\mu;\tilde{u})\varphi=\beta\varphi_0. \tag{3.11}$$

Comparing (3.11) with (3.5) we see that $\varphi=\beta\dfrac{\partial\tilde{u}}{\partial\alpha}$ (see remark below) so that (3.10) contradicts (3.8). All of the above arguments being valid for μ sufficiently close to μ_0, the proof of Lemma 4.1 is complete.

Remark. The operator on the left in (3.11) and (3.5) is uniquely invertible for small α and τ, since $N_u'(\mu;\tilde{u})$ tends to zero as $\mu\to\mu_0$. Equation (3.5), considered as a linear inhomogeneous equation for $\dfrac{\partial u}{\partial\alpha}$, therefore has a unique solution.

4. Stability and Leray-Schauder Degree

We begin with a brief summary of the Leray-Schauder degree of mapping theory. Let Φ denote a completely continuous transformation on a Banach space $\mathscr{B}$ to itself. Let Ω be a bounded open set in $\mathscr{B}$ with boundary $\partial\Omega$.

1. If Φ has no fixed points on $\partial\Omega$, then the Leray-Schauder degree $d\{I-\Phi, \Omega\}$ is defined and is an integer. (Remark: We omit the customary notation $\deg\{I-\Phi, \Omega, p\}$ since we are only concerned with fixed points of Φ, that is, solutions of $u-\Phi(u)=0$.)

2. Let Φ_t be a family of completely continuous operators for $a \leqq t \leqq b$ which is continuous from $[a, b]\times\Omega$ to $\mathscr{B}$. If Φ_t has no fixed points on $\partial\Omega$ for $a \leqq t \leqq b$, then $d\{I-\Phi_a, \Omega\}=d\{I-\Phi_b, \Omega\}$.

3. If u_0 is an isolated fixed point of Φ in Ω, then the *index* of u_0, denoted by $i[\Phi, u_0]$, is $d\{I-\Phi, \Omega_0\}$, where $\Omega_0 \subset \Omega$ is a neighborhood of u_0 containing no other fixed points of Φ. If Φ has a finite number of fixed points, $u_1, \ldots, u_n$, in Ω then $d\{I-\Phi, \Omega\}=\sum_{k=1}^{n} i[\Phi, u_k]$.

4. Let u_0 be a fixed point of Φ and let $A=\Phi'(u_0)$ be the Fréchet derivative of Φ at u_0. Then A is compact. If $I-A$ is invertible, then u_0 is an isolated fixed point of Φ, and $i[\Phi, u_0]=d\{I-A, \Omega\}$.

5. Let A have real eigenvalues $1<\mu_1<\mu_2<\cdots<\mu_n$ of multiplicities $m_1, \ldots, m_n$, respectively, and let $(I-A)$ be invertible. Then $d\{I-A, \Omega\}=(-1)^m$ where $m=m_1+\cdots+m_n$.

For a more complete treatment of the Leray-Schauder degree theory, see [8], [15].

Take $\Phi_\mu=-\mu T+N(\mu; u)$. We can find a $\delta>0$ and a bounded neighborhood Ω of the origin such that for $|\mu-\mu_0|\leqq\delta$, Ω contains only the solutions of (3.1) which bifurcate at $\mu=\mu_0$ and no fixed points on its boundary. (See the argument below.) In what follows μ is to be restricted to this interval.

We can now give some motivation for the use of topological degree theory. The index of the zero solution (which is a fixed point of Φ_μ) is $i[\Phi_\mu, 0]=d\{I-\mu T, \Omega\}$ since $N_u'(\mu; 0)=0$. Since $\lambda=0$ is a simple eigenvalue of $L-\mu_0 B$, $-1/\mu_0$ is a simple eigenvalue of T. By property 5 of the degree of a mapping, the index of the zero solution is therefore $+1$ for $\mu<\mu_0$ and -1 for $\mu>\mu_0$. Since the change in index of the zero solution is so intimately associated with its becoming unstable, one is led to ask whether a similar relationship holds between the index and the stability of the bifurcating solutions near $\mu=\mu_0$.

