Overlapping Batches for the Assessment of Solution Quality in Stochastic Programs

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Problem Description

$$z^* = \min_{\mathbf{x} \in \mathcal{X}} \mathbb{E} f(\mathbf{x}, \tilde{\xi}) \qquad \text{(SP)}$$

$$\mathbf{x}^* \in \operatorname*{argmin}_{\mathbf{x} \in \mathcal{X}} \mathbb{E} f(\mathbf{x}, \tilde{\xi})$$

$\mathbf{x}$ Vector of decision variables.

$\tilde{\xi}$ Vector of problem parameters.

f Objective function, may depend on $\tilde{\xi}$.

- Lower semicontinuous on x a.s.
- $\mathbb{E} f(\mathbf{x}, \tilde{\xi})$ defined and finite for all x.

$\mathcal{X}$ Feasible region.

- Nonempty, closed, compact, independent of $\tilde{\xi}$.
- e.g., $\mathcal{X}(\tilde{\xi}) = \left\{ \mathbf{x} \in \mathbb{R}^n : g_i(x, \tilde{\xi}) \leq 0, i = 1, \ldots, m, \mathbf{x} \geq 0 \right\}$.

Project Goal

- We will combine two previously published results
  - One from stochastic programming
  - One from simulation
- To develop a new variation on answering the question:

Given a candidate solution, $\hat{\mathbf{x}}$, to the stochastic problem, how can we estimate how good it is, compared to the optimal solution?

Defining a “Good” Candidate Solution

Suppose we have a candidate solution $\hat{\mathbf{x}}$ (maybe we solved (SP_n)).
How good is it?

- We can look at its “optimality gap”,

$$\mathbb{E}f(\hat{\mathbf{x}}, \tilde{\xi}) - \mathbb{E}f(\mathbf{x}^*, \tilde{\xi}) = \mathbb{E}f(\hat{\mathbf{x}}, \tilde{\xi}) - z^*.$$

- Clearly is positive.
- Want a statistical estimate on size.

Approximating Problem

Following Mak, Morton and Wood (1999), we set up the approximating problem

$$z_n^* = \min_{\mathbf{x} \in \mathcal{X}} \frac{1}{n} \sum_{i=1}^{n} f(\mathbf{x}, \tilde{\xi}^i) \qquad (\mathrm{SP}_n)$$

$$\mathbf{x}_n^* \in \operatorname*{argmin}_{\mathbf{x} \in \mathcal{X}} \frac{1}{n} \sum_{i=1}^{n} f(\mathbf{x}, \tilde{\xi}^i)$$

where

$\tilde{\xi}^i$,	i.i.d. samples from the distribution of $\tilde{\xi}$, $i = 1, \ldots, n$
z_n^*,	optimal objective value of the approximating problem
$\mathbf{x}_n^*$,	optimal solution of the approximating problem

Properties of (SP_n)

Properties of Approximating Problem

1. Sample Approximating Problem is biased

$$\mathbb{E}z_n^* = \mathbb{E}\min_{\mathbf{x}\in\mathcal{X}} \frac{1}{n}\sum_{i=1}^{n} f(\mathbf{x}, \tilde{\xi}^i) \leq \mathbb{E}\min_{\mathbf{x}\in\mathcal{X}} f(\mathbf{x}, \tilde{\xi}) = \mathbb{E}z^*$$

2. Bias is uniformly nonincreasing with increasing sample size

$$\mathbb{E}z_n^* \leq \mathbb{E}z_{n+1}^* \leq \mathbb{E}z^*$$

Optimality Gap Estimation

The optimality gap $\mathbb{E}f(\hat{\mathbf{x}}, \tilde{\xi}) - \min_{\mathbf{x}\in\mathcal{X}} \mathbb{E}f(\mathbf{x}, \tilde{\xi})$ is approximated by

$$\frac{1}{M}\sum_{i=1}^{M} f(\hat{\mathbf{x}}, \tilde{\xi}^i) - \min_{\mathbf{x}\in X} \frac{1}{M}\sum_{i=1}^{M} f(\mathbf{x}, \tilde{\xi}^i)$$

- Since the approximated optimal solution is negatively biased, the gap estimator is positively biased.
- Bias monotonically nonincreasing as batch size M increases.

