MATH 254

Course Summary. Trial Final. Trial Exam.

Course Summary

1. **Definitions, Classifications**. Section 1.1,1.2, Lectures 1-2.
   Type: ode, pde, system of ode's, pde's.
   For ode's: order, linear or nonlinear, autonomous or nonautonomous.
   Solution: Implicit. Explicit. How to find latter from former. Initial Value problem.
   First order ode's (1) $\frac{dy}{dx} = f(x, y) = \frac{G(x, y)}{F(x, y)} = \frac{\text{numerator}}{\text{denominator}}$

   Methods: Direction fields. `SLOPES`. §1.3, Lectures 1-2, 3-4. Linear. $f(x, y) = -p(x)y + q(x)$. Int. Factor §2.3, Lectures 5-6. Separable. $f(x, y) = f(x)\, h(y)$. §2.2, Lectures 1-2, 3-4. Autonomous. Graphical. $f(x, y) = f(y)$. Lectures 3-4. Notion of equilibrium point; y_e, $f(y_e) = 0$. Lectures 3-4. Lyapunov stable, unstable, asymptotically stable equilibrium points. Basin of attraction. Stable & unstable manifolds. Lectures 3-4.

   Existence & Uniqueness of solutions to $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$. §1.2. Lectures 1-2.

   Applications to population growth, radioactive decay, salt mixing, heating/cooling, falling bodies, LR & RC circuits. Chapter 3. Numerical: Euler scheme. §1.4. Lectures 5-6, 7-10.

2. **Linear 2nd order ode's**
   (2) $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = g(x), x \in I$, p, q, g cns. on I.

   See Chapter §4.1, §4.2, §4.3, §4.4, §4.5, §4.6. + Notes on complex arithmetic, §4.8, §4.9 + Lecture Notes 11-15, 16-21.
   General solution of (2) = general solution of (3) $y'' + p(x)y' + q(x)y = 0$ + particular solution of (2).
   General solution of (3) = lin. combination of 2 l.i. solutions of (3).
   Notions of l.d. and l.i. of functions on intervals and the connection with Wronskians.

   Equivalence of l.i. (l.d.) of $y_1, \cdots y_n$, solutions of (4) $a_n y^{(n)} + \cdots a_1 y' + a_0 y = 0$ and nonvanishing (vanishing) of $W[y_1, \cdots y_n]$, the Wronskian of $y_1, \cdots y_n$.
   For (3), $W[y_1, y_2](x) = W[y_1, y_2](x_0) \exp - \int_{x_0}^{x} p(s)ds$.
   Methods for solving (3),(4) with constant coefficients and (1) with constant coefficients and periodic forcing.
   Undetermined coefficients.
   Mass-Spring/LRC circuits. Notion of resonance and tuned oscillator/circuit. Applications.
   Variation of parameter and reduction of order methods. §4.8, §4.9, §5.6.

3. **2nd order autonomous systems & phase plane analysis**.
   Chapter §5.1-5.6, Lectures 22-30.
   Equilibrium solutions x_e, y_e of (5), $\frac{dx}{dt} = F(x, y)$, $\frac{dy}{dt} = G(x, y)$
   Their stability (Lyapunov, asymptotic) properties, basins of attraction, stable and unstable manifolds.
   Connection with solutions of (6), $\frac{dy}{dx} = \frac{G(x, y)}{F(x, y)}$.
   Applications to pendulum, predator-prey.

   Use of `SLOPES` & `SYSTEMS`.

Q1. Type (ode, system of ode's, pde, system of pde's) the following differential equations. For ode's, give the order of the ode or the order of the system.

(1) $\frac{d^2x}{dt^4} + x^2 = 0$

(2) $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 50x = 0$

(3) $\frac{dx}{dt} = y,\ \frac{dy}{dt} = -2y - 50x$

(What is the connection between 3 & 2?)

(4) $\frac{dx}{dt} = \sigma(y - x),$

$\frac{dy}{dx} = rx - y - xz, \frac{dz}{dt} = -bz + xy,$

σ, r, b are positive constants

(Lorenz equations!)

(5) $\frac{\partial u}{\partial t}(x,t) = K\frac{\partial^2 u(x,t)}{\partial x^2}$ Heat Equation

(6) $\frac{\partial u(x,t)}{\partial t} + u(x,t)\frac{\partial u}{\partial x}(x,t) = v\frac{\partial^2 u(x,t)}{\partial x^2}$ Burgers' equation

(7) $\frac{\partial u}{\partial t}(x,t) = D_1\frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t) - u(x,t)u^2(x,t),\ \frac{\partial v}{\partial t}(x,t) = D_2\frac{\partial^2 v}{\partial x^2}(x,t) + u(x,t)v(x,t)$

(8) Given $u(x,y,z,t)$, $v(x,y,z,t)$, $w(x,y,z,t)$, $p(x,y,z,t)$, ρ is constant, ν is constant

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial y} = -\frac{1}{p}\frac{\partial p}{\partial x} + v\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{p}\frac{\partial p}{\partial y} + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{p}\frac{\partial p}{\partial z} + v\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right).$$

By showing $\begin{cases} u(x,y,z,t) \\ v \\ w \end{cases}$ exist for all time (or by showing they don't), you could win you $1M !!!