In order to be able to apply degree theory to the bifurcation at $(\mu_0, 0)$ we must show there exists an open domain $\Omega=\{u: \|u\|<\varepsilon\}$ such that, for small $|\mu-\mu_0|$, no solutions of (3.1) lie on $\partial\Omega$. This is a standard step in the application of degree theory to bifurcation problems. The proof below follows that of VELTE [16], pp. 117–118. Here it suffices to assume that there is no vertical bifurcation. In that case there is an $\varepsilon>0$ such that no non-trivial solutions of (3.1) with $\|u\|\leqq\varepsilon$ exist for $\mu=\mu_0$. We then claim there exists a $\delta>0$ such that for $|\mu-\mu_0|\leqq\delta$ there exist no solutions of (3.1) of norm $\|u\|=\varepsilon$. In the contrary case there would exist a sequence (u_k, μ_k) of solutions of (3.1) with $\mu_k\to\mu_0$ and $\|u_k\|=\varepsilon$. But then, as in [16], the complete continuity of T and N, and the uniform continuity of N in μ imply the existence of a solution of (3.1) of norm ε for $\mu=\mu_0$, contrary to assumption.

Since no solutions of (3.1) lie on $\partial\Omega$ for $|\mu-\mu_0|\leqq\delta$, $d\{I-\Phi_\mu,\Omega\}$ remains constant for $\mu_0-\delta\leqq\mu\leqq\mu_0+\delta$ by property 2. We therefore have

Lemma 4.1. *Let μ_0 be a regular bifurcation point. Then for a sufficiently small neighborhood of the origin in $\mathscr{B}$ and for μ close to μ_0 there are, besides the trivial solution of* (3.1), *either*

(i) *two solutions of index $+1$ for $\mu>\mu_0$ and none for $\mu<\mu_0$*

or

(ii) *one solution of index -1 for $\mu<\mu_0$ and one of index $+1$ for $\mu>\mu_0$*

or

(iii) *two solutions of index -1 for $\mu<\mu_0$ and none for $\mu>\mu_0$.*

Proof. The nature of the bifurcating solutions has already been determined by the Lyapounov-Schmidt method. It remains to compute the indices of the bifurcating solutions in each case.

For example, consider case (i) in which two solutions bifurcate for $\mu>\mu_0$. For $\mu<\mu_0$ zero is the only fixed point of Φ_μ in Ω, so $d\{I-\Phi_\mu,\Omega\}=+1$ by property 3. For $\mu>\mu_0$ we have three solutions, with the index of the trivial solution equal to -1. The other two solutions must have index $+1$ in order for $d\{I-\Phi_\mu,\Omega\}$ to be equal to $+1$. The other two cases are treated similarly.

We can now prove

Theorem 4.2. *Let $(\mu_0, 0)$ be a regular bifurcation point of* (3.1) *and let N be twice continuously Fréchet differentiable, with $N(\mu,\alpha u)=\alpha^2 N_1(\mu;u;\alpha)$ where N_1 is Fréchet differentiable in μ, u and α. Then solutions which bifurcate above criticality are stable and subcritical bifurcating solutions are unstable.*

Proof. The operator $\mu T-N'_u(\mu;\tilde{u})$ has a simple eigenvalue ζ in the vicinity of $\zeta=1$. Since $(\mu_0,0)$ is a regular bifurcation point, we can assume that $\zeta\neq 1$ when $\mu\neq\mu_0$, $|\mu-\mu_0|$ sufficiently small. Property 5 of the degree of mapping theory then shows that $\zeta<1$ if and only if $i[\Phi,\tilde{u}]=+1$.

Note that it is only necessary to consider solutions of index $+1$ for $\mu>\mu_0$ and of index -1 for $\mu<\mu_0$. Accordingly let $\mu>\mu_0$ and let $\tilde{u}$ be a non-trivial solution of (3.1) with index $+1$. Consider the operators

$$D(s,\mu)=L-s\mu_0 B-s[\tau B-F'_u(\mu;\tilde{u})],$$

$$\Phi(s,\mu)=I-s[\mu T-N'_u(\mu,\tilde{u})]$$

where $\tau=\mu-\mu_0$.

