Laser Cavity Simulation

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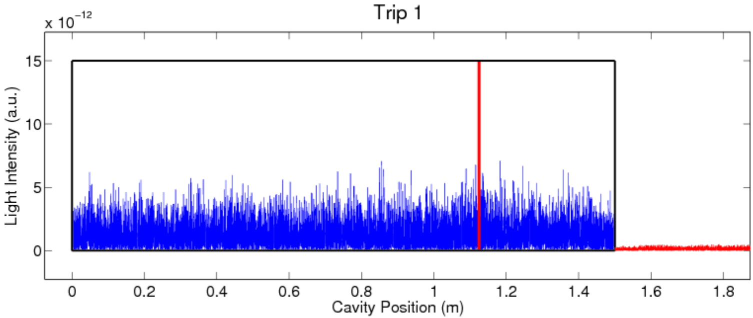
Talk Outline

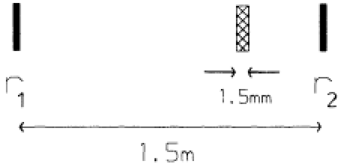
1. What is it (physically)?
2. What is it (mathematically)?
3. How is it done?
4. Results

Progress

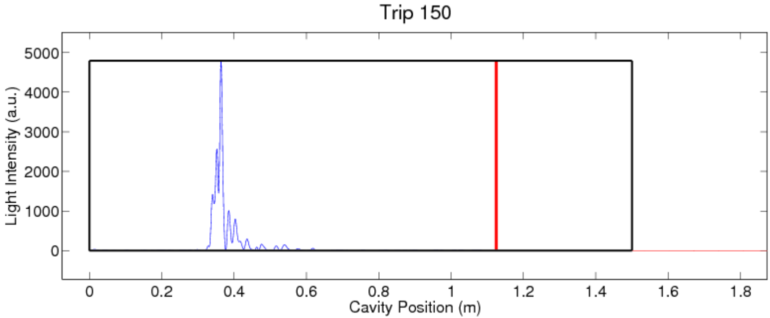
1. What is it (physically)?
   - An evolution example
   - Some physics involved
2. What is it (mathematically)?
   - Physical equations
   - Approximations
   - Analysis of equations
3. How is it done?
   - Computer setup
   - Numerical equations
4. Results

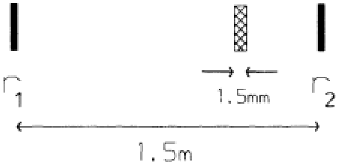
Early Time Evolution



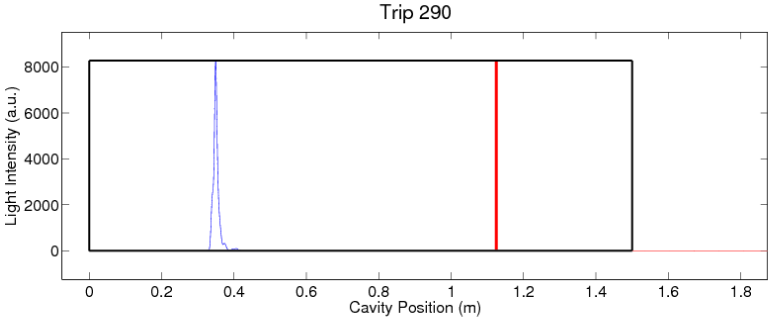


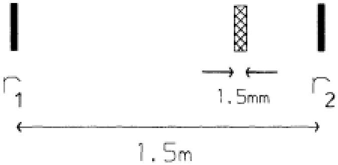
Middle Time Evolution





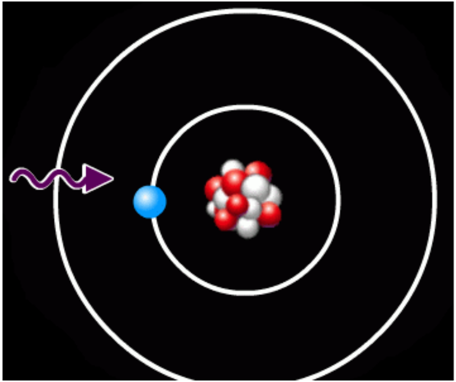
Late Time Evolution





Two Level Atom Model

- Every atom has one conduction electron and two energy levels, $E_2 > E_1$
- Each emits light at angular frequency $\omega = \dfrac{E_2 - E_1}{\hbar}$
- Initially, all atoms are identical (same energy levels)



