A Data-driven Method for Robust Water Allocation under Uncertainty

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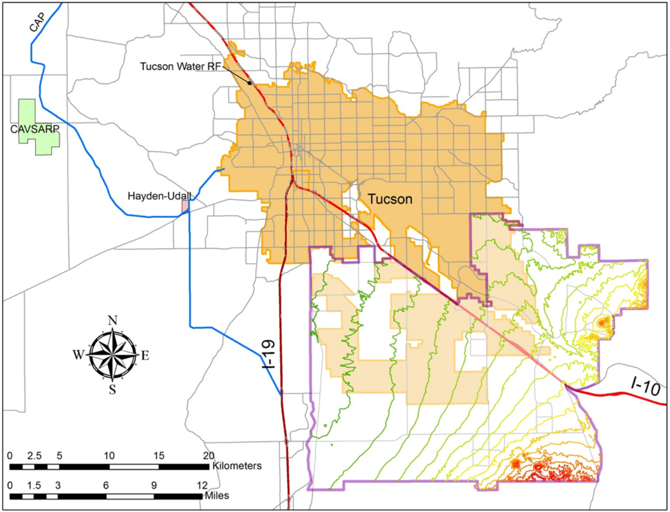
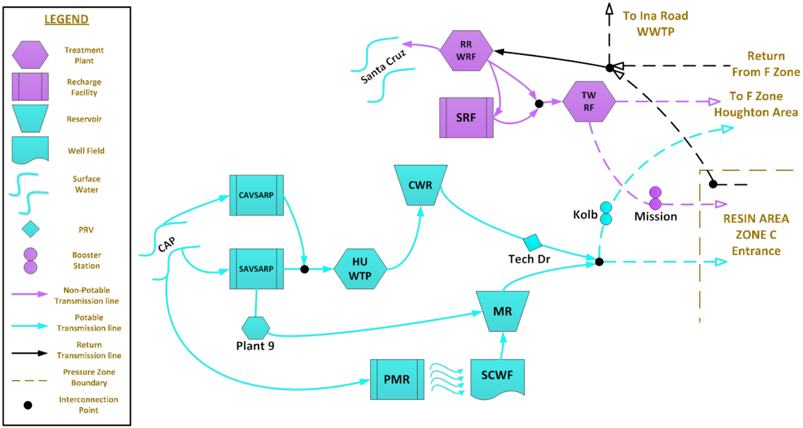
Tucson Topographical Map



Flow Model Description



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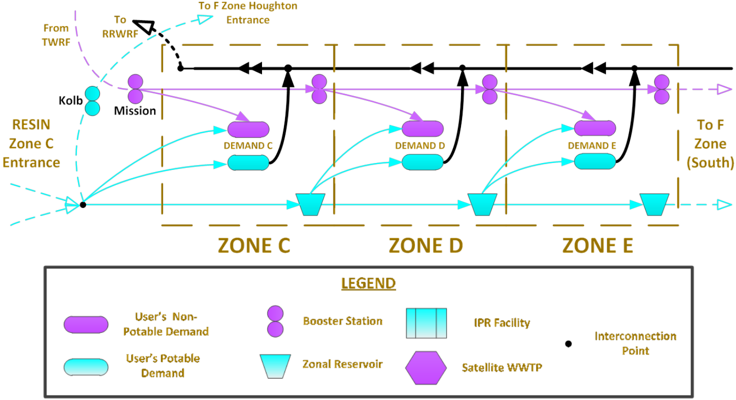