Common Random Numbers

- Note the use of common random numbers.
  - Decreases variance, as usual.
  - Also ensures that gap estimate is always positive.

$$\frac{1}{M}\sum_{i=1}^{M} f(\hat{\mathbf{x}},\tilde{\xi}^i) - \min_{\mathbf{x}\in X}\frac{1}{M}\sum_{i=1}^{M} f(\mathbf{x},\tilde{\xi}^i)$$

Multiple Replications Procedure

$$N = 12 \quad \text{Total sample size}$$
$$M = 4 \quad \text{Batch size}$$
$$k = \frac{N}{M} = 3 \quad \text{Number of batches}$$

$\tilde{\xi}^1$ $\tilde{\xi}^2$ $\tilde{\xi}^3$ $\tilde{\xi}^4$ $\tilde{\xi}^5$ $\tilde{\xi}^6$ $\tilde{\xi}^7$ $\tilde{\xi}^8$ $\tilde{\xi}^9$ $\tilde{\xi}^{10}$ $\tilde{\xi}^{11}$ $\tilde{\xi}^{12}$

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$[\tilde{\xi}^1, \ldots, \tilde{\xi}^4] \rightarrow \bar{G}_1$

$$\bar{G}_1 = \frac{1}{M}\sum_{i=1}^{M} f(\hat{\mathbf{x}}, \tilde{\xi}^{M(1-1)+i}) - \min_{\mathbf{x}\in\mathcal{X}} \frac{1}{M}\sum_{i=1}^{M} f(\mathbf{x}, \tilde{\xi}^{M(1-1)+i})$$

Multiple Replications Procedure

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[———— $\bar{G}_1$ ————] [———— $\bar{G}_2$ ————]

$$\bar{G}_2 = \frac{1}{M}\sum_{i=1}^{M} f(\hat{\mathbf{x}}, \tilde{\xi}^{M(2-1)+i}) - \min_{\mathbf{x}\in\mathcal{X}} \frac{1}{M}\sum_{i=1}^{M} f(\mathbf{x}, \tilde{\xi}^{M(2-1)+i})$$

Multiple Replications Procedure

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$\tilde{\xi}^9 \quad \tilde{\xi}^{10} \quad \tilde{\xi}^{11} \quad \tilde{\xi}^{12}$ — $\bar{G}_3$

$$\bar{G}_3 = \frac{1}{M}\sum_{i=1}^{M} f(\hat{\mathbf{x}}, \tilde{\xi}^{M(3-1)+i}) - \min_{\mathbf{x}\in\mathcal{X}} \frac{1}{M}\sum_{i=1}^{M} f(\mathbf{x}, \tilde{\xi}^{M(3-1)+i})$$

Multiple Replications Procedure

$$\begin{aligned} N &= 12 && \text{Total sample size} \\ M &= 4 && \text{Batch size} \\ k = \tfrac{N}{M} &= 3 && \text{Number of batches} \end{aligned}$$

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$$\bar{G}_j = \frac{1}{M} \sum_{i=1}^{M} f(\hat{\mathbf{x}}, \tilde{\xi}^{M(j-1)+i}) - \min_{\mathbf{x} \in \mathcal{X}} \frac{1}{M} \sum_{i=1}^{M} f(\mathbf{x}, \tilde{\xi}^{M(j-1)+i})$$

$$VG = \frac{1}{(k-1)k} \sum_{j=1}^{k} (\bar{G}_j - \bar{\bar{G}})^2 \qquad \bar{\bar{G}} = \frac{1}{k} \sum_{j=1}^{k} \bar{G}_j$$

Batch Means

- This situation is much like that of simulation:
  - Partition simulation output into batches.
  - Do statistics on the batches.
- Meketon & Schmeiser (1984) proposed that batches could be overlapped.