Relevant Physical Quantities

Density Matrix stores relevant material variables

- ρ_{11}, ρ_{22}: Fraction of atoms in ground and excited states
- $\rho_{12} = \rho_{21}^{*}$: Dipole of the material

Polarization is the mechanism that connects the propagating light to the atoms in the active medium. Electric fields induce polarization in the medium, and polarization amplifies the electric field

Population Inversion describes how many of the atoms are in the ground and excited state. Given as $\rho_{22} - \rho_{11}$

Progress

Equations of Light

Maxwell's Equations for plane wave in a dispersionless, non-conducting medium

$$\frac{\epsilon}{c}\frac{\partial E_x}{\partial t} = -\frac{\partial H_y}{\partial z} - \frac{4\pi}{c}\frac{\partial P}{\partial t}$$
$$\frac{\mu}{c}\frac{\partial H_y}{\partial t} = -\frac{\partial E_x}{\partial z}$$

Variables:

- E_x Electric Field (in x-direction)
- H_y Magnetic Field (in y-direction)
- P Polarization
- ϵ Electric permittivity of the material
- μ Magnetic permeability of material
- c Speed of light

Material Equations

$$\frac{\partial \rho_{12}}{\partial t} + \left(\frac{1}{t_2} - i\omega_0\right)\rho_{12} = -\frac{i}{\hbar}\bar{\mu}(\rho_{11} - \rho_{22})E_x$$

$$\frac{\partial(\rho_{22} - \rho_{11})}{\partial t} = \frac{1}{t_1}\left[(\rho_{22} - \rho_{11})^0 - (\rho_{22} - \rho_{11})\right] + \frac{2i}{\hbar}\bar{\mu}(\rho_{12} - \rho_{21})E_x$$

- t_1 Transverse relaxation time. Time for population to relax to base state
- t_2 Longitudinal relaxation time. Time for polarization to vanish
- ω_0 Angular frequency of emitted photons
- $\bar{\mu}$ Mean component of atomic dipole matrix element
- $\hbar$ Planck's constant

Variable Relationships

Dipole Matrix Relationships

$$P = N\bar{\mu}(\rho_{12} + \rho_{21}) \qquad P_c = N\bar{\mu}\rho_{12} \qquad n = N(\rho_{22} - \rho_{11})$$

N Density of atoms in material

n Population inversion

Variable Relationships

Dipole Matrix Relationships

$$P = N\bar{\mu}(\rho_{12} + \rho_{21}) \qquad P_c = N\bar{\mu}\rho_{12} \qquad n = N(\rho_{22} - \rho_{11})$$

Define variables

$$E^{\pm} = \sqrt{\epsilon}E_x \pm \sqrt{\mu}H_y$$

to get Maxwell equations

$$\frac{\eta}{c}\frac{\partial E^{\pm}}{\partial z} \pm \frac{\partial E^{\pm}}{\partial z} = -\frac{4\pi\sqrt{\mu}}{c}\frac{\partial P}{\partial t}$$

- N Density of atoms in material
- n Population inversion
- η Refractive index

Physical System

Physical Equations

$$\frac{\eta}{c}\frac{\partial E^{\pm}}{\partial z} \pm \frac{\partial E^{\pm}}{\partial z} = -\frac{4\pi\sqrt{\mu}}{c}\frac{\partial P}{\partial t}$$

$$\frac{\partial P_c}{\partial t} + \left(\frac{1}{t_2} - i\omega_0\right) P_c = \frac{i\bar{\mu}^2}{\hbar} n \frac{E^+ + E^-}{2\sqrt{\epsilon}}$$

$$\frac{\partial n}{\partial t} + \frac{1}{t_1}(n - n^0) = \frac{2i}{\hbar}(P_c - P_c^*)\frac{E^+ + E^-}{2\sqrt{\epsilon}}$$

Boundary Conditions

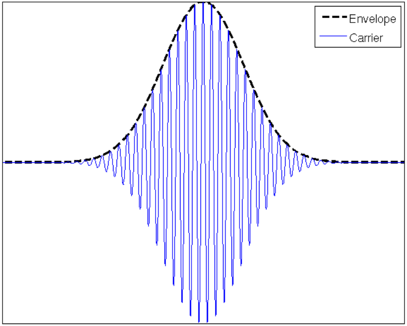
$$E^+(z_1, t) = -r_1 E^-(z_1, t) \qquad E^-(z_2, t) = -r_2 E^+(z_2, t)$$

Slowly Varying Envelope Approximation

Envelope Approximation

$$E^{\pm}(z,t) = \mathcal{E}^{\pm}(z,t)e^{i(\omega_0 t - kz)} + \mathcal{E}^{\pm *}(z,t)e^{-i(\omega_0 t - kz)}$$
$$P(z,t) = p(z,t)e^{i\omega_0 t}$$

- Decompose the variables
  - Envelope $\mathcal{E}^{\pm}$ and p
  - Carrier $e^{i(\omega t - kz)}$
- Assume that envelope varies much slower than carrier



Fourier Approximation

Fourier Approximation

$$p(z,t) = e^{-ikz} \sum_{j=0}^{\infty} p_j^{+} e^{-2ijkz} + e^{ikz} \sum_{j=0}^{\infty} p_j^{-} e^{2ijkz}$$

$$n(z,t) = n_0 + \sum_{j=1}^{\infty} \left(n_j e^{-2ijkz} + n_j^{*} e^{2ijkz} \right)$$

- Retain only the zeroth level terms in the model.

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- Retain only the zeroth level terms in the model.
- And then do a lot of renormalizing variables...

Final Equations of the Model

Final Equations

$$\frac{\partial E^{\pm}}{\partial t'} \pm \frac{\partial E^{\pm}}{\partial z'} = -P_0^{\pm}$$
$$\frac{\partial P_0^{\pm}}{\partial t'} = -\Gamma_2(P_0^{\pm} + E^{\pm} D_0) + S^{\pm}$$
$$\frac{\partial D_0}{\partial t'} = -\Gamma_1 \left[D_0 - A(t) - \mathrm{Re}\{E^{+*}P_0^{+} + E^{-*}P_0^{-}\}\right]$$

- $E^{\pm}$ Normalized left- and right-moving electric fields
- $P^{\pm}$ Normalized polarization for left- and right-moving fields
- D_0 Normalized population inversion
- $S^{\pm}$ Spontaneous emission term
- $A(t)$ Pumping function of the laser

In The Air

In the air our equations reduce to

Free Space Equations

$$\frac{\partial E^+}{\partial t'} + \frac{\partial E^+}{\partial z'} = 0$$
$$\frac{\partial E^-}{\partial t'} - \frac{\partial E^-}{\partial z'} = 0$$

which are just a pair of transport equations!

Material Model Equations

Equations

$$\frac{\partial E^{\pm}}{\partial t'} \pm \frac{\partial E^{\pm}}{\partial z'} = -P_0^{\pm}$$

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- A pair of transport equations with sources in the material.

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- A pair of transport equations with sources in the material.
- With no light, $P_0^{\pm}$ and D_0 decay exponentially to equilibrium states.
- Both exhibit nonlinear coupling to the other variables.

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Storing Data in a Computer

- Discretize the active medium by $n_z = 15$ points, into $n_z - 1 = 14$ regions
- Regions left and right are given $n_l = 7136$ and $n_r = 2378$ points
- A total of $N = 19056$ data points are stored in a circular array-pointer system (right)
- Light propagates counter-clockwise along the array

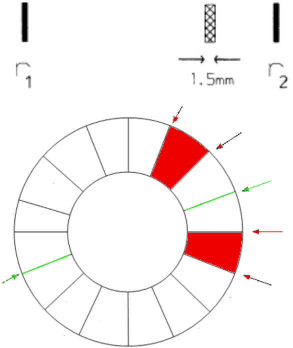

Figure: Circular Array Setup
Red regions and pointers indicate the active region, greed lines and pointers indicate mirrors

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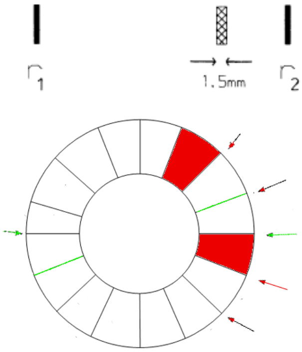

Figure: Circular Array Setup
Light is propagated by advancing each of the pointers one step in the clockwise direction

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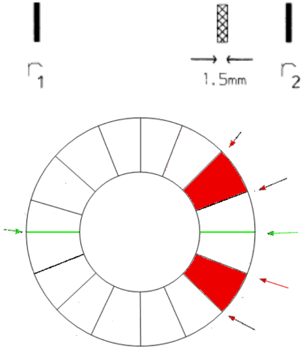

Figure: Circular Array Setup

Then the boundary conditions and material differential equations are applied

Numerical Schemes

Two Types of Equations

$$\frac{\partial x(z,t)}{\partial t} + \Gamma x(z,t) = f(z,t)$$

$$\pm\frac{\partial y(z,t)}{\partial z} + \frac{\partial y(z,t)}{\partial t} = g(z,t)$$

Numerical Scheme for Material Equations

$$\frac{\partial x(z,t)}{\partial t} + \Gamma x(z,t) = f(z,t)$$

- Apply the integrating factor and integrate

$$x_j^{m+1} = x_j^m e^{-\Gamma \Delta t} + \int_{t_m}^{t_{m+1}} e^{-(t-t')\Gamma} f(t') dt'$$

- Assume that f is linear to get

$$x_j^{m+1} = C(\Gamma \Delta t) x_j^m + \frac{A(\Gamma \Delta t)}{\Gamma} f_j^m + \frac{B(\Gamma \Delta t)}{\Gamma} f_j^{m+1} + O(\Delta t^2)$$

Numerical Scheme for Propagating Equations

$$\pm\frac{\partial y(z,t)}{\partial z}+\frac{\partial y(z,t)}{\partial t}=g(z,t)$$

- Then simply integrate along the characteristic lines $\Delta x = \Delta t$ to get
$$y_j^{n+1}-y_{j\mp1}^{n}=\frac{1}{2}\Delta z\left(g_j^{n+1}+g_{j\mp1}^{n}\right)+O(\Delta z^3)$$

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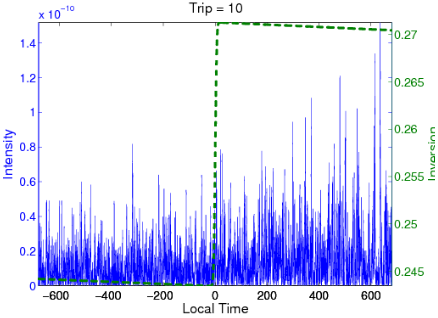
Another Look at Pulse Evolution



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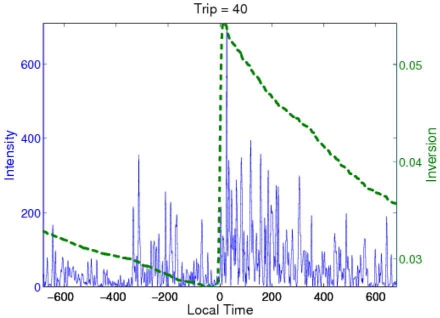

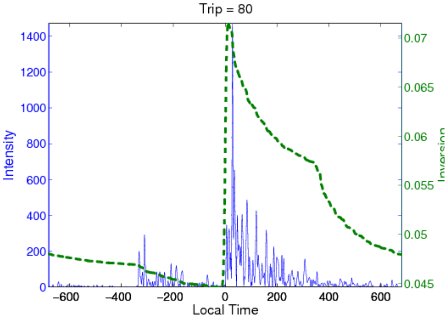

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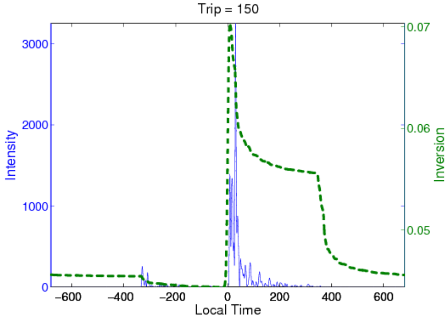

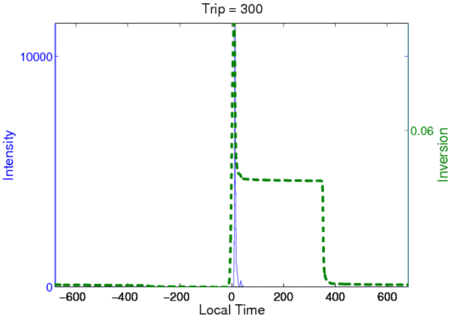

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- I would also like to thank Moysey Brio, Miroslav Kolesik and Jörg Hader for their help.

Progress

5. Variations on the theme
    - Inhomogeneous broadening
    - Other methods of pumping

Non-identical Atoms

Real materials have atoms with varying energy levels.

- Active medium variables must be indexed according to their energy levels
- Fraction with central frequency between ω^ξ and $\omega^\xi + d\omega^\xi$ is $p(\omega^\xi)d\omega^\xi$

Non-identical Atoms

Real materials have atoms with varying energy levels.

Inhomogeneously Broadened Equations

$$\frac{\partial E^{\pm}}{\partial t'} \pm \frac{\partial E^{\pm}}{\partial z'} = -P_0^{\pm}$$

$$\frac{\partial P_0^{\pm,\xi}}{\partial t'} = -\Gamma_2^{\xi}(P_0^{\pm,\xi} + E^{\pm} D_0^{\xi}) + S^{\pm}$$

$$\frac{\partial D_0^{\xi}}{\partial t'} = -\Gamma_1 \left[D_0^{\xi} - A(t) - \mathrm{Re}\{E^{+*} P_0^{+,\xi} + E^{-*} P_0^{-,\xi}\} \right]$$

- Active medium variables must be indexed according to their energy levels
- Fraction with central frequency between ω^{ξ} and $\omega^{\xi} + d\omega^{\xi}$ is $p(\omega^{\xi}) d\omega^{\xi}$
- Total polarization $P_0^{\pm} = \int_{-\infty}^{\infty} P_0^{\pm,\xi} p(\omega^{\xi}) d\omega^{\xi}$

The Problem with Inhomogeneous Broadening

$$\frac{\partial D_0^{\xi}}{\partial t'} = -\Gamma_1 \left[D_0^{\xi} - A(t) - \mathrm{Re}\{ E^{+*} P_0^{+,\xi} + E^{-*} P_0^{-,\xi} \} \right]$$

- Leaving $A(t)$ alone causes all atoms to be pumped equally
- Pumping should depend on the frequency of the atoms and the spectral width of the pump pulse
- Thus $A(t) \to D_0^0$, and we must find another method of pumping the laser

Pumping in the Polarization

We attempted two method of pumping

$$\frac{\partial P_0^\pm}{\partial t'} = -\Gamma_2(P_0^\pm + E^\pm D_0) + S^\pm + A(t)$$
$$\frac{\partial P_0^\pm}{\partial t'} = -\Gamma_2(P_0^\pm + (E^\pm + A(t))D_0) + S^\pm$$

but neither produced workable results.

Pumping Through the Boundaries

We then tried to alter the pumping through the non-outcoupling mirror, modifying the boundary condition

$$E^{+}(z_1, t) = -r_1 E^{-}(z_1, t) + A(t)$$

but this created similar problems.