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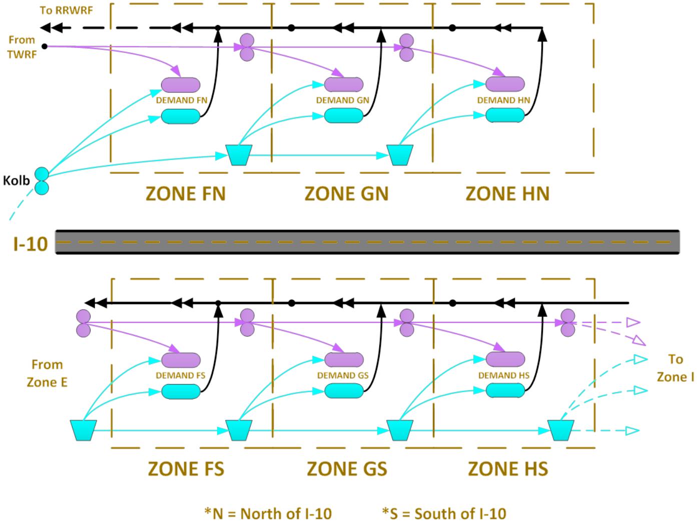
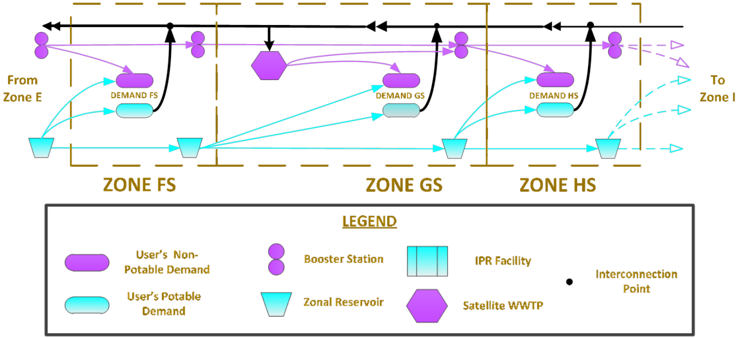


Decentralized Treatment Possibilities



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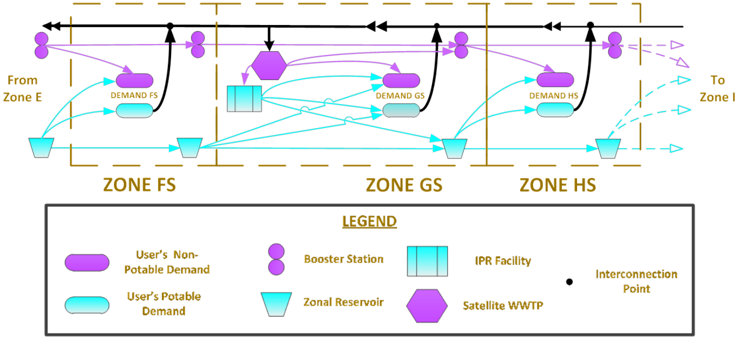

Other Model Details

- Losses identical on all arcs, percentage chosen for 10% loss from system.
- Models years 2010–2050.
- Models energy and treatment costs.
- Future costs given 4% discount rate.
- Generalized Network Model:

$$\min_{x,s} \sum_t \sum_{i,j} c_{ijt} x_{ijt} + \sum_t \sum_i c'_{it} s_{it}$$

$$\text{s.t. } s_{it} + (1-L) \sum_{j:(j,i)\in A} x_{jit} - \sum_{j:(i,j)\in A} x_{ijt} - s_{i,t+1} = d_{it} \ \forall i, t$$

$$l_{ijt} \leq x_{ijt} \leq u_{ijt}, \ l'_{it} \leq s_{it} \leq u'_{it} \ \forall ijt$$

Model Scenarios

Scenario	Population	Supply	Decentralized	IPR
1	High	High	No	No
2	High	Low	No	No
3	Low	High	No	No
4	Low	Low	No	No
5	High	High	Yes	No
6	High	Low	Yes	No
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8	Low	Low	Yes	No
9	High	High	Yes	Yes
10	High	Low	Yes	Yes
11	Low	High	Yes	Yes
12	Low	Low	Yes	Yes

	Population		Supply
Estimate	2010	2050	(CAP/yr)
Low	33,304	371,804	130,000
High	40,705	762,427	140,000

Ambiguous Distributions

- Distributions of supply and population are unknown.
- Leave distribution unknown, use worst-case from “ambiguity” set.
- “Likelihood Robust Optimization” by Zizhuo Wang, Peter W. Glynn and Yinyu Ye.
  - Working paper “Likelihood Robust Optimization for Data-Driven Newsvendor,” 2010.

Empirical Likelihood Functions

- Model uncertainty by a multinomial distribution:
  - Scenarios, enumerated by $i = 1, \ldots, n$.
  - Probabilities $p_1, \ldots, p_n$, $\sum_{i=1}^n p_i = 1, p_i \geq 0$.
  - Each scenario observed $N_i \geq 0$ times, total $N = \sum_{i=1}^n N_i$.
  - Empirical likelihood function $L = \prod_{i=1}^n p_i^{N_i}$
- Log-likelihood

$$\log L = \sum_{i=1}^{n} N_i \log p_i$$

- Maximum likelihood estimator
  - $p_i = \dfrac{N_i}{N} \ \forall i$

Likelihood Ambiguity Set

- Likelihood parameter γ

$$\log L = \sum_{i=1}^{n} N_i \log p_i \geq \gamma$$

$$L = \prod_{i=1}^{n} p_i^{N_i} \geq e^{\gamma}$$

- Describe with $0 < \gamma' < 1$, proportion of maximum likelihood.

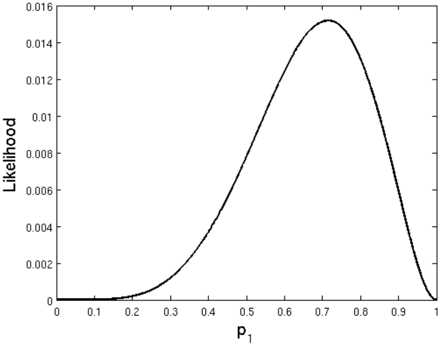

Figure : Empirical likelihood of two scenarios with observations 5,2

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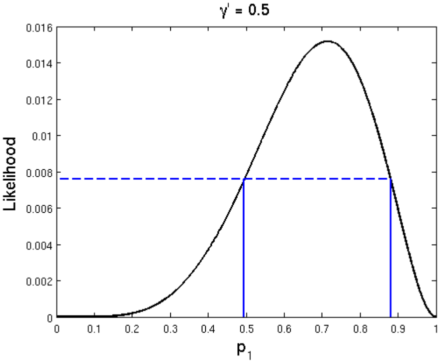

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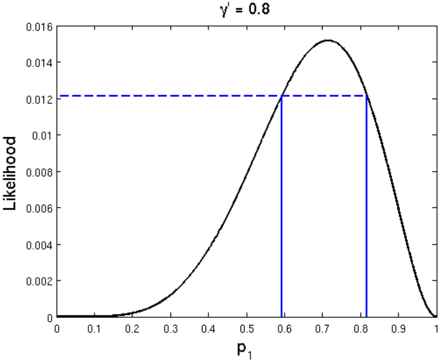

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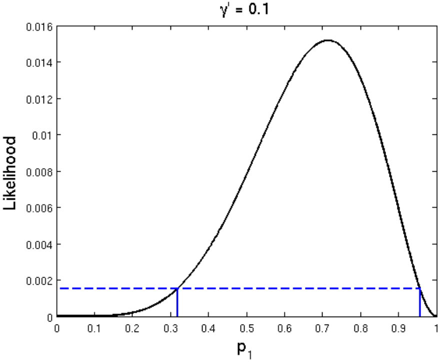

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Likelihood-Robust Optimization

Two-stage generalized network water model:

$$\begin{aligned} \min_{x \in X} \; cx + \; \max_{p} \; & \sum_{i=1}^{n} p_i h_i(x) & \\ \text{s.t.} \; & \sum_{i=1}^{n} N_i \log p_i \geq \gamma & (\lambda) \\ & \sum_{i=1}^{n} p_i = 1 & (\mu) \\ & p_i \geq 0 & \end{aligned}$$

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- Expected value with worst-case distribution.
- Likelihood ambiguity set.
- Probability distribution.

Full LRO Problem

Take dual of maximization, combine minimization terms:

$$\begin{aligned} \min_{x,\lambda,\mu}\ & cx + \mu + \bar{N}\lambda + N\lambda\log\lambda - \sum_{i=1}^{n} \lambda N_i \log\left(\mu - h_i(x)\right) \\ \text{s.t. }\ & \lambda \geq 0 \\ & \mu \geq h_i(x)\ \forall i \\ & x \in X \end{aligned}$$

- Where $\bar{N} = N(\log N - 1) - \log\gamma'$.
- Worst-case distribution:

$$p_i = \frac{\lambda N_i}{\mu - h_i(x)}$$

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Some Features

1. LRO is convex.
2. Easy to convert derivatives of $h_i(x)$ to derivatives of LRO.
3. LRO formulation keeps same time-structure as the original problem.
4. LRO can be written as a coherent risk measure.

Value of Additional Data

- Want to study the (potential) value of an extra observation.
  - E.g., another simulation of population growth, or more weather data.
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  - If we know which observations will result in worst-case cost decrease, what is the probability of selecting one of those scenarios?
- Could run another optimization, but computationally expensive.
- Want an easy to calculate test.

