

An Introduction to Geophysical Fluid Dynamics

GFD is a BFD

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Graduate Applied Math Brown Bag Seminar

September 3, 2010

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- Geophysical fluid dynamics is the study of natural, global fluid bodies: ocean and atmosphere.
- Focus is on large-scale attributes, such as hurricanes, eddies, global scale wind and ocean currents.
- GFD differs from classical fluid dynamics in two ways: Earth's rotation (resulting in Coriolis force), and stratification due to density differences.
- The governing equations form basis upon which GCMs (global circulation models) and weather models are built.

Changing Frames of Reference

Two sets of reference frames:

- ① Lagrangian (moving with flow) vs. Eulerian (flow moves by)
- ② rotating vs. inertial

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Laws of physics are expressed in:

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Laws of physics are expressed in:

- ① Lagrangian
- ② inertial

Eulerian to Lagrangian

- Position $\mathbf{X} = (x, y, z) = (x(t), y(t), z(t))$
- Velocity $\mathbf{U} = (u, v, w) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$
- For a function $f(x, y, z, t) = f(x(t), y(t), z(t), t), :$

$$\left(\frac{df}{dt}\right)_L = \left(\frac{\partial f}{\partial t}\right)_E + \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial t}$$

This includes local time rate of change and advective terms.

Material Derivative Operator

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla$$

Rotating to Inertial

In 2 dimensions, rotation by angle ωt (moving from \mathbf{I}, \mathbf{J} to \mathbf{i}, \mathbf{j}) is given by

$$\begin{pmatrix} \mathbf{i} \\ \mathbf{j} \end{pmatrix} = \begin{pmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{pmatrix} \begin{pmatrix} \mathbf{I} \\ \mathbf{J} \end{pmatrix}$$

$$x = X \cos \omega t + Y \sin \omega t$$

$$y = -X \sin \omega t + Y \cos \omega t$$

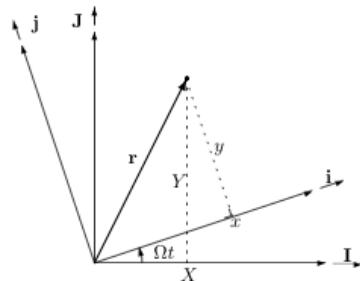


Figure: Changing frames of reference

In 3 dimensions

$$\left(\frac{d}{dt}\right)_I = \left(\frac{d}{dt}\right)_R + \Omega \times$$

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Taking d/dt in rotating frame:

$$\begin{aligned} \dot{x} &= \dot{X} \cos \omega t + \dot{Y} \sin \omega t \\ &\quad -\omega X \sin \omega t + \omega Y \cos \omega t \\ \dot{y} &= -\dot{X} \sin \omega t + \dot{Y} \cos \omega t \\ &\quad -\omega X \cos \omega t + \omega Y \sin \omega t \end{aligned}$$

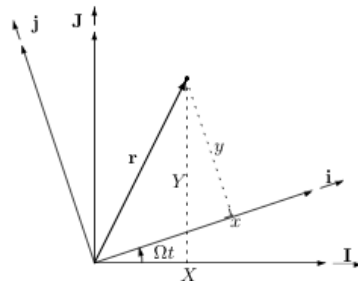


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Taking d/dt in rotating frame:

$$\dot{x} = \dot{X} \cos \omega t + \dot{Y} \sin \omega t + \omega y$$

$$\dot{y} = -\dot{X} \sin \omega t + \dot{Y} \cos \omega t - \omega x$$

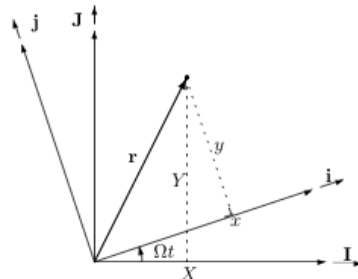


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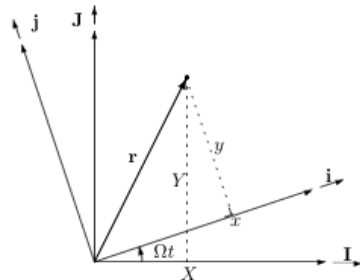
Call \mathbf{u} velocity in moving frame:

$$\mathbf{u} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} = u\mathbf{i} + v\mathbf{j}$$

