Class 15: When σ is Not Known: The T-Distribution (Text: Section 7.1)

Statistical Inference

We take a sample to learn about a population.

- We have made estimates using **confidence intervals.**
- We have decided whether to believe statements using **Hypothesis Testing**:

In both cases, we assumed we knew the population standard deviation, σ. What if we don't know σ?

Standard Error: What if we don't know σ for population?

CLT says Standard error $= \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$. If we don't know σ, we use s, the standard deviation of the sample (instead of the population), so

$$\text{Standard error} \approx \frac{s}{\sqrt{n}}$$

This expression for the standard error is an *estimate* of standard deviation of the sampling distribution, $\sigma_{\bar{x}}$.

Central Limit Theorem: We know Z is distributed normally with mean 0, standard deviation 1, that is

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim \text{N}(0,1)$$

T-distribution

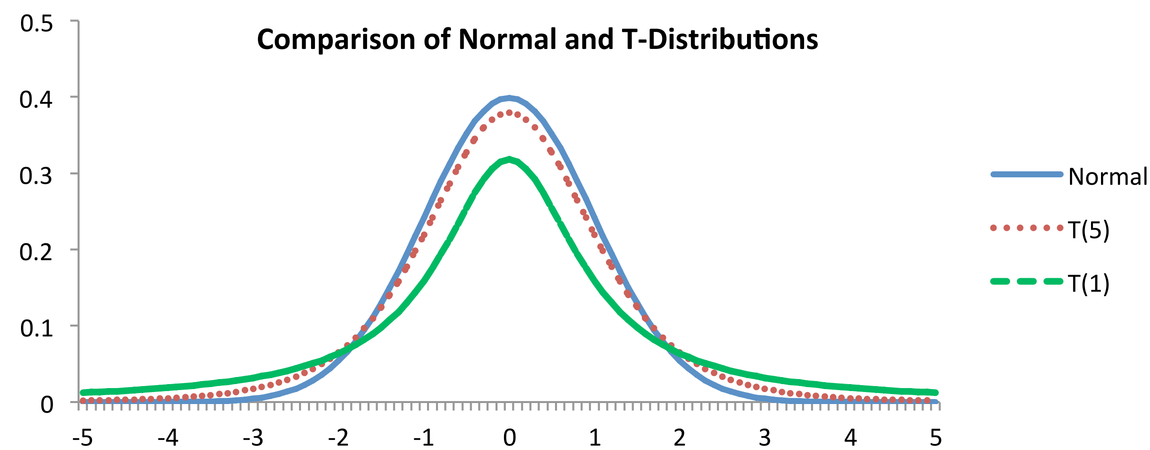
If we replace σ by the sample standard deviation, how is the test statistic distributed?
If the original distribution is *norma*l, the test statistic has the T-distribution.

If population is $N(\mu, \sigma)$, for a SRS of size n, we calculate a statistic that is very like z:

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}} \sim \text{T}(\text{n} - 1)$$

This variable has the T-distribution with $(n - 1)$ degrees freedom.

Ex:. Graph $N(0,1), T(5), T(1)$ on $5 \leq x \leq 5, 0 \leq y \leq 0.5$.



What is the T-distribution? What is a degree of freedom?

- The T-distributions is similar to normal, but has thicker tails, since there is more variability in T than in Z because s is variable, whereas σ was not.
- For large sample sizes, the T-distribution is approximately equal to the standard normal. For $n > 100$, you can use Z instead of T.
- A *degree of freedom* (df) is a number that tells us how variable the T-distribution is. In the case of Z, there was only one standard normal distribution (with $\mu = 0, \sigma = 1$). In the case of T, there is a whole family of T-distributions, one for each sample size. The larger the sample size, the less s is likely to vary (because bigger samples give better estimates of s), so the less T varies.
  --No matter what the size of the sample:
  $$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim \mathrm{N}\,(0,1)$$
  --The distribution of the t statistic depends on n. The df $= n - 1$:
  $$t = \frac{\bar{x} - \mu}{s/\sqrt{n}} \sim \mathrm{T}\,(\mathrm{n} - 1)$$

Reading the T-table

In Z-table, probabilities measured from the left are in body of table.
In T-table, probabilities are across the top, and they are measured from the *right*. The T-values are in the body of the table. Down the left are the degrees of freedom—take the closest one if the one you want is not there. The bottom row is Z scores, corresponding to an infinite df.