(See http://www.claymath.org/Millenium_Prize_Problems/).

Q.2. For each ordinary differential equation below, give its order and state whether it is linear or nonlinear (L/NL), autonomous or non-autonomous (A/NA).

	Equation	Order	L/NL	A/NA
(i)	$\frac{d^2x}{dt^2} = x^2$			
(ii)	$\frac{d^2x}{dt^2} + x - t\cos x = 0$			
(iii)	$\frac{d^3x}{dt^3} + 2x\frac{dx}{dt} = 0$			
(iv)	$\frac{d^4x}{dt^4} = 0$			
(v)	$\frac{dx}{dt} = t^2 + x$			
(vi)	$2x\frac{dx}{dt} + \frac{d^2x}{dt^2} = 0$			
(vii)	$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F}{m}\cos\omega t$			
(viii)	$\frac{d^2y}{dx^2} + y = 0$			
(ix)	$\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$			
(x)	$\frac{d^4x}{dt^4} + x = t$			
(xi)	$\frac{d^2x}{dt^2} + 0.2(x^2-1)\frac{dx}{dt} + x = 0$			

Solve

(i) with i.c. $x(1) = 2$

(iv) given $x(0) = 5,\ \frac{dx}{dt}(0) = 1,\ \frac{d^2x}{dt^2}(0) = 4,\ \frac{d^3x}{dt^3}(0) = 7.$

(v) with i.c. $x(0) = 2$

(vii) Solve for $\frac{b}{m} = 2,\ \frac{k}{m} = 5,\ \frac{F}{m} = 1,\ x(0) = 0,\ \frac{dx}{dt}(0) = 0.$

(viii) given $y(0) = 1,\ \frac{dy}{dt}(0) = 2$

(ix) given $y(0) = 1,\ \frac{dy}{dx}(0) = 0$

(x) given $x(0) = \frac{dx}{dt}(0) = \frac{d^2x}{dt^2}(0) = \frac{d^3x}{dt^3}(0) = 0.$

(xi) by writing it as a system and using `SYSTEMS`.

Extra difficult

(iii) Solve given $x(0) = E = 1,\ \frac{dx(0)}{dt} = V,\ \frac{d^2x}{dt^2}(0) = 0$, where V is arbitrary.

(vi) Solve given $x(0) = E,\ \frac{dx}{dt} = V$.

Q3. Solve the following initial value problems

(i) $\frac{dy}{dt} + 2ty = e^{-t^2} \qquad y(1) = 2$

(ii) $\frac{dy}{dt} = y(1-y) \qquad y(0) = 2$

(iii) $\frac{dy}{dx} = -\frac{x}{y} \qquad y(3) = -4$ Find the branch through $(3, -4)$.

(iv) $\frac{dy}{dx} = \frac{-\sin x}{y} \qquad y(0) = -2.$ Find the branch through $(0, -2)$.

(v) $\frac{dy}{dx} = \sqrt{y}, \qquad y \geq 0$

Find all solutions with $y(0) = 0$
Find all solutions with $y(0) = 1$.

(vi) $\frac{dy}{dt} = -y \ln y, \qquad y(0) = e$

(vii) $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 50x = 0, \qquad x(0) = 1,\ \frac{dx}{dt}(0) = 0$

(viii) $\frac{d^2x}{dt^2} + 0.2\frac{dx}{dt} + 9.01x = \cos 3t \qquad x(0) = \frac{dx}{dt}(0) = 0$

(ix) $\frac{d^2x}{dt^2} - x = te^t \qquad x(0),\ \frac{dx}{dt}(0) = 0.$

(x) $t^2\frac{d^2x}{dt^2} + t\frac{dx}{dt} - x = t, \qquad x(1) = 0,\ \frac{dx(1)}{dt} = 0.$

(xi) $(t^2+1)\frac{d^2x}{dt^2} - 2t\frac{dx}{dt} + 2x = 2, \qquad x(0) = \frac{dx}{dt}(0) = 0.$

(xii) $t^2\frac{d^2x}{dt^2} - t(t+2)\frac{dx}{dt} + (t+2)x = 2t^3, \qquad x(1) = 0, \frac{dx(1)}{dt} = 0.$

