The Mechanics of Euler's Disk

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Outline

Euler's Disk

- We wish to study the motion of a spinning disk, such as a coin, on a surface.
- In observation, the coin spins with increasing frequency until it abruptly comes to rest on the surface in finite time.
- In classical theory, this frequency remains constant and motions persists forever.

Previous Work

- Moffatt (2000) - Suggested main dissipative mechanism that creates finite time singularity is air viscosity.
- Van den Engh (2000) & Petrie (2002) - Experiments performed in a vacuum do not agree with Moffatt's predictions.
- Bildsten (2002) - Main dissipative mechanism attributed to rolling friction from point in contact.

Informal Definitions

- A *manifold* M is geometric object that locally is homeomorphic to an open subset of $\mathbb{R}^n$.
- For a point p on a manifold M and a curve γ on M such that $\gamma(t_0) = p$, $\dot{\gamma}(t_0) = v$ is a *tangent vector* at p.
- The *tangent space* TM_p is the vector space of all tangent vectors at p.
- The *cotangent space* T^*M_p is the dual space to TM_p. Its' elements $\alpha : TM_p \to \mathbb{R}$ are called *covectors* or *1-forms*.

Tangent & Cotangent Bundles

- The *tangent bundle* TM of a manifold M is the collection of all tangent spaces TM_p. Elements in TM are of the form $(p, v), p \in M, v \in TM_p$.
- The *cotangent bundle* T^*M is the collection of all cotangent spaces T^*M_p.
- If M is n dimensional, TM and T^*M are $2n$ dimensional manifolds.

Lie Groups & Lie Algebras

- A *Lie group* is a smooth manifold G with a group action, such that the group action $G \times G \to G$ and inversion $g \mapsto g^{-1}$ are differentiable.
- A *Lie algebra* is a real vector space $\mathcal{G}$ with a bilinear operation $[\cdot, \cdot]$ called the *Lie bracket*, that satisfies certain properties.
- For matrix groups, the Lie bracket is the commutator $[A, B] = AB - BA$.
- The Lie algebra $\mathcal{G}$ associated to a Lie group G is the tangent space TG_e to the identity element.

Outline

Reference Frames

- Consider a rigid body D moving in 3-space.
- There are two coordinate systems, the inertial *space frame*, and the *body frame*, a coordinate system that is fixed with D and moves with the rigid body.
- Recall rotation matrices $R \in SO(3)$ are real matrices that satisfy $R^T = R^{-1}$ and $\det(R) = 1$.
- The linear transformation that moves from the body frame to the space frame is given by

$$x' = x'_c + Rx$$

where x'_c is the center of mass of D (*coordinates in the space frame*)

Configuration Manifold

- To describe the motion of the rigid body, we need to describe R and $\dot{R}$.
- The geometric object in which we wish to describe our dynamics is the *configuration manifold*, and is $TSO(3) = \mathbb{R}^3 \times SO(3)$.

Lagrangian

- The *Lagrangian* is a time-dependent function that lives on the configuration manifold with coordinates $(q, \dot{q})$ of the form

$$\mathcal{L} = T(q, \dot{q}) - V(q)$$

- For our problem:

$$\mathcal{L} = \underbrace{\frac{1}{2}M\dot{x'}^2 + \frac{1}{2}\mathrm{tr}(\dot{R}K\dot{R}^T)}_{\text{kinetic energy}} - \underbrace{U(r') - V(R)}_{\text{potential energy}} + \underbrace{\mathrm{tr}(\Lambda R^T R - \Lambda)}_{\text{Lagrange multiplier term}}$$

Euler-Lagrange Equation

- The general Euler-Lagrange equation is

$$\frac{\partial \mathcal{L}}{\partial q} + \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}$$

- Treating R as our coordinates for $SO(3)$, the E-L becomes

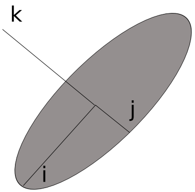
$$R^T \ddot{R} K - K \ddot{R}^T R = \frac{\partial V}{\partial R^T} R - R^T \frac{\partial V}{\partial R}$$

- Using $R^T \dot{R} = \Omega \in so(3)$, the Lie algebra of skew-symmetric matrices associated to $SO(3)$, this becomes

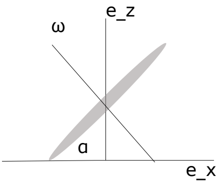
$$K\dot{\Omega} + \dot{\Omega} K + \Omega^2 K - K\Omega^2 = \frac{\partial V}{\partial R^T} R - R^T \frac{\partial V}{\partial R}$$