Estimating the Value of Data

- $f_N(\mu, \lambda) = \mu + \bar{N}\lambda + N\lambda \log \lambda - \sum_i \lambda N_i \log(\mu - h_i)$.
- Let μ^*, λ^* solve $z_N = \min_{\mu,\lambda} f_N(\mu, \lambda)$.

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    - $f_{N+1}(\mu^*,\lambda^*) \geq \min_{\mu,\lambda} f_{N+1}(\mu,\lambda) = z_{N+1}$
- Check for $0 < z_N - f_{N+1}(\mu^*,\lambda^*) \leq z_N - z_{N+1}$.

Estimating Continued

- An observation of scenario k will decrease the worst-case cost if

Worst-Case Cost Decrease Condition

$$\frac{N_k}{N} \geq \left(\frac{N+1}{N}\right) p_k$$

- With worst-case $p_k = \frac{\lambda^* N_{i^*}}{\mu^* - h_k}$:
- Relation between maximum-likelihood and worst-case distributions!

Probability of Worst-Case Cost Decrease

- Let $D = \left\{ i : \frac{N_i}{N} \geq \left(\frac{N+1}{N}\right) p_i \right\}$
    - $N_D = \sum_{i \in D} N_i$
    - $N_{D^C} = N - N_D$
- Solve

$$\begin{aligned} \min_p \quad & \sum_{i \in D} p_i \\ \text{s.t.} \quad & \sum_{i=1}^{n} N_i \log p_i \geq \gamma, \sum_{i=1}^{n} p_i = 1, p_i \geq 0 \end{aligned}$$

Probability Continued

Dual form

$$-\min_{\lambda,\mu\geq 0} \mu + \bar{N}\lambda + N\lambda\log\lambda - \lambda\left(N_D\log(\mu+1) + N_{DC}\log\mu\right)$$

$$\bar{N} = N(\log N - 1) - \log\gamma'$$

- Investigate as a function of parameter γ'

Value of Data in a Problem

1. Create the problem:
    1. A set of scenario costs.
    2. The (true) probability of each scenario.
2. Sample from the distribution.
3. For each value of γ':
    1. Solve the LRO. Get N_D.
    2. Find the probability of cost decrease.
4. Results:

Value of Data Computations

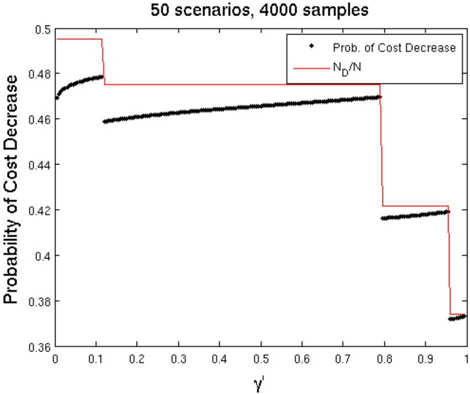


Figure : For a given (randomly generated) problem, computed the probability of solution decrease against normalized likelihood parameter γ'.

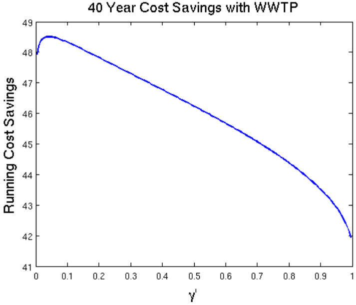
Scenarios Revisited

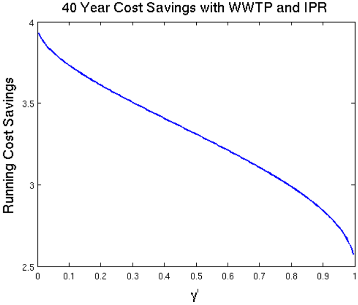
Scenario	Population	Supply	Decentralized	IPR	Cost (M$)
1	High	High	No	No	435
2	High	Low	No	No	437
3	Low	High	No	No	225
4	Low	Low	No	No	220
5	High	High	Yes	No	391
6	High	Low	Yes	No	380
7	Low	High	Yes	No	193
8	Low	Low	Yes	No	187
9	High	High	Yes	Yes	387
10	High	Low	Yes	Yes	376
11	Low	High	Yes	Yes	192
12	Low	Low	Yes	Yes	186

Programming setup

3 blocks of 4 scenarios, one observation per scenario.

Cost Savings of Facilities





Acknowledgments

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