Beginning with $\mu=\mu_0$ take s so close to (but less than) 1 that $D(s,\mu_0)$ has an eigenvalue $\lambda_1=\lambda_1(s,\mu_0)$ just to the right of zero and the rest of its spectrum to the right of $\operatorname{Re}\lambda\geqq\sigma>0$. (Recall that $\tilde{u}(\mu_0)=0$ so that $D(s,\mu_0)=L-s\mu_0 B$.) Now let μ increase slightly above μ_0; $\lambda_1(s,\mu)$ varies continuously with μ so it will not cross the origin for small variations in μ. But then the operator $\mu T-N'_u(\mu;\tilde{u})$ has a simple eigenvalue $\zeta<1$ by our remarks above. Therefore s can be increased to 1 without the eigenvalue of $I - s\mu T + sN'_u \mu, \tilde{u}(\mu)$ ever crossing the origin. Since this operator has a zero eigenvalue if and only if $D(s, \mu)$ does, we see that as s is increased to 1, the eigenvalue of $L - s\mu B + sF'_u(\mu, \tilde{\mu})$ can never cross zero. Since it must remain real throughout the homotopy the eigenvalue of $L - \mu B + F'_u(\mu; \tilde{u})$ must be positive.

The above argument holds for μ near μ_0. Now, however, we can continue out along the branch $\alpha = \alpha(\tau)$ to larger values of μ so long as f_α does not become zero. (Of course, at such a point, the bifurcating solution would again resume marginal stability.)

To prove that a solution for $\mu < \mu_0$ with index -1 is unstable, repeat the above argument only this time letting s be slightly greater than 1. Then the eigenvalue in question is negative throughout the homotopy.

5. Examples and Applications

The bifurcation and stability of stationary hydrodynamic flows has been extensively discussed in the literature, so we only indicate here briefly two important examples to which Theorem 4.2 applies.

A. The Bénard Problem. (Convection in a layer of fluid heated from below.) In this case μ is related to the Rayleigh number R, which is directly proportional to the temperature difference across the layer. The simplicity of the critical value has been established in a number of particular cases of interest [10], [16], [18], [19]. The non-linear term in fluid flow problems is $u \cdot \operatorname{grad} u$, a quadratic operator which is therefore analytic. It can be easily established [4], [16] that the pure conduction state (trivial solution) is the only possible stationary flow below criticality. Theorem 4.2 then implies the existence of two stable non-trivial flows (convection flows) above criticality. (See also [14], [20].)

B. The Taylor Problem. (Flow between rotating coaxial cylinders.) The Taylor problem is more difficult because the linearized eigenvalue problem is not self adjoint, as it is in the case of the Bénard problem. The linearized eigenvalue problem has been investigated in [6], [17], [21]. The bifurcation and stability of stationary flows have been analyzed by KIRCHGÄSSNER & SORGER [7]. Using numerical methods, they have shown that the critical eigenvalue is simple for an appropriate choice of wave number α. By numerical calculation they determine that both solutions bifurcate above criticality. Hence these flows are stable by Theorem 4.2 (1), as KIRCHGÄSSNER & SORGER prove using analytic perturbation theory.

C. Finally, let us consider some simple examples illustrating the three types of behavior at criticality. Consider the boundary value problem

$$\begin{aligned} -(\Delta + \mu)u + g(u) &= 0 \quad \text{in } D, \\ u &= 0 \quad \text{on } \partial D \end{aligned} \tag{5.1}$$

where D is a bounded domain in E^n with smooth boundary. We assume $g(0) = g'(0) = 0$. The critical values are simply the eigenvalues of the Laplacian, $0 < \mu_0 < \mu_1, \ldots$; as is well known, μ_0 is simple.

First assume that $ug(u) \geqq 0$. Then no solutions are possible for $\mu < \mu_0$. In fact, multiplying (5.1) by u and integrating over D, we get

$$\int_D [(\nabla u)^2 - \mu u^2]\, dx \leqq 0;$$

this is impossible if $\mu < \mu_0$, since μ_0 is characterized by the variational problem

$$\mu_0 = \inf \frac{\int (\nabla u)^2\, dx}{\int u^2\, dx}.$$

If g is such that μ_0 is a regular bifurcation point—for example, if g is analytic—then two stable solutions must bifurcate above $\mu = \mu_0$. Taking $g(u) = u^3$, we can check this result easily by analytic perturbation theory.

Similarly, taking $g = -u^3$, one can show by Schmidt-Lyapounov and perturbation methods that two unstable solutions exist just below criticality and tend to zero as $\mu \to \mu_0$.

Finally, taking $g = -u^2$, one obtains case (ii): One unstable solution below criticality and one stable one above criticality. If one specializes to one space dimension, *viz.*

$$u'' = \mu u - u^2, \qquad u(0) = u(l) = 0, \tag{5.2}$$

the problem can be solved exactly by quadratures. Then one sees that for $\mu < \mu_0$ there exists a convex, negative (unstable) solution which tends uniformly to zero as $\mu \to \mu_0^-$. For $\mu > \mu_0$ the non-trivial solution is concave, positive stable, and increases with μ.

Supercriticality. In general, supercritical values of μ exist; that is, if μ is increased beyond μ_0 there are larger values of μ for which an eigenvalue of $L - \mu B$ crosses through the origin.

Solutions may bifurcate at supercritical values of the parameter, but they may be unstable. This is the case in (5.1) above. If a solution $\tilde{u}$ bifurcates at $\mu = \mu_n$, then for μ close to μ_n, $\tilde{u}$ is small in norm; hence the spectrum of the derivative operator $-(\Delta + \mu) + g'(\tilde{u})$ is close to that of $-(\Delta + \mu)$. But the latter operator has negative eigenvalues at $\mu_1 - \mu$, $\mu_2 - \mu, \ldots, \mu_{n-1} - \mu$. This phenomenon may be typical of many physical problems.

Finally, let us close with a simple example which illustrates that the conclusion of Theorem 4.2 is false in the case of a double eigenvalue. Consider the equations

$$(1 - \mu) u + v^3 = 0,$$

$$(1 - \mu) v + u^3 = 0.$$

For $\mu = 1$ there are no real solutions except the null solution. For $\mu > 1$ we get two unstable solutions with index -1, as the reader may check for himself. Thus in the case of a double eigenvalue solutions bifurcating above criticality need not be stable, and indeed it may be, as here, that no solutions bifurcating above criticality are stable.

Addendum

The results in this paper were obtained in the summer of 1969. Since then, the book of G. R. GAVALAS, "Nonlinear Differential Equations of Chemically Reacting Systems", Springer, New York, 1968, has come to my attention. This book contains an extensive discussion of the relationship between degree theory and stability, and the results complement those in the present paper. In particular, there is a discussion of degree and stability theory for ordinary differential equations. The relationship between index and stability is much clearer in the finite dimensional case, and the interested reader is advised to read GAVALAS' treatment (*cf.* Theorem 1.7.1, p. 25).

GAVALAS assumes the Jacobian of the non-linear mapping is invertible at the equilibrium solution and shows that index -1 implies instability—a result more general than ours. On the other hand, index $+1$ does not imply stability in general. (See his discussion on p. 27.) By considering the stability question in the vicinity of a bifurcation at a simple eigenvalue, we have been able to prove that the relevant Jacobian (in the infinite dimensional case this role is played by the Fréchet differential) is invertible along a bifurcating curve. This is the content of Lemmas 3.2 and 3.3. We also see that the stability is completely determined by the index in the vicinity of a bifurcation at a simple eigenvalue. GAVALAS also considers the infinite dimensional case (Theorem 2.9.2, p. 91) and shows that index -1 implies instability of the corresponding stationary solution for a particular system of partial differential equations.

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