Q4. Using graphical methods, discuss the long time behavior of solutions to

$$\frac{dx}{dt} = -x(1-x)(1+x)(2-x).$$

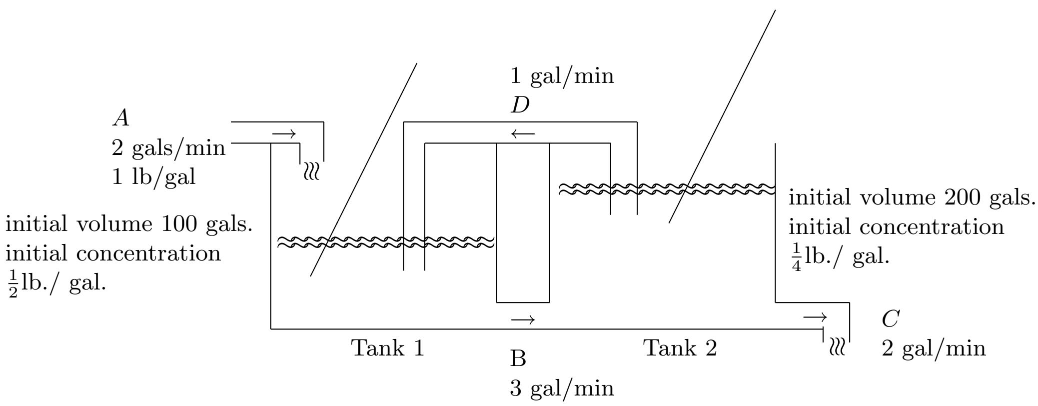
Identify
(a) the equilibrium points
(b) say whether each is stable or unstable
(c) say if stable whether asymptotically stable
(d) identify the basins of attraction for the asymptotically stable equilibria

Q5. At time $t = 0$, a 200 gallon tank initially containing 50 gallons of pure water is filled with a brine solution containing $\frac{1}{2}$lb. of salt per gallon at a rate of 4 gals/min. It is drained at the rate of 3 gals./min. and the tank is kept well stirred throughout. If $x(t)$ lbs. is the amount of salt in the tank at time t minutes, write an ode for $x(t)$. What is $x(0)$? What is $C(t)$, the concentration of brine in the tank at time t, in terms of $x(t)$ and the volume $V(t)$ of fluid in the tank at time t? What is $V(t)$? After how many minutes will the tank overflow? Show that $C(0) = 0$, $C(50) = \frac{15}{32}$, $C(100) = \frac{40}{81}$ and $C(150) = \frac{255}{512}$ lbs/gal. Draw the graph of $C(t)$ vs t for $0 < t \leq t_1$, $t_1 =$ overflow time.

Q6. A cake is taken out of an oven at 300°F and allowed to cool in a room with temperature 60°F. At 3pm, its temperature had dropped to 160°F and an hour later had dropped to 110°F. At what time was it taken out of the oven?

Q7. Consider the two tank arrangement shown in the Figure below. At time $t = 0$, the valves A, B, C, D are simultaneously opened. Find the amounts of salt $x_j(t)$, $j = 1, 2$ and concentrations $C_j(t)$, $j = 1, 2$ of salt in tanks 1 and 2 as a function of time.

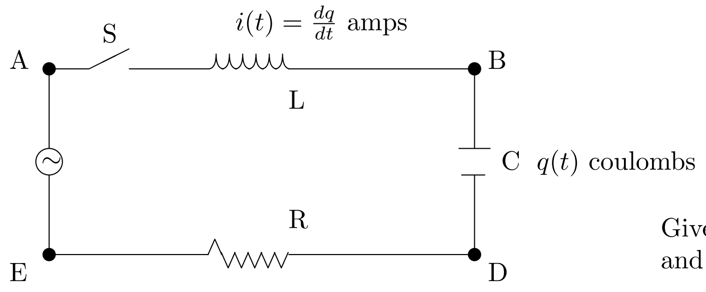
Draw the graphs of $C_1(t)$ and $C_2(t)$.



Q8. A body of mass 10kgs. falls from rest under the joint influences of gravity ($g = 10$ m/sec^2– approximately) and friction (the friction force is 0.01 times the square of the velocity). Show that the velocity after $5 \ln 2$ seconds is -33.33 m/sec (We measure $x(t)$ and $v(t) = \frac{dx}{dt}$ —distance and velocity-positive upwards). To what value does $v(t)$ tend as $t \to \infty$. Calculate $x(10 \ln 2)$.

Q9. A very light spring (its extension under its own weight is negligible) hangs from a hinge. A weight of 2 kgs. is added to the pan and the spring is stretched by 2 cms. If g, gravity, is taken to be 10 m/sec^2 and if the friction force experienced by the mass-string arrangement is numerically equal to 4 times the velocity, find the equation which the displacement from equilibrium $x(t)$ satisfies. Given $x(0) = 1$ cm. and $\frac{dx}{dt} = 0$ m/sec., write $x(t)$ in the form $Ce^{at}\cos(bt - \varphi)$. Determine $C, a, b,$ and $\cos\varphi,\ \sin\varphi$.

Q10. Consider the LRC circuit shown below.



Given $V_E - V_A = -V_0 \cos \omega t$
and $q(0) = \frac{dq}{dt}(0) = 0$.

At time $t = 0$, the switch is closed and the emf starts current flowing in the circuit.
Solve for both $q(t)$ and $i(t) = \frac{dq}{dt}$ and in particular show that

$$\begin{aligned} i(t) &= e^{at}(A_1 \cos bt + A_2 \sin bt) \\ &+ \frac{V_0}{Z} \cos(\omega t - \varphi). \end{aligned}$$

Write expressions for $a, b, \frac{1}{Z}$ and $\cos \varphi$, $\sin \varphi$ in terms of L, R, C, ω. Also find A_1, A_2. For $L = C = 1$, draw the graphs (carefully) of $\frac{1}{Z(L,R,C;\omega)}$ versus ω for the values $R = 1$, $R = .2$, $R = .1$.

At what value of ω is $\frac{1}{Z}$ maximum? What is its value there? For this value what is $V_0/Z \cos(\omega t - \varphi)$ equal to?

Q11. Consider the 2nd order autonomous system

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x + x^3. \end{aligned}$$

What are the equilibrium points (x_e, y_e)? Are they stable or unstable? If stable, are they Lyapunov stable or asymptotically stable? If unstable, find the equation for the graphs of their stable and unstable manifolds near (x_e, y_e).

The solution curves $x(t), y(t)$ in the (x, y) plane obey the first order ode

$$\frac{dy}{dx} = \frac{-x + x^3}{y}.$$

(a) By direct integration, compute the integral curves. Write expressions for the stable and unstable manifolds through the unstable equilibrium points (x_e, y_e)?

(b) By hand, sketch out direction fields; namely calculate $\theta(\tan \theta = \frac{dy}{dx}$ for several points such as $(0, \pm\frac{1}{2})$, $(0, \pm 2)$, $(\pm\frac{1}{2}, 0)$, $(\pm 2, 0)$, $(\pm 1, \pm 1)$

(c) Use `SLOPES` to do the same thing. Use `SYSTEMS` to find solutions $x(t), y(t)$ starting out with initial conditions
(i) $x(0) = 0, y(0) = \pm.5$ (ii) $x(0) = 0, y(0) = \pm 1$ (iii) $x(0) = 0$, $y(0) = \pm\frac{1}{\sqrt{2}}$.

Q12. By adding a damping term to Q.11,

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -by - x + x^3,$$

with $b = 1, 0.1$, recalculate phase plane.

Q13. Draw the phase plane for the undamped pendulum

$$\frac{d^2x}{dt^2} + \sin x = 0.$$

Hint: Set $\frac{dx}{dt} = y$.

KNOW THIS WELL
You could explain this to your mother using a meter stick, right?

Q14. Consider

(*) $a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_1 \frac{dx}{dt} + a_0 x = 0.$

Given the following roots of the characteristic equation $a_n r^n + \cdots + a_1 r + a_0 = 0$, write down the general solution of (*).

(a) $r = 0, 0, 0, 1, 1, -1-i, -1+i$

(b) $r = 1, 1, 1, 1, 2, -1+2i, -1-2i, -1+2i, -1-2i$

(c) $r = 0, 0, 2, 2, i, -i, i, -i$

(d) $r = 1, 2, 3, 4, 5, i, -i$

Q15. Find the general solution for

(a) $t^2 \frac{d^2x}{dt^2} + t\frac{dx}{dt} = 0$

(b) $\frac{d^2x}{dt^2} + \omega_0^2 x = \cos \omega t \qquad \omega^2 \neq \omega_0^2$

(c) $\frac{d^2x}{dt^2} + \omega^2 x = \cos \omega t$

(d) $\frac{d^3x}{dt^3} - 3\frac{d^2x}{dt^2} + 3\frac{dx}{dt} - x = e^t$

(e) $\frac{d^4x}{dt^4} = -x$

Hint: the roots of $r^4 = -1$ are $r = \frac{-1+i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}}, \frac{1+i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}}$. Check!

(f) $\frac{d^4x}{dt^4} = x.$

(g) Given $x(0) = \frac{dx}{dt}(0) = \frac{d^2x}{dt^2}(0), \frac{d^3x}{dt^3}(0) = 1$ in f. Show that $x(t = \frac{\pi}{2}) = \frac{1}{4}e^{\pi/2} - \frac{1}{4}e^{-\pi/2} - \frac{1}{2}$.