Overlapping Batches

$$\begin{aligned} N = 12 &\quad \text{Total sample size} \\ M = 4 &\quad \text{Batch size} \\ k = \left\lfloor \frac{N}{M} \right\rfloor = 3 &\quad \text{Number of nonoverlapping batches} \end{aligned}$$

$\tilde{\xi}^1 \quad \tilde{\xi}^2 \quad \tilde{\xi}^3 \quad \tilde{\xi}^4 \quad \tilde{\xi}^5 \quad \tilde{\xi}^6 \quad \tilde{\xi}^7 \quad \tilde{\xi}^8 \quad \tilde{\xi}^9 \quad \tilde{\xi}^{10} \quad \tilde{\xi}^{11} \quad \tilde{\xi}^{12}$

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$$[\text{———}\bar{G}_1\text{———}]$$

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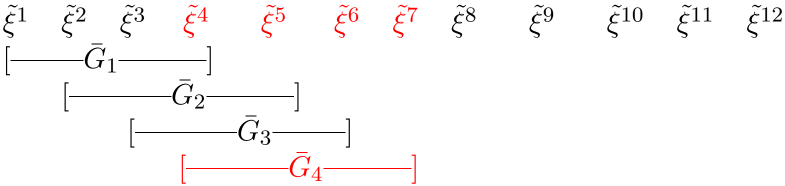

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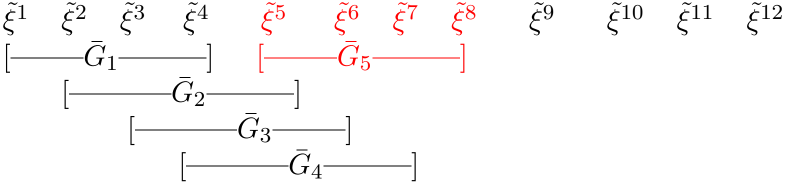

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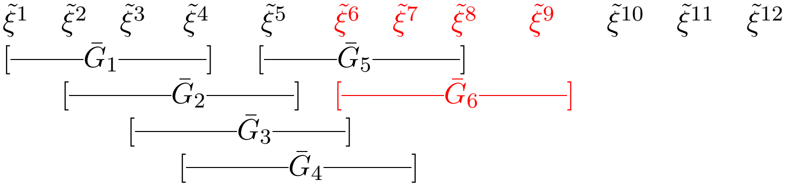

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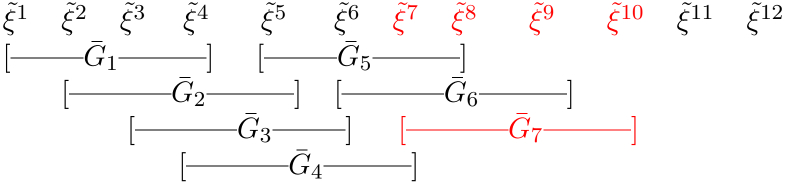

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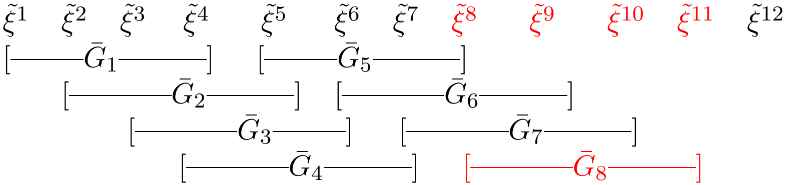

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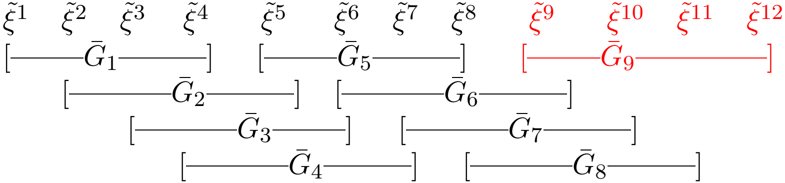