Outline

Reference Frames



body frame



space frame

$$
\begin{aligned}
\cos\theta\vec{i} + \sin\theta\vec{j} &= \vec{b} = \cos\phi\cos\alpha\vec{e}_x + \sin\phi\cos\alpha\vec{e}_y - \sin\alpha\vec{e}_z \\
-\sin\theta\vec{i} + \cos\theta\vec{j} &= \vec{n}\times\vec{b} = -\sin\phi\vec{e}_x + \cos\phi\vec{e}_y \\
\vec{k} &= \vec{n} = \sin\alpha\cos\phi\vec{e}_x + \sin\alpha\sin\phi\vec{e}_y + \cos\alpha\vec{e}_z
\end{aligned}
$$

Rotation Matrix

Calculating R, Ω, we have

$$R = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \cos\phi\cos\alpha\cos\theta + \sin\phi\sin\theta & \cos\phi\cos\alpha\sin\theta - \sin\phi\cos\theta & \cos\phi\sin\alpha \\ -\cos\phi\sin\theta + \sin\phi\cos\alpha\cos\theta & \cos\phi\cos\theta + \sin\phi\cos\alpha\sin\theta & \sin\phi\sin\alpha \\ -\sin\alpha\cos\theta & -\sin\alpha\sin\theta & \cos\alpha \end{pmatrix}$$

$$\Omega = \begin{pmatrix} 0 & \dot\theta\cos\alpha - \dot\phi & \dot\alpha\cos\phi - \dot\theta\sin\phi\sin\alpha \\ \dot\phi - \dot\theta\cos\alpha & 0 & \dot\alpha\sin\phi + \dot\theta\cos\phi\sin\alpha \\ \dot\theta\sin\phi\sin\alpha - \dot\alpha\cos\phi & -\dot\alpha\sin\phi - \dot\theta\cos\phi\sin\alpha & 0 \end{pmatrix}$$

K, V

$$K = \begin{pmatrix} \pi/4 & 0 & 0 \\ 0 & \pi/4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- $V = \sin\alpha$, so we can express this in terms of the entries of R by

$$V = \sqrt{1 - R_{33}^2}$$

$$\frac{\partial V}{\partial R} = \frac{\partial V}{\partial R^T} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{R_{33}}{\sqrt{1-R_{33}^2}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\cot\alpha \end{pmatrix}$$

No Slip Condition

- The *no slip condition* is that the instantaneous velocity of the point in contact $\vec{x}_p' = (\cos\theta, \sin\theta, 0)^T$ (*with respect to the space frame*) is zero:

$$\vec{0} = \dot{\vec{x}}_p = \dot{R}\vec{x}_p'$$

- This gives us $\dot{\alpha} = 0$ and $\dot{\theta} = \cos\alpha\dot{\phi} = \beta$ for some constant β.
- Integrate to get

$$\theta = \beta t \qquad \phi = \frac{\beta}{\cos\alpha} t$$

- The point in contact traces a circle on the surface at rate $\beta/\cos\alpha$, faster than the rate at which the point in contact on the disk is moving.

Euler-Lagrange Equation

- Using these conditions when $\dot{\alpha} = 0$, the E-L equation reduces to

$$\begin{pmatrix} 0 & 0 & -\frac{1}{4}\beta^2 \cos\theta \tan\alpha \\ 0 & 0 & -\frac{1}{4}\beta^2 \pi \sin\theta \tan\alpha \\ \frac{1}{4}\beta^2 \cos\theta \tan\alpha & \frac{1}{4}\beta^2 \pi \sin\theta \tan\alpha & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -\cos\alpha\cos\theta \\ 0 & 0 & -\cos\alpha\sin\theta \\ \cos\alpha\cos\theta & \cos\alpha\sin\theta & 0 \end{pmatrix}$$

- This gives us a relation where we can calculate the constant β, and obtain the result

$$\beta = 2\sqrt{\frac{\cos\alpha}{\pi\tan\alpha}} \qquad \dot{\phi}^2 \sin\alpha = \frac{4}{\pi}$$

Outline

Numerical Issues

- The system has two disparate time scales - the frequency of the spinning disk is quite fast compared to the slow rate at which the disk settles on the surface.
- Our equations in $\dot{\alpha}, \dot{\theta}, \dot{\phi}$ are quite cumbersome, and rotation matrices have numerically unstable properties $(R^T = R^{-1}, \det(R) = 1)$
- We want to use the matrices R themselves as coordinates of the Lie group $SO(3)$, and we want to parameterize these coordinates by a vector space, the natural choice being the Lie algebra $so(3)$.

Cayley Transform

- The *Cayley Transform* is a map $so(3) \rightarrow SO(3)$ given by

$$R = (I - Q)(I + Q)^{-1}$$

- The Cayley transform will become singular if Q has eigenvalues close to -1.
- We cannot solve for $\dot{\Omega}$ in the E-L equation since $K, \dot{\Omega}$ do not commute, so we also need to introduce

$$\tilde{\dot{\Omega}} \mapsto K\dot{\Omega} + \dot{\Omega}K$$

  This map is nonsingular, so we will be able to recover $\dot{\Omega}$ from $\tilde{\dot{\Omega}}$

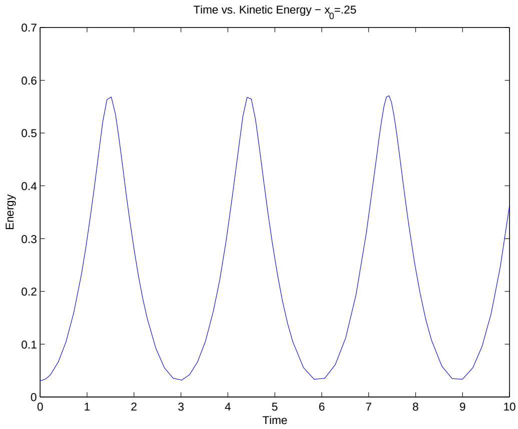
Numerical Integration & Dissipation

- The system of matrix ODEs we want to solve is

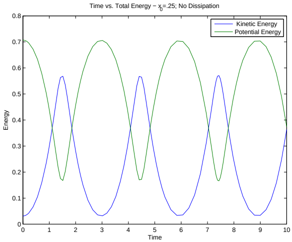
$$\begin{aligned}\dot{\tilde{\Omega}} &= -\Omega^2 K + K\Omega^2 + \frac{\partial V}{\partial R^T} R - R^T \frac{\partial V}{\partial R} \\ \dot{Q} &= -\frac{1}{2}\left(\Omega + [\Omega, Q] - Q\Omega Q\right)\end{aligned}$$

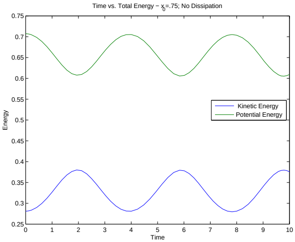
- We can numerically add dissipation to the system by including a term $-a\Omega$ in the second equation, resulting in exponential decay. This corresponds to a decrease in angular momentum, the result of a dissipative mechanism acting on the entire system, such as air viscosity.

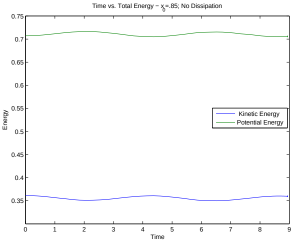
No Dissipation



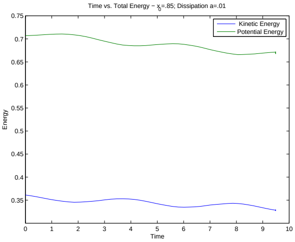
No Dissipation

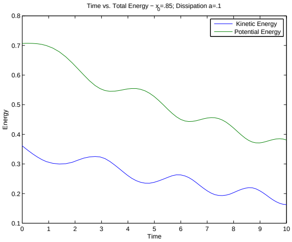


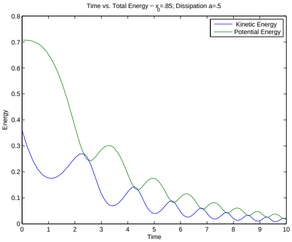




With Dissipation







Outline

Summary

- Classical theory predicts Euler's disk will continue its motion indefinitely, in contrast with the observable 'finite-time' singularity.
- Though the Lagrangian formulation lives on the tangent bundle of $SO(3)$, standard parameterization of the manifold in terms of angles α, θ, ϕ yields a complex and numerically unstable space in which to compute.
- Reparameterizing the Lie group $SO(3)$ using a vector space, with the natural association being its Lie algebra $so(3)$, via the Cayley transform allows for simpler and more stable computations.

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Questions, Acknowledgements

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