Ex. For 1 df, what does the 15.89 in top row mean?

Ex. For 1 df, what does the 96% at the bottom of this column mean?

Ex. For 3 df, find $P(T > 1.638) = 0.10 = 10\%$
For 3df, find $P(T > 6) \approx 0.004 = 0.4\%$.
For 3df, find $P(T < -1.638) = 10\%$,
For 3 df, find $P(-1.638 < T < 1.638) = 80\%$.

Ex. For 10 df, find t for an upper tail probability of 2.5%. What confidence interval does this give you?

Ex. For 10 df, find the T-values for a 99% confidence interval

With TI-83/84

Ex: For 1 df: find $P(|T|) < 15.89) = \text{tcdf}(-15.89, 15.89, 1) = 0.96$
For 3 df: find $P(T > 1.638) = \text{tcdf}(1.638, 100, 3) = 0.10$
For 10 df: find$P(T > 2.228) = \text{tcdf}(2.228, 100, 10) = 0.025$

$\text{tcdf}(a, b, n) = P(a < X < b)$ if X is distributed according to t-distribution with n degrees of freedom

Confidence intervals for mean μ using the T-Distribution

If σ is not known, and s calculated from data, replace z-values by t-values. Confidence interval is

$$\left(\bar{x} - t\frac{s}{\sqrt{n}}, \bar{x} + t\frac{s}{\sqrt{n}}\right)$$

The margin of error is $ME = t\frac{s}{\sqrt{n}}$. The value of t depends on the confidence level.

Ex. **A coaching service claims to raise SAT scores.** Without coaching, scores are normally distributed with mean of 475. If a random sample of 40 students who are coached has mean score 500 with standard deviation 90:

(a) Find a 95% confidence interval for the mean of all coached students.
(b) Decide if the mean of the coached group is significantly different than the mean, 475.

A confidence interval can always be used to do a hypothesis test:
—If the mean of the population is *outside* the interval, result is *significant*
—If the mean of the population is *inside* the interval, result is *not significant*

Ex: If we take 100 samples and the find 100 corresponding 95% confidence intervals, how often do we expect the true mean to be in the confidence interval?

Hypothesis Testing
Ex: Use the same SAT coaching data. Use a hypothesis test to decide: **Does coaching service *change* scores?** Give null and alternative hypothesis; P-value. Use 5% and 1% significance levels.

Step 1:

Step 2:

Step 3:

Step 4:

Interpretation: If the mean is unchanged by coaching,

Ex. Use the same SAT coaching data. Use a hypothesis test to decide: **Does coaching service raise scores?** Give null and alternative hypothesis; p-value. Use 5% and 1% significance levels.

Step 1:

Step 2:

Step 3:

Step 4:

Ex: Compare the results of a one sided two-sided test for the previous example.

Notice that it is possible to draw one conclusion from a two sided test and a different one from a one-sided test.

Ex. **Alternate method: Instead of P-values, use table to compare critical t-values**

The T-values in the table are called *critical values*. Instead of using P-values, we can compare the t-value we get with the critical values.

For 39 ≈ 40 df,

For $p = 0.05$, critical $t =$

For $p = 0.01$, critical $t =$

If our t-value is *larger* in magnitude than the critical value corresponding to the significance level, the result is significant.

Ex. A school wonders whether its SAT scores are comparable to the national average, 475. A SRS of 70 students in this school has mean $\mu = 460$ and standard deviation $s = 80$.
Does the school have a significantly different SAT average? Test at 5% level, 1% level, 10% level.

Ex. **The same school wonders if its scores are lower than national average.**

Notice again that we have one conclusion from a two sided test and a different one from a one sided test.