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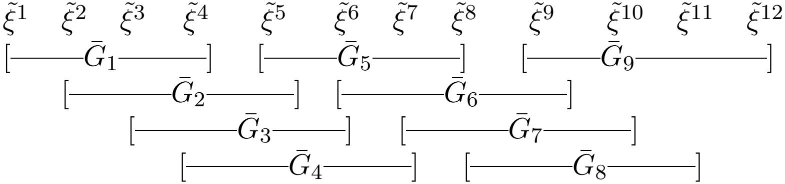
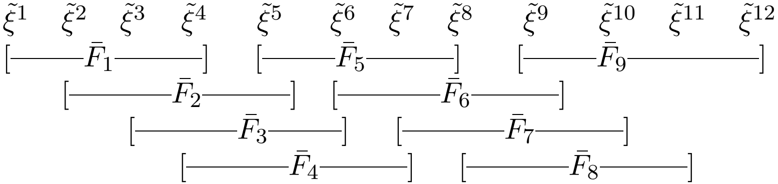


$$VG = \frac{1}{(k-1)(N-M+1)} \sum_{j=1}^{N-M+1} (\bar{G}_j - \bar{\bar{G}})^2$$

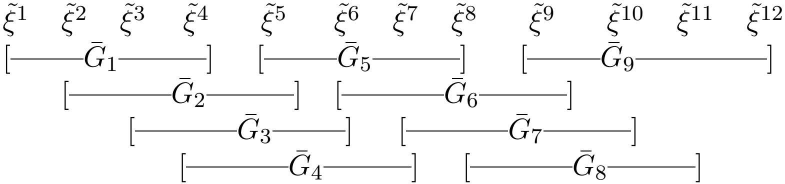
What About the Grand Average?



- A small excursion to simulation, and suppose we wish to estimate $\mathbb{E}f(\hat{\mathbf{x}}, \tilde{\xi})$ (rather than $\mathbb{E}f(\hat{\mathbf{x}}, \tilde{\xi}) - z^*$).
- Meketon and Schmeiser simply average all observations:

$$\bar{F}_j = \frac{1}{M}\sum_{i=1}^{M} f(\hat{\mathbf{x}}, \tilde{\xi}^{M(j-1)+i}) \qquad \bar{\bar{F}} = \frac{1}{N}\sum_{i=1}^{N} f(\hat{\mathbf{x}}, \tilde{\xi}^{i})$$

Extending to Stochastic Programming



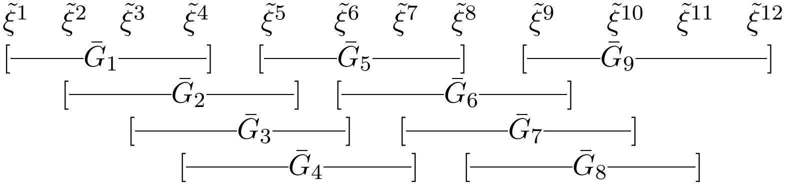
- Our batch average

$$\bar{G}_j = \frac{1}{M} \sum_{i=1}^{M} \left[f(\hat{\mathbf{x}}, \tilde{\xi}^{M(j-1)+i}) - f(\mathbf{x}^*_{M,j}, \tilde{\xi}^{M(j-1)+i}) \right]$$

- But what would the grand average be?

$$\bar{\bar{G}} = \frac{1}{N} \sum_{i=1}^{N} \left[f(\hat{\mathbf{x}}, \tilde{\xi}^{i}) - f(??, \tilde{\xi}^{i}) \right]$$

A Small Problem



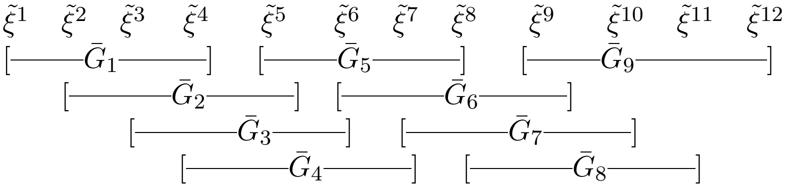
Problem The internal minimization of

$$\bar{G}_j = \frac{1}{M}\sum_{i=1}^{M} f(\hat{\mathbf{x}}, \tilde{\xi}^{M(j-1)+i}) - \min_{\mathbf{x}\in\mathcal{X}} \frac{1}{M}\sum_{i=1}^{M} f(\mathbf{x}, \tilde{\xi}^{M(j-1)+i})$$

means the batches have individual properties, making it difficult to take the grand average.

Solution We must average in two directions.

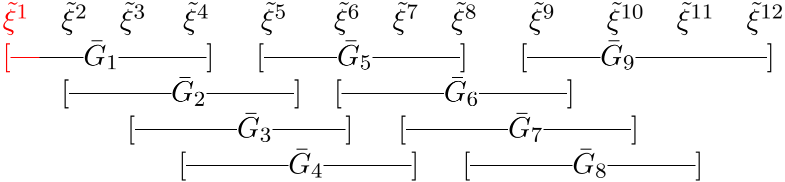
A Small Problem



$$\bar{\bar{G}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{|N(i)|} \sum_{j \in N(i)} \left[f(\hat{\mathbf{x}}, \tilde{\xi}^i) - f(\mathbf{x}^*_{M,j}, \tilde{\xi}^i) \right]$$

1. For each sample $\tilde{\xi}^i$, average the gap over batches in which $\tilde{\xi}^i$ appears.
2. Then average over the values of $\tilde{\xi}^i$.

A Small Problem



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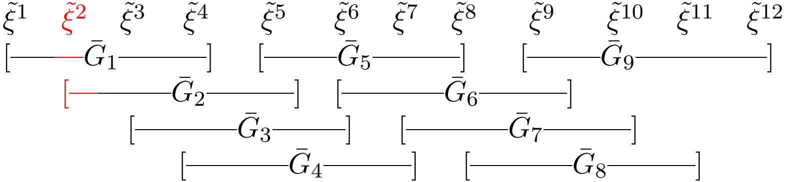

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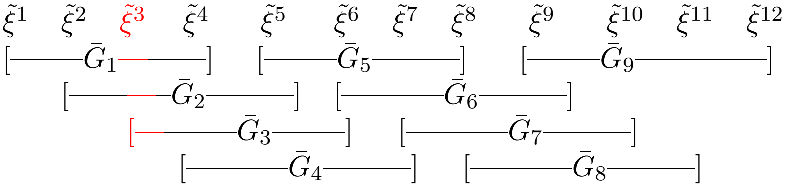

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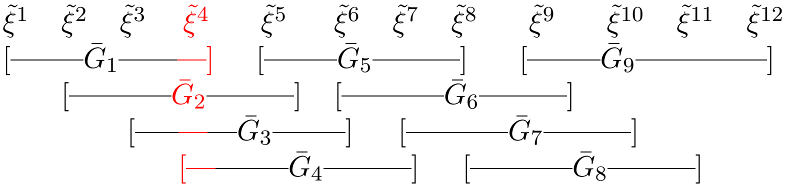

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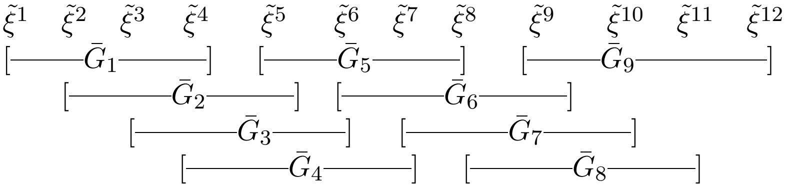

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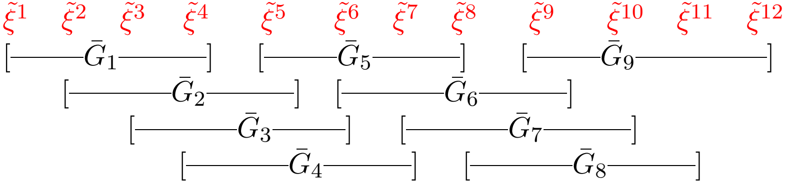

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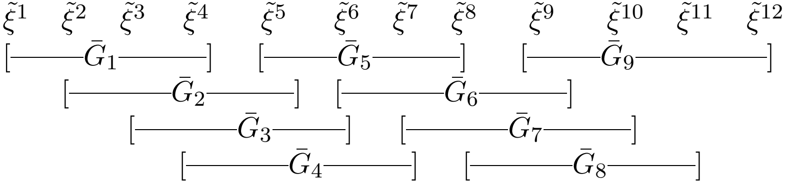

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Estimators of OMRP

Batch Mean $\bar{G}_j = \frac{1}{M} \sum_{i=1}^{M} \left[f(\hat{\mathbf{x}}, \tilde{\xi}^{M(j-1)+i}) - f(\mathbf{x}^*_{M,j}, \tilde{\xi}^{M(j-1)+i}) \right]$

Overall Mean $\bar{\bar{G}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{|N(i)|} \sum_{j \in N(i)} \left[f(\hat{\mathbf{x}}, \tilde{\xi}^{i}) - f(\mathbf{x}^*_{M,j}, \tilde{\xi}^{i}) \right]$

Nonoverlapping Variance $VG(\text{NOL}) = \frac{1}{(k-1)k} \sum_{j=1}^{k} (\bar{G}_j - \bar{\bar{G}})^2$

Overlapping Variance $VG(\text{OL}) = \frac{1}{(k-1)(N-M+1)} \sum_{j=1}^{N-M+1} (\bar{G}_j - \bar{\bar{G}})^2$

Assumptions

A1 Samples of the random variable $\tilde{\xi}$ are i.i.d.

A2 $z_n^* \to z^*$ a.s.

A3 $\exists \epsilon > 0$ such that $\mathbb{E}\left[(\sup_{x \in X} f(x, \tilde{\xi}))^{4+\epsilon}\right] < \infty$.

Important (Computational) Result

- Bias in the variance estimator is essentially unaffected by the decision to overlap.
- Both overlapping and nonoverlapping estimators are unbiased in the limit as $N, M, k \to \infty$.
- Overlapping batches lowers the variance of $VG(\text{OL})$, the variance estimator of the optimality gap.

$$\lim_{N,M,k\to\infty} \frac{\text{Var}\left(VG(\text{OL})\right)}{\text{Var}\left(VG(\text{NOL})\right)} = \frac{2}{3}$$

Another Small Problem

Problem The internal minimization of

$$\bar{G}_j = \frac{1}{M}\sum_{i=1}^{M} f(\hat{\mathbf{x}}, \tilde{\xi}^{M(j-1)+i}) - \min_{\mathbf{x}\in\mathcal{X}} \frac{1}{M}\sum_{i=1}^{M} f(\mathbf{x}, \tilde{\xi}^{M(j-1)+i})$$

means we do not have the nice properties we want.

Solution Introduce a sequence of “non-minimized” gap estimators, $\bar{D}_j$. Show convergence between these two series.

Estimators for Proof

Unbiased Gap Estimator

- $\bar{D}_j = \frac{1}{M} \sum_{i=1}^{M} \left[f(\hat{\mathbf{x}}, \tilde{\xi}^{M(j-1)+i}) - f(\mathbf{x}^*, \tilde{\xi}^{M(j-1)+i}) \right]$
- $\bar{\bar{D}}_j = \frac{1}{N} \sum_{i=1}^{N} \left[f(\hat{\mathbf{x}}, \tilde{\xi}^{i}) - f(\mathbf{x}^*, \tilde{\xi}^{i}) \right]$
- $VD_j(\text{OL}) = \frac{1}{(k-1)(N-M+1)} \sum_{j=1}^{N-M+1} (\bar{D}_j - \bar{\bar{D}})^2$