Call \mathbf{U} velocity in fixed frame:

$$\mathbf{U} = \dot{X}\mathbf{I} + \dot{Y}\mathbf{J} = U\mathbf{I} + V\mathbf{J}$$

Figure: Changing frames of reference



In 3 dimensions

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Rotating to Inertial

In 2 dimensions, rotation by angle ωt (moving from \mathbf{I}, \mathbf{J} to \mathbf{i}, \mathbf{j}) is given by

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Now expressing \mathbf{U} in terms of rotating units \mathbf{i}, \mathbf{j} :

$$\begin{aligned} \mathbf{U} &= (\dot{X} \cos \omega t + \dot{Y} \sin \omega t) \mathbf{i} \\ &+ (-\dot{X} \sin \omega t + \dot{Y} \cos \omega t) \mathbf{j} \end{aligned}$$

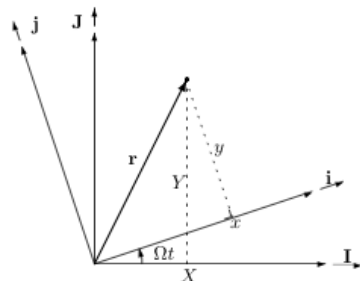


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Velocities between frames are related by

$$U = u - \omega y$$

$$V = v + \omega x$$

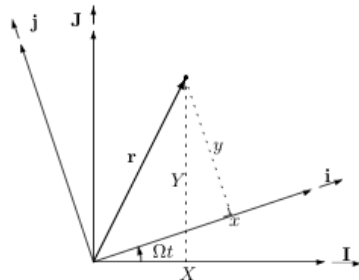


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In 3 dimensions

$$\left(\frac{d}{dt} \right)_I = \left(\frac{d}{dt} \right)_R + \Omega \times$$

Rotating to Inertial

Applying this operator to the position \mathbf{X} :

$$\mathbf{U}_I = \mathbf{U}_R + \boldsymbol{\Omega} \times \mathbf{X}$$

Calculate acceleration (in inertial frame):

$$\begin{aligned} \left(\frac{d\mathbf{U}_I}{dt}\right)_I &= \left(\frac{d}{dt}\right)_I (\mathbf{U}_R + \boldsymbol{\Omega} \times \mathbf{X}) \\ &= \left(\frac{d}{dt}\right)_R (\mathbf{U}_R + \boldsymbol{\Omega} \times \mathbf{X}) + \boldsymbol{\Omega} \times (\mathbf{U}_R + \boldsymbol{\Omega} \times \mathbf{X}) \\ &= \left(\frac{d\mathbf{U}_R}{dt}\right)_R + \boldsymbol{\Omega} \times \left(\frac{d\mathbf{X}}{dt}\right)_R + \boldsymbol{\Omega} \times \mathbf{U}_R + \boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \mathbf{X} \\ &= \left(\frac{d\mathbf{U}_R}{dt}\right)_R + \underbrace{2\boldsymbol{\Omega} \times \mathbf{U}_R}_{\text{Coriolis}} + \underbrace{\boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \mathbf{X}}_{\text{centrifugal}} \end{aligned}$$

Coriolis force

- "Force" resulting from rotating fluid body. For the earth, the magnitude of rotation is $|\Omega| = \omega \approx 7.29 \times 10^{-5} \text{rad/s}$
- Free particles deflect to the right in the northern hemisphere, to the left in southern hemisphere.
- This creates inertial oscillations on the scale of $2\pi/f$, where $f = 2\omega \sin \phi$ is the Coriolis parameter, ϕ is latitude. This is about 12 hours near the poles, approaches ∞ at the equator.

Centrifugal force

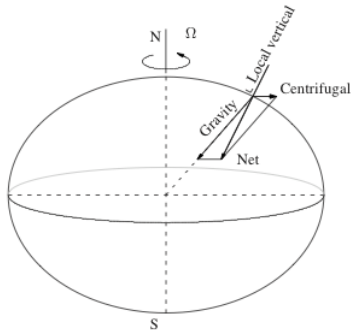


Figure: Centrifugal force points directly outward, distorting spherical shape. Then gravity is not exactly normal to the surface, but the amount of flattening is such that, when centrifugal and gravity forces are combined, the net force is local downward. Centrifugal force is ignored, and gravity is taken as pointing directly down.

Conservation of Momentum

Newton's 2nd Law

$$F = ma$$

$$\rