THE MECHANICS OF EULER'S DISK

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ABSTRACT. This paper presents a model of the mechanics of a spinning disk on a surface using the Lagrangian formulation and the Euler-Lagrange equation on the tangent bundle $TSO(3)$. Numerical issues on computing in such a space are addressed, and the Cayley transform is used to allow us to parameterize the Lie group $SO(3)$ by its Lie algebra $so(3)$.

1. Introduction

The problem we wish to address is understanding the motion of a disk spinning on its side, such as a coin spinning on a table, commonly referred to as Euler's disk. As we shall see, the classical mechanics predict the motion will continue forever, in contrast to the observation that the angle between the disk and the coin decreases in time, until the coin eventually settles rather abruptly on the table. The problem has had a renewed interest in the last several years, after an explanation was offered by Moffatt [6] in 2000 that the air viscosity between the disk and the table is sufficient to explain the abrupt settling of the coin. Experiments conducted later that year, performed in a vacuum, did not agree with Moffatt's predictions, and it was subsequently suggested in 2002 by Bildsten [1], among others, that the dominant dissipative force is rolling friction from the changing contact point.

In section 2 we will discuss the necessary geometric background to study this problem. The fundamental geometric structure we will be working on are smooth manifolds M, which, for the unfamiliar reader, have the structure that locally, there exists a coordinate chart $\mathbf{x}$ that maps an open subset of M homeomorphically to an open subset of Euclidean space. In section 3 we move on to general rigid body mechanics, including the Lagrangian and Hamiltonian formulations as well as Euler's equations for a rigid body. In section 4 we consider our specific problem and develop the equations that govern the motion of Euler's disk, and finally in section 5 we address the issues that arise when one wishes to perform numerical analysis on this system.

2. Geometric Setting

2.1. **Preliminary Notions.** We begin with a few fundamental definitions. A *smooth* manifold is a manifold where the change of coordinate map is always smooth, and a map between manifolds is called smooth if the induced map in coordinates between Euclidean spaces is smooth.

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Definition 2.1. *Let $\gamma(t)$ be a parameterized curve on an M dimensional smooth manifold, such that $\gamma(t_0) = p \in M$. We call $\dot{\gamma}(t_0) = v$ a* tangent vector *to a point p, and the collection of all such tangent vectors v at p forms the* tangent space TM_p. *The* cotangent space *is the dual space T^*M_p, whose elements $\alpha : TM_p \to \mathbb{R}$ we call* covectors *or* 1-forms.

For a local coordinate chart $\mathbf{x}$, we can also identify a tangent vector as a first order differential operator, and as such we have $\frac{\partial}{\partial x_i}$ form a basis for TM_p. We also recall that for a smooth map between manifolds $\phi : M \to N$, the differential of ϕ at p is the map $d\phi_p : TM_p \to TN_{\phi(p)}$ where $d\phi(v)$ is a tangent vector to $\phi(p)$; more precisely if γ is a curve in M such that $\gamma(t_0) = p, \dot{\gamma}(t_0) = v$, then $\phi(\dot{\gamma}(t_0) = d\phi(v)$. We will later use the notation ϕ_* to denote the differential map. Then $\{dx^i\}$ form a basis for T^*M_p, where dx^i is the differential of the ith coordinate map.

Definition 2.2. *The* tangent bundle TM *for a manifold M is the collection of all tangent spaces TM_p, whose elements are of the form (p, v), $p \in M, v \in TM_p$. The* cotangent bundle T^*M *is the collection of all co-tangent spaces, whose elements are of the form $(p, \alpha), p \in M, \alpha \in T^*M_p$.*

We note if M is n dimensional, both the tangent bundle and cotangent bundle are $2n$ dimensional manifolds, and also have vector space structure.

Definition 2.3. *A* smooth derivation δ *on M is a smooth mapping $\delta : C^\infty(M) \to C^\infty(M)$ that satisfies:*

(1) $\delta(f + \lambda g) = \delta(f) + \lambda\delta(g)$
(2) $\delta(fg) = f\delta(g) + g\delta(f)$

for all $f, g \in C^\infty(M), \lambda \in \mathbb{R}$.

A smooth vector field $\mathbf{v}$ on an open subset U of an n-dimensional manifold M is a smooth assignment of a tangent vector v to each point $p \in U$; we can define a covector field analogously. There is a one-to-one correspondence between smooth vector fields and smooth derivations: given a smooth vector field $\mathbf{v}$, we construct a derivation by $\delta_{\mathbf{v}}(f) = v \cdot \nabla(f)$[1], or in a local coordinate chart $\mathbf{x}$:

$$\delta_{\mathbf{v}}(f) = \sum_{i=1}^{n} v_i \frac{\partial f}{\partial x_i}$$

where $v = (v_1, \ldots, v_n) = \mathbf{v}(p)$. For the opposite identification, we have given a derivation δ, that $\delta(\mathbf{x})$ provides a vector field in local coordinates $\mathbf{x}$.

Given two smooth vector fields $\mathbf{u}, \mathbf{v}$, we can construct a derivation by

$$\delta_{[\mathbf{u},\mathbf{v}]}(f) = \delta_{\mathbf{u}}\left(\delta_{\mathbf{v}}(f)\right) - \delta_{\mathbf{v}}\left(\delta_{\mathbf{u}}(f)\right)$$

If we use our above identification between derivations and vector fields, then we can write out the vector field $[\mathbf{u}, \mathbf{v}]$ directly in terms of $\mathbf{u}, \mathbf{v}$. The kth component of $[\mathbf{u}, \mathbf{v}]$ in local coordinates $\mathbf{x}$ would be

$$[\mathbf{u}, \mathbf{v}]_k = \sum_{j=1}^{n} u_j \frac{\partial v_k}{\partial x_j} - \sum_{j=1}^{n} v_j \frac{\partial u_k}{\partial x_j}$$

Definition 2.4. *The operation $[\mathbf{u}, \mathbf{v}]$ is called the* Lie Bracket *of the vector fields $\mathbf{u}, \mathbf{v}$.*

[1] We actually must be careful here, as we need the additional structure of a Riemannian metric to talk about the gradient of f

Let $\phi : M \to N$ be a smooth map with $\phi(x) = y$, and $\phi_* : TM_x \to TN_y$ be the differential of ϕ.

Definition 2.5. *The* pullback ϕ^* *of* ϕ *at* x *is the linear transformation taking covectors at* $\phi(x) = y$ *to covectors at* x

$$\phi^* : T^*N_y \to T^*M_x$$

$$\phi^*(\alpha)(v) := \alpha(\phi_*(v))$$

for all covectors $\alpha \in T^*N_y$ *and vectors* $v \in TM_x$*.*

If ϕ is not one-to-one, then ϕ_* will not push forward a vector field $\mathbf{v}$ in M to a well-defined vector field in N. However the pull back ϕ^* will take a covector field α on N to a well-defined covector field $\phi^*(\alpha)$ on M. This motivates the following result:

Theorem 2.6. *There exists a globally defined 1-form* λ *on every cotangent bundle* T^*M *called the Poincaré 1-form . In local coordinates* $(\mathbf{q}, \mathbf{p})$ *it is given by*

$$\lambda = \sum_i p_i dq^i$$

For a proof, we refer the reader to Frankel [4]. We can interpret the Poincaré 1-form as follows: let $a \in T^*M$, so $a = (x, \alpha)$ is some 1-form α at a point $x \in M$. Let $\pi : T^*M \to M$ be the projection map, so that $\pi(a) = x$. The pullback $\pi^*\alpha$ gives a one form at each $\pi^{-1}(x)$, specifically at a. The Poincaré 1-form λ at a is exactly this form $\pi^*\alpha$.

2.2. Lie Groups.

Definition 2.7. *A* Lie group *is smooth manifold* G *with a group action* $G \times G \to G$ *such that the group action and the inversion map* $g \to g^{-1}$ *are differentiable.*

Matrix groups under matrix multiplication are common examples of Lie groups, and in particular where our attention will lie.

Definition 2.8. *A* Lie algebra *is a vector space* $\mathcal{G}$ *with a bilinear operation* $[\cdot, \cdot]$*, called the* Lie bracket*, that satisfies the following properties:*

(1) Skew-symmetry $[X, Y] = -[Y, X]$
(2) Jacobi Identity $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

for all $X, Y, Z \in \mathcal{G}$ *and scalars* α, β*.*

For matrix groups, the Lie bracket operation is the commutator $[A, B] = AB - BA$.

Remark 2.9. *The Lie algebra* $\mathcal{G}$ *associated to a Lie group* G *is the tangent space* TG_e *to the identity element.*

One can move between the Lie algebra and the Lie group via the following result:

Theorem 2.10. *The exponential map* $\exp : \mathcal{G} \to G$*,* $\exp(A) = e^A$ *is a diffeomorphism of some neighborhood of* $0 \in \mathcal{G}$ *onto a neighborhood of the identity* $e \in G$*.*

For a proof of this result, we refer the reader to [4].

3. Rigid Body Mechanics

3.1. **Preliminary Setup.** We want to consider the general case of a rigid body D moving in 3-space. Motion will be described by a point $\mathbf{x} \in \mathbb{R}^3$ in space, and how that point moves in space in time t, which can be described by a rotation matrix $R \in SO(3)$, a real matrix such that $R^T = R^{-1}$ and $\det(R) = 1$. We assume the motion has an inertial point $\vec{x}_c$ of D that either remains fixed (a pivot point), or moves with constant velocity. Then there exists an instantaneous axis of rotation $\tilde{\omega}$ that passes through this point $\vec{x}_c$. In this inertial frame, any point $\vec{x} \neq \vec{x}_c$ then moves with instantaneous speed ωr_x, where r_x is the minimum distance from $\vec{x}$ to the instantaneous axis of rotation $\tilde{\omega}$. The instantaneous angular velocity vector $\vec{\omega}$ is then the vector of magnitude ω positioned along $\tilde{\omega}$. We note these quantities all have a time dependence.

We have two orthogonal coordinate systems that we wish to move between: an inertial system which we call the *space frame*, and an non-inertial coordinate system that is fixed with D and moves with the body, appropriately called the *body frame*. We will denote a vector belonging to the *space frame* with $'$, and will temporarily drop the arrow notation to denote a vector. Then the linear transformation that allows us to move between the body and space frames will be given by rotation matrix:

$$x' = x'_c + Rx \tag{1}$$

where x_c is the position vector of the inertial point in the space frame. This constraint between the two frames gives rise to our *configuration manifold*, meaning the space in which the motion lives and on which we wish to describe our dynamics. We need to describe how a position vector $\vec{x} \in D$ in the body frame moves to the space frame by $R \in SO(3)$ the group of 3-dimensional rotation matrices. To describe the motion we will need to describe R and $\dot{R}$, so our configuration manifold is $TSO(3) = SO(3) \times \mathbb{R}^3$.

Let ρ be our mass density function, and then the kinetic energy of the system will be given by

$$T = \frac{1}{2}\int_D \rho(x')\|\dot{x'}\|^2 dx' \tag{2}$$

Changing the integration variable from x' to x (with the determinant of the Jacobian $\det R = 1$), we have

$$T = \frac{1}{2}\int_D \rho(x)\|\dot{x'}_c + \dot{R}x\|^2 dx \tag{3}$$

$$= \frac{1}{2}\|\dot{x'}\|^2 \underbrace{\int_D \rho dx}_{\text{mass=M}} + \underbrace{2\rho\dot{x'}\dot{R}\int x dx}_{=0} + \frac{1}{2}\rho\int \operatorname{tr}\left(\dot{R}xx^T\dot{R}^T\right) dx \tag{4}$$

This second integral is zero as we are integrating density over the center of mass. Let K be the matrix given by $K_{kl} = \int_D \rho x_k x_l dk$. The kinetic energy then simplifies to

$$T = \frac{1}{2}M\dot{x'}^2 + \frac{1}{2}\operatorname{tr}(\dot{R}K\dot{R}^T) \tag{5}$$

3.2. **Lagrangian Formulation.** We consider the Lagrangian $\mathcal{L}(q,\dot{q})$, a time dependent function that lives on the configuration manifold, itself a tangent bundle TM. This is often of the form (and will be for our case)

$$\mathcal{L}(q,\dot{q}) = T(q,\dot{q}) - V(q) \tag{6}$$

where T is kinetic energy and V is potential energy. We have already found T, and for V we assume that the potential energy can be decomposed into a piece that is dependent only on the inertial point $\vec{x}_c$, and a piece that is dependent on R, an assumption that restricts the set of rigid body problems we consider, but does apply to a rigid body rotating about a pivot point, as well as the problem of intereset, Euler's disk. We also need to impose a Lagrange multiplier Λ to handle the additional constraint that $R \in SO(3)$, so that $R^T R = I$. This gives us

$$\mathcal{L} = \underbrace{\frac{1}{2}M\dot{x'}^2 + \frac{1}{2}\mathrm{tr}(\dot{R}K\dot{R}^T)}_{\text{kinetic energy}} - \underbrace{U(r') - V(R)}_{\text{potential energy}} + \underbrace{\mathrm{tr}(\Lambda R^T R - \Lambda)}_{\text{Lagrange multiplier term}} \tag{7}$$

3.3. **Euler-Lagrange Equation.** We want to motivate the Euler-Lagrange equation using Hamilton's variational principle. Let $\gamma : [a, b] \to \mathbb{R}^2$ be a parameterized curve, and consider integrating the Lagrangian of this curve over $[a, b]$:

$$A(\gamma) = \int_a^b \mathcal{L}(\gamma, \dot{\gamma}, t)dt \tag{8}$$

We want to find an "optimal" such γ, subject to any constraints that may be imposed by the Lagrangian. We write $\gamma + \epsilon\eta$, where $\eta : [a, b] \to \mathbb{R}^2$ is our *variation*, as in Figure 1. We want the value of $\gamma + \epsilon\eta$ to agree with γ at the endpoints, so we only consider η that satisfy $\eta(a) = \eta(b) = 0$. We now consider $A(\gamma + \epsilon\eta)$, take the derivative with respect to ϵ, and

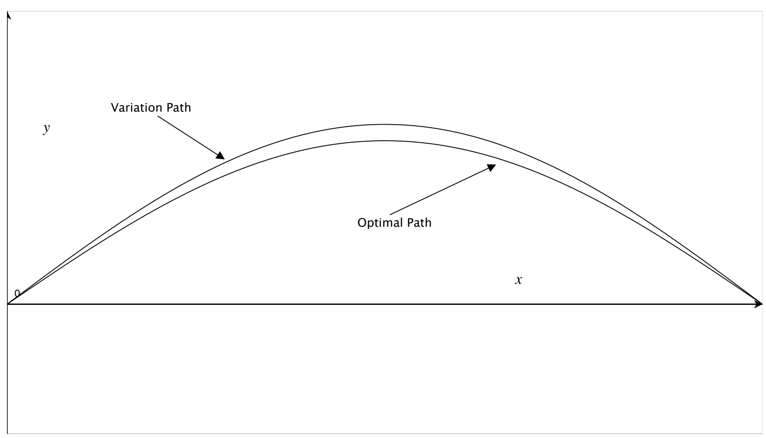

FIGURE 1. An optimal γ, and a variation $\gamma + \epsilon\eta$

evaluate at $\epsilon = 0$ to make use of γ being an "optimal" function or path from a to b:

$$\left.\frac{dA}{d\epsilon}\right|_{\epsilon=0} = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \int_a^b \mathcal{L}\left(\gamma + \epsilon\eta, \dot{\gamma} + \epsilon\dot{\eta}, t\right) dt \tag{9}$$

$$= \int_a^b \left[\frac{\partial \mathcal{L}}{\partial \gamma}\eta + \frac{\partial \mathcal{L}}{\dot{\gamma}}\dot{\eta}\right] dt \tag{10}$$

$$= \int_a^b \left[\frac{\partial \mathcal{L}}{\partial \gamma} + \frac{d}{dt}\frac{\partial \mathcal{L}}{\dot{\gamma}}\right] \eta dt \tag{11}$$

Now we want this quantity to be 0 for all admissible η, so therefore we need

$$\frac{\partial \mathcal{L}}{\partial \gamma} + \frac{d}{dt}\frac{\partial \mathcal{L}}{\dot{\gamma}} = 0 \tag{12}$$

This is the general Euler-Lagrange equation. For our setting of rigid body motion, we treat R as our coordinates. We then differentiate the trace of matrices by the rules $\frac{\partial \mathrm{tr}(AB)}{\partial A} = B^T$, $\frac{\partial \mathrm{tr}(AB)}{\partial A^T} = B$, and using the symmetry of K, we have

$$\frac{\partial L}{\partial \dot{R}} = \frac{1}{2}\frac{\partial \mathrm{tr}(\dot{R}K\dot{R}^T)}{\partial \dot{R}} = \frac{1}{2}[(K\dot{R}^T)^T + \dot{R}K] = \dot{R}K \tag{13}$$

Then taking a time derivative, we have $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{R}} = \ddot{R}K$. We then have

$$\frac{\partial \mathcal{L}}{\partial R} = -\frac{\partial V}{\partial R} + \frac{\partial \mathrm{tr}(\Lambda R^T R - \Lambda)}{\partial R} = -\frac{\partial V}{\partial R} + 2RV \tag{14}$$

Then our Euler-Lagrange equation is

$$\ddot{R}K + \frac{\partial V}{\partial R} = 2RV \quad \Rightarrow \quad R^T\ddot{R}K + R^T\frac{\partial V}{\partial R} = 2\Lambda \tag{15}$$

Since Λ is symmetric, we can eliminate this term by subtracting the transpose of both sides.

$$R^T\ddot{R}K + R^T\frac{\partial V}{\partial R} - (R^T\ddot{R}K)^T - (R^T\frac{\partial V}{\partial R})^T = 0 \tag{16}$$

$$\Rightarrow R^T\ddot{R}K - K\ddot{R}^T R = \frac{\partial V}{\partial R^T}R - R^T\frac{\partial V}{\partial R} \tag{17}$$

It will be useful to write this in terms of $\Omega = R^T\dot{R}$, a skew-symmetric element of the Lie algebra of $SO(3)$, in which case the Euler-Lagrange becomes

$$K\dot{\Omega} + \dot{\Omega}K + \Omega^2 K - K\Omega^2 = \frac{\partial V}{\partial R^T}R - R^T\frac{\partial V}{\partial R} \tag{18}$$

Remark 3.1. *By the association of a skew-symmetric matric Ω with a vector $\omega \in \mathbb{R}^3$, this Ω corresponds to the angular velocity vector ω in our rigid body motion.*

3.4. **Hamiltonian Formulation.** We briefly mention the Hamiltonian $\mathcal{H}(q,p) = p_i\dot{q}^i - \mathcal{L}(q,\dot{q})$ formulation, an alternative formulation on which to describe dynamics. The geometric object on which the Hamiltonian lives is the cotangent bundle T^*M, the phase space, and satisfies Hamilton's equations

$$\frac{dq^i}{dt} = \frac{\partial \mathcal{H}}{\partial p_i} \qquad \frac{dp_i}{dt} = -\frac{\partial \mathcal{H}}{\partial q^i} \tag{19}$$

as well as Hamilton's equations

$$\dot{q}^\beta = \frac{\partial \mathcal{H}}{\partial p_\beta} \qquad \dot{p}_\beta = -\frac{-\partial \mathcal{H}}{\partial q^\beta} \tag{20}$$

where p_α is the generalized momentum $p_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha}$

4. Euler's Disk Problem

4.1. **Problem Setup.** We now consider the motion of a spinning disk. We want to determine a rotation matrix R that will allow us to move from the space frame, for which we will use orthogonal basis vectors $\vec{e}_x, \vec{e}_y, \vec{e}_z$, to the body frame of the disk, for which we'll use a basis $\vec{i}, \vec{j}, \vec{k}$, where $\vec{i}, \vec{j}$ lie in the plane of the disk, and $\vec{k}$ is the outward normal. That is, we want an R such that $\vec{x} = R\vec{x}' + \vec{x}_c$, where $\vec{x}$ is a vector in the space frame, $\vec{x}'$ is in the body frame, and $\vec{x}_c$ is the center of mass of the disk. We will parameterize a point on the disk by θ. If we call the radius of the disk one unit, then the vector $\vec{b} = \cos\theta\vec{i} + \sin\theta\vec{j}$ will be a point on the edge of the disk. Let α be the angle the disk makes with the surface on which it is spinning, and parameterize the path traced by the point in contact with the surface by ϕ. Then we can represent $\vec{b}$ in the space frame, along with the orthogonal vectors $\vec{n}$ and $\vec{b} \times \vec{n}$:

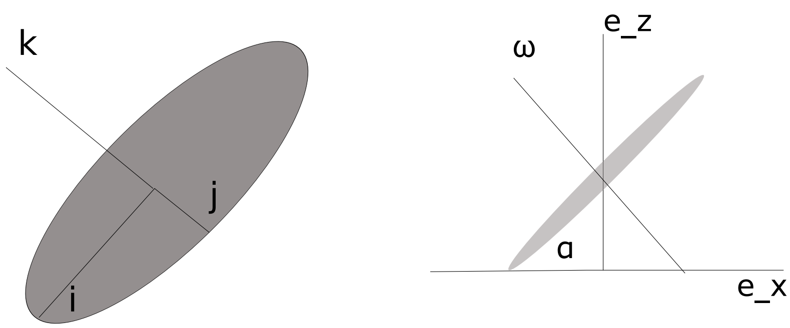

Figure 2. Coordinates in the body (*left*) and space (*right*) frame. $\vec{e}_y$ is in the plane orthogonal to $\vec{e}_x$, and $\vec{k}$ is meant to be the outward normal to the disk

$$\begin{array}{ccccc} \cos\theta\vec{i} + \sin\theta\vec{j} & = & \vec{b} & = & \cos\phi\cos\alpha\vec{e}_x + \sin\phi\cos\alpha\vec{e}_y - \sin\alpha\vec{e}_z \\ -\sin\theta\vec{i} + \cos\theta\vec{j} & = & \vec{n} \times \vec{b} & = & -\sin\phi\vec{e}_x + \cos\phi\vec{e}_y \\ \vec{k} & = & \vec{n} & = & \sin\alpha\cos\phi\vec{e}_x + \sin\alpha\sin\phi\vec{e}_y + \cos\alpha\vec{e}_z \end{array}$$

Now ignoring the $\vec{b}, \vec{n}, \vec{n} \times \vec{b}$ vectors, we see a rotation matrix in θ on the left, and a product of rotation matrices in ϕ, α on the left. Calculating our rotation matrix R, we get:

$$R = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{21}$$

$$= \begin{pmatrix} \cos\phi\cos\alpha\cos\theta + \sin\phi\sin\theta & \cos\phi\cos\alpha\sin\theta - \sin\phi\cos\theta & \cos\phi\sin\alpha \\ -\cos\phi\sin\theta + \sin\phi\cos\alpha\cos\theta & \cos\phi\cos\theta + \sin\phi\cos\alpha\sin\theta & \sin\phi\sin\alpha \\ -\sin\alpha\cos\theta & -\sin\alpha\sin\theta & \cos\alpha \end{pmatrix} \tag{22}$$

We now can use R to calculate the skew-symmetric $\Omega = R^T\dot{R}$

$$\Omega = \begin{pmatrix} 0 & \dot{\theta}\cos\alpha - \dot{\phi} & \dot{\alpha}\cos\phi - \dot{\theta}\sin\phi\sin\alpha \\ \dot{\phi} - \dot{\theta}\cos\alpha & 0 & \dot{\alpha}\sin\phi + \dot{\theta}\cos\phi\sin\alpha \\ \dot{\theta}\sin\phi\sin\alpha - \dot{\alpha}\cos\phi & -\dot{\alpha}\sin\phi - \dot{\theta}\cos\phi\sin\alpha & 0 \end{pmatrix} \tag{23}$$

Now through the identification of an anti-symmetric matrix with a vector, we have the Ω corresponds to

$$\omega = \begin{pmatrix} -\dot{\alpha}\sin\phi - \dot{\theta}\cos\phi\sin\alpha \\ \dot{\alpha}\cos\phi - \dot{\theta}\sin\phi\sin\alpha \\ \dot{\phi} - \dot{\theta}\cos\alpha \end{pmatrix} \tag{24}$$

where ω is the angular velocity vector (in the body frame).

From our Lagrangian for rigid body motion, we had the assumption that we could decompose potential energy into $U(r')$, which depended only on the inertial point r', and $V(R)$, which depended on our coordinates R. There will be no potential energy dependence on r', and $V(R)$ will be mg times the height of the center of mass, which is $\sin\alpha$. Representing $V(R)$ in terms of the entries of R, we have the $3,3$ entry is $\cos\alpha$, so we can use $V(R) = \sqrt{1 - R_{33}^2}$. Then differentiating we have

$$V = \sqrt{1 - R_{33}^2} \tag{25}$$

where we have normalized out mg. Since $V(R)$ depends only on a diagonal entry of R, we have

$$\frac{\partial V}{\partial R} = \frac{\partial V}{\partial R^T} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{R_{33}}{\sqrt{1-R_{33}^2}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\cot\alpha \end{pmatrix} \tag{26}$$

Lastly we need to calculate the matrix K of second moments of mass distribution, whose entries are $K_{mn} = K_{nm} = \int_D \mu(\vec{x}) x_m x_n d\vec{x}$. Since we are assuming uniform mass distribution on our disk D, the entries off the diagonal will be 0. Moreover if we assume the disk is infinitesimally thin, the third row and column will be identically 0. So with $x_1 = r\cos\theta, x_2 = r\sin\theta$, we have

$$K_{11} = \int_D x_1^2 d\vec{x} = \int_0^{2\pi}\int_0^1 r^2\cos^2\theta r dr d\theta = \frac{\pi}{4} = \int_D x_2^2 d\vec{x} = K_{22} \tag{27}$$

$$\Rightarrow K = \begin{pmatrix} \pi/4 & 0 & 0 \\ 0 & \pi/4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{28}$$

4.2. **No Slip Condition.** We now want to focus our attention on the point on the disk in contact with the surface, so $\vec{x}_p' = (\cos\theta, \sin\theta, 0)^T$. We'll first consider the case where the center of mass is fixed. The *no slip condition* is then that the instantaneous velocity of this point (with respect to the space frame) is zero: $\vec{0} = \dot{\vec{x}}_p = \dot{R}\vec{x}_p'$. This gives us

$$\dot{R}\vec{x}_p' = \begin{pmatrix} -\dot{\alpha}\cos\phi\sin\alpha + \dot{\theta}\sin\phi - \dot{\phi}\sin\phi\cos\alpha \\ -\dot{\theta}\cos\phi + \dot{\phi}\cos\phi\cos\alpha - \dot{\alpha}\sin\phi\sin\alpha \\ -\dot{\alpha}\cos\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{29}$$

If $\alpha \neq 90^\circ$, we have $\dot{\alpha} = 0$, which then gives us $\cos\phi(\dot{\phi}\cos\alpha - \dot{\theta}) = 0$, so we have the relation

$$\dot{\theta} = \cos\alpha\dot{\phi} = \beta \tag{30}$$

for some constant β. Integrating gives us

$$\theta = \beta t \qquad \phi = \frac{\beta}{\cos\alpha} t \tag{31}$$

Equations 31 tell us that the point on the disk in contact moves linearly with time with speed β, and the path traced by the point on the surface also moves linearly with speed $\beta/\cos\alpha$, a rate faster than the point moving on the disk.

The condition that $\dot{\alpha} = 0$ says the angle between the disk and the surface should not change in time, which does not agree with the obvious observation that the spinning disk eventually settles on the surface.

If $\alpha = 90°$, the above equations give us that $\dot{\theta} = 0$, which is consistent since then the disk is spinning on its side, so the point in contact on the disk would remain fixed.

If we now look at the more general case where we add in the center of mass $\vec{x}_c = (x_c, y_c, \sin\alpha)^T$ and it is free to move, so that $\dot{\vec{x}}_c$ is not assumed to be $\vec{0}, \dot{R}\vec{x}'_p + \dot{\vec{x}}_c = \vec{0}$ gives us two equations in five unknowns, which are

$$\dot{x}_c = \dot{\alpha}\cos\phi\sin\alpha - \dot{\theta}\sin\phi + \dot{\phi}\sin\phi\cos\alpha \tag{32}$$

$$\dot{y}_c = \dot{\alpha}\sin\phi\sin\alpha + \dot{\theta}\cos\phi - \dot{\phi}\cos\phi\cos\alpha \tag{33}$$

If we consider the case where the coin is spinning on its side, $\dot{\theta} = \dot{\alpha} = 0, \alpha = 90°$, then from 32, 33 we get $\dot{x_c} = \dot{y_c} = 0$, which would indicate the coin would remain in one location as it spins, whereas in observation the coin certainly processes about the surface before toppling over.

4.3. **Euler-Lagrange Equation.** Using our relations (31), the Euler-Lagrange equation (18) becomes considerably simpler, and we can calculate the angular velocities in the space frame, $\dot{\phi}$ (the velocity of the point in contact moving on the circle traced on the surface), and the body frame, ω (24). The Ω_s term (the subscript is to denote these restrictions) becomes

$$\Omega_s = \begin{pmatrix} 0 & 0 & -\beta\tan\alpha\sin\theta \\ 0 & 0 & \beta\tan\alpha\cos\theta \\ \beta\tan\alpha\sin\theta & -\beta\tan\alpha\cos\theta & 0 \end{pmatrix} \tag{34}$$

Then the Euler-Lagrange reduces to

$$K\dot{\Omega}_s + \dot{\Omega}_s K = \frac{\partial V}{\partial R^T}R - R^T\frac{\partial V}{\partial R} \tag{35}$$

We note the $\Omega_s^2 K - K\Omega_s^2$ term has reduced to 0. This gives us:

$$\begin{pmatrix} 0 & 0 & -\frac{1}{4}\beta^2\cos\theta\tan\alpha \\ 0 & 0 & -\frac{1}{4}\beta^2\pi\sin\theta\tan\alpha \\ \frac{1}{4}\beta^2\cos\theta\tan\alpha & \frac{1}{4}\beta^2\pi\sin\theta\tan\alpha & 0 \end{pmatrix} \tag{36}$$

$$= \begin{pmatrix} 0 & 0 & -\cos\alpha\cos\theta \\ 0 & 0 & -\cos\alpha\sin\theta \\ \cos\alpha\cos\theta & \cos\alpha\sin\theta & 0 \end{pmatrix} \tag{37}$$

So for this equation to be consistent, we

$$\cos\alpha - \frac{1}{4}\beta^2\pi\tan\alpha = 0 \tag{38}$$

$$\Rightarrow \beta^2 = \frac{4}{\pi}\frac{\cos\alpha}{\tan\alpha} \tag{39}$$

Plugging β into $\dot{\phi} = \beta/\cos\alpha$, we have

$$\dot{\phi}^2\sin\alpha = \frac{4}{\pi} \tag{40}$$

which is in agreement with Moffatt's [6] calculations. Our angular velocity ω_s under these conditions is

$$\omega_s = \begin{pmatrix} \beta\cos\phi\sin\alpha \\ -\beta\sin\phi\sin\alpha \\ \frac{\beta}{\cos\alpha} - b\cos\alpha \end{pmatrix} \tag{41}$$

$$= 2\sqrt{\frac{\sin\alpha}{\pi}} \begin{pmatrix} \cos\alpha\cos\phi \\ -\cos\alpha\sin\phi \\ 1 \end{pmatrix} \tag{42}$$

5. Geometric Numerical Integration

5.1. **Cayley Transform.** One issue with performing numerical work on this model is the two disparate time scales; the frequency of the spinning disk is quite fast, compared to the slow rate at which the disk settles on the surface, and will result in a stiff system of matrix differential equations. There is also the difficulty of working on our configuration manifold $TSO(3)$; it is very easy for round off or truncation error to quickly move us out of the space of rotation matrices with the numerically unstable properties that $R^T = R^{-1}$ and $\det(R) = 1$. Instead of parameterizing $SO(3)$ by the angles α, ϕ, θ that we used to define our rotation matrices, we instead wish to parameterize the coordinates on our manifold by something that lives in a vector space. By the natural association of the Lie algebra to a Lie group, we wish to instead work with $so(3)$.

Definition 5.1. *The* Cayley transform *is a map from skew-symmetric matrices (Q) to special orthogonal matrices (R), given by*

$$R = (I-Q)(I+Q)^{-1} \tag{43}$$

So in our setting, the Cayley transform is an alternative way to move from the Lie algebra $so(3)$ to our Lie group $SO(3)$. We note that when Q has an eigenvalue approaching -1, the transform becomes singular. To get around this, one could do a similar transform of $(I+Q)(I-Q)^{-1}$.

We want to compute $\dot{Q}$:

$$R(I+Q) = I - Q \quad \Rightarrow \quad \dot{R} + \dot{R}Q + R\dot{Q} = -\dot{Q} \tag{44}$$

$$\Rightarrow \quad \dot{Q} = -(I+R)^{-1}\dot{R}(I+Q) \tag{45}$$

Making use of $\dot{R} = R\Omega = (I-Q)(I+Q)^{-1}\Omega$, we have

$$\dot{Q} = -\left(I + (I-Q)(I+Q)^{-1}\right)^{-1}(I-Q)(I+Q)^{-1}\Omega(I+Q) \tag{46}$$

$$= -\left(2(I+Q)^{-1}\right)^{-1}(I-Q)(I+Q)^{-1}\Omega(I+Q) \tag{47}$$

Now we can commute the $(I-Q)(I+Q)^{-1}$ term to get

$$\dot{Q} = -\frac{1}{2}(I-Q)\Omega(I+Q) = -\frac{1}{2}\left(\Omega + \Omega Q - Q\Omega - Q\Omega Q\right) \tag{48}$$

$$\Rightarrow \quad \dot{Q} = -\frac{1}{2}\left(\Omega + [\Omega, Q] - Q\Omega Q\right) \tag{49}$$

5.2. **Numerical Integration & Dissipation.** (49) gives us one matrix differential equation we wish to solve. We also want to solve for $\dot{\Omega}$ in the Euler-Lagrange equation, however since $K\dot{\Omega}$ do not commute, we use the map

$$\tilde{\dot{\Omega}} \mapsto K\dot{\Omega} + \dot{\Omega}K$$

We need this map to be nonsingular, so that given $\tilde{\dot{\Omega}}$, we can recover $\dot{\Omega}$. If we scale the $\pi/4$ out of K, and label the elements of $\dot{\Omega} \in so(3)$ by

$$\dot{\Omega} = \begin{pmatrix} 0 & -\dot{x_3} & \dot{x_2} \\ \dot{x_3} & 0 & -\dot{x_1} \\ -\dot{x_2} & \dot{x_1} & 0 \end{pmatrix}$$

Then we can easily compute $K\dot{\Omega} + \dot{\Omega}K$:

$$K\dot{\Omega} + \dot{\Omega}K = \begin{pmatrix} 0 & -2\dot{x_3} & \dot{x_2} \\ 2\dot{x_3} & 0 & -\dot{x_1} \\ -\dot{x_2} & \dot{x_1} & 0 \end{pmatrix}$$

which is nonsingular. Then our system of matrix differential equations we wish to solve is

$$\tilde{\dot{\Omega}} = -\Omega^2 K + K\Omega^2 + \frac{\partial V}{\partial R^T} R - R^T \frac{\partial V}{\partial R} \tag{50}$$

$$\dot{Q} = -\frac{1}{2}\left(\Omega + [\Omega, Q] - Q\Omega Q\right) \tag{51}$$

Of course when computing (50), we use the Cayley transform for R, and compute $\frac{\partial V}{\partial R}$ by the transformed entries of R.

We can now easily include a dissipation term by adding $-a\Omega$ to (50), which will result in exponential decay. This corresponds to a decrease in angular momentum, the result of a dissipative mechanism acting on the entire system, such as air viscosity.

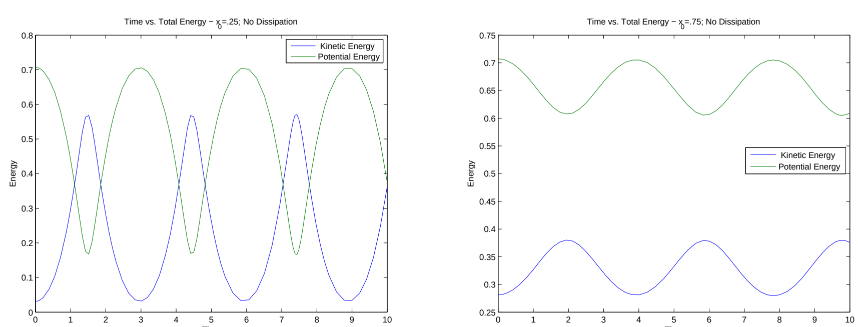

FIGURE 3. Initial conditions resulting in precession. No dissipation has been added

5.3. **Numerical Results.** In Figure 3, we have imposed initial conditions that result in the spinning disk precessing, where $\dot{\alpha}$ would not be constant, but rather α would oscillate in a certain range. In Figure 4, we have imposed initial conditions closer to the steady state where $\dot{\alpha} = 0$, and we see the kinetic and potential energy remain relatively constant. We also note the numerical scheme was cut short, as an eigenvalue of Q was approaching -1. In both of these scenarios, no dissipation was imposed, and we note that energy is not lost

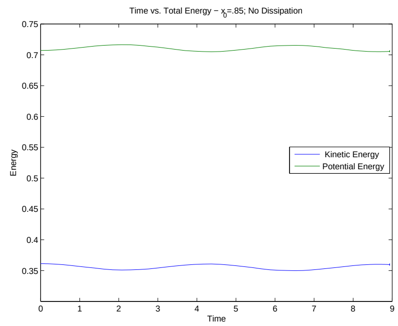

FIGURE 4. Initial conditions resulting in close to steady state, where $\dot{\alpha} = 0$, also without dissipation.

over the time shown. In Figure 5 we show the effects of adding the dissipation term $-a\Omega$ to

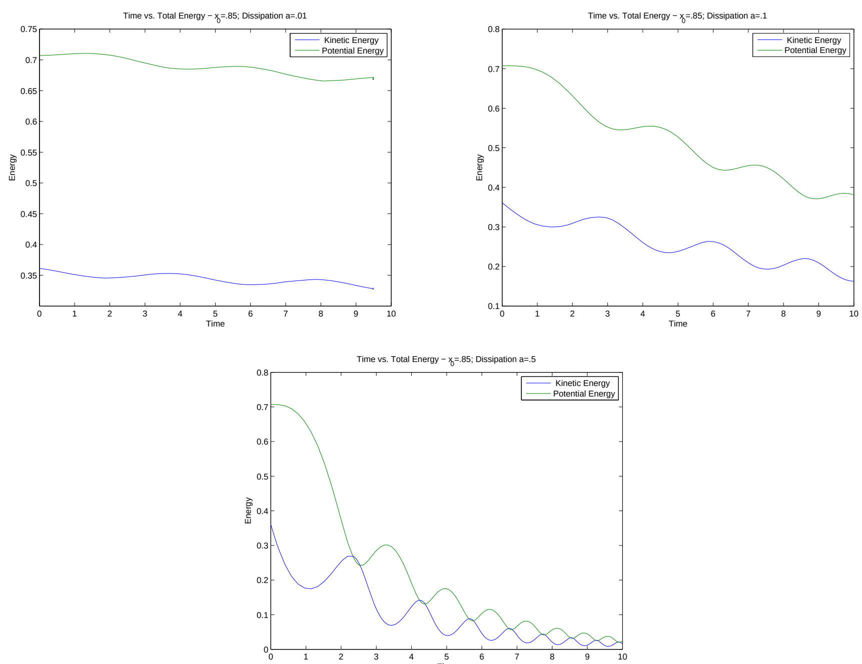

FIGURE 5. Dissipation added to the numerical scheme by $-a\Omega$.

(50), with the initial conditions the same as in 4. We can begin to see the dissipative effects with $a = .01$, and a dramatic loss of energy with $a = .5$.

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