Proposed Gap Estimator

- $\bar{G}_j = \frac{1}{M} \sum_{i=1}^{M} \left[f(\hat{\mathbf{x}}, \tilde{\xi}^{M(j-1)+i}) - f(\mathbf{x}^*_{M,j}, \tilde{\xi}^{M(j-1)+i}) \right]$
- $\bar{\bar{G}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{|N(i)|} \sum_{j \in N(i)} \left[f(\hat{\mathbf{x}}, \tilde{\xi}^{i}) - f(\mathbf{x}^*_{M,j}, \tilde{\xi}^{i}) \right]$
- $VG(\text{OL}) = \frac{1}{(k-1)(N-M+1)} \sum_{j=1}^{N-M+1} (\bar{G}_j - \bar{\bar{G}})^2$

Partial Overlap

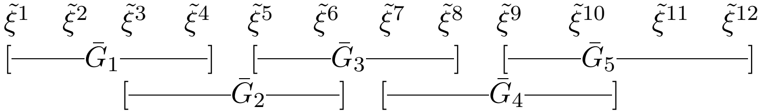


Figure: Graphical representation of $\gamma = 2$.

Let γ be the batch nonoverlap parameter.

- γ gives the number of new samples added to each new batch.
- $\gamma = 1$, maximally overlapping case discussed above.
- $\gamma = M$, classical nonoverlapping case.
- More useful: $\dfrac{\gamma}{M} \in (0, 1]$.

Result

Variance reduction of Sample Variance (P. D. Welch 1987)

- $\gamma \equiv 1$, variance is reduced to $\frac{2}{3}$ of original.
- $\gamma = \frac{1}{3}M$, variance is reduced to $\frac{19}{27}$ of original.
- $\gamma = \frac{1}{2}M$, variance is reduced to $\frac{3}{4}$ of original.
- $\gamma = \frac{1}{L}M$, variance is reduced to $\frac{2+\frac{1}{L^2}}{3}$ of original.

Newsvendor Problem

- You purchase newspapers from a supplier at cost c, and sell them to the public at price r.
- Number of customers $\tilde{d} \geq 0$ is a random variable.
- Purchase x papers every day.
- Maximize expected profit $\mathbb{E}\left[-cx + r\min\left\{x, \tilde{d}\right\}\right]$.

Optimal solution occurs when $\mathbb{P}\left[\tilde{d} \leq x\right] = \frac{r-c}{r}$.

Newsvendor Problem

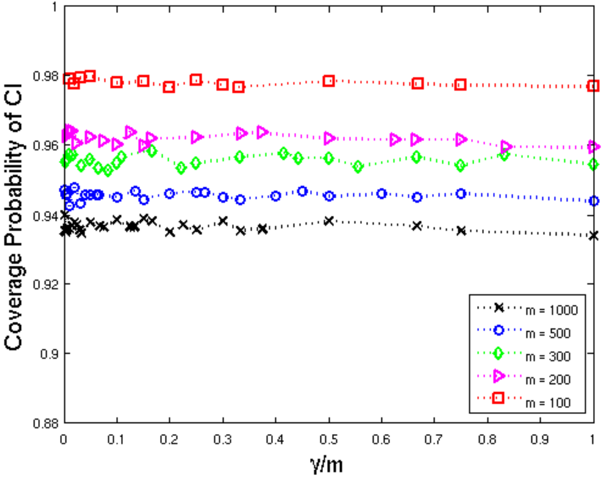
We tested the computational properties of our method on the newsvendor problem, with

- Selling price $r = 15$.
- Cost of paper $c = 5$.
- Optimal quantile $\frac{r-c}{r} = \frac{2}{3}$.
- Daily demand of newspapers is $\tilde{d} \sim U(0, 10)$.
- $\hat{\mathbf{x}} = 8.7749$.

Computational Method

- Take a variety of batch sizes M.
- For each, choose sample size $N = 30M$.
- Compute confidence intervals for γ/M
    - Near 0, for fully overlapping
    - $\gamma/M = 1$ for nonoverlapping
- Repeat each computation $10,000$ times for estimates.
- Construct a 90% confidence interval.

Coverage Probability



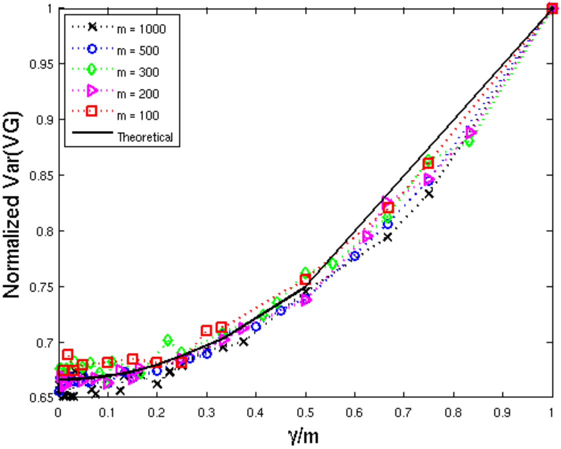
Variance Reduction

- The main result:

$$\lim_{N,M,k\to\infty} \frac{\mathrm{Var}\left(VG(\frac{\gamma}{M})\right)}{\mathrm{Var}\left(VG(\mathrm{NOL})\right)}$$

is reduced by overlapping.

Variance Reduction of Variance Estimator



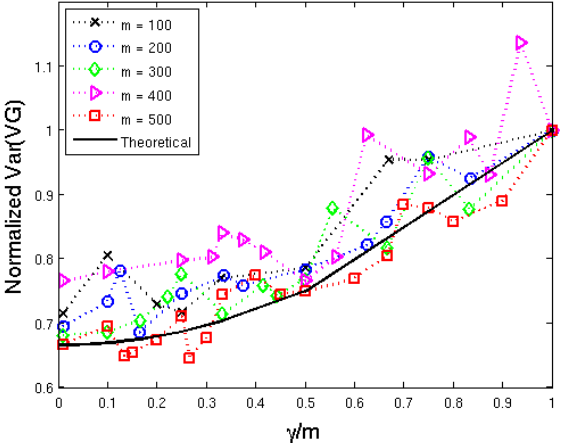
SLP-2 Test Problems

- Also tested OMRP on several Two-Stage Stochastic Linear Programs with Recourse.
- Solved with the regularized decomposition algorithm of Ruszczyński and Świetanowski.
- Results for APL1P and CEP1 presented here.

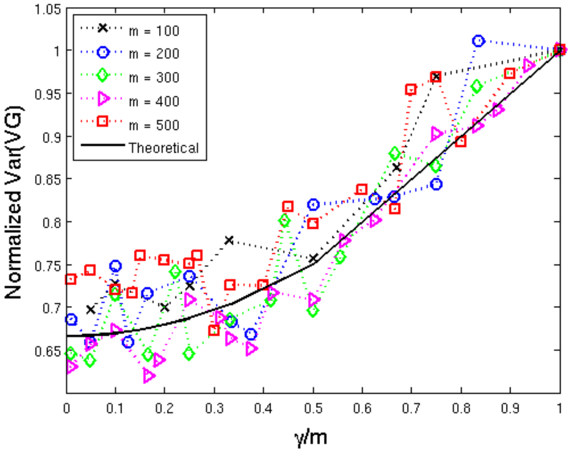
  APL1P Power expansion planning problem with 5 independent stochastic parameters and 1280 total realizations.

  CEP1 Capacity expansion problem, 3 independent stochastic parameters and 216 total realizations.
- Similar results on confidence interval coverage probability.

Variance Reduction in APL1P



Variance Reduction in CEP1



Conclusions

- Began with old results:
  - Variance reduction by overlapping batches.
  - Solution quality assessment by MRP.
- I have shown that these results can be combined in stochastic programming.
- Computational results show small-sample behavior is in close agreement with asymptotic results.

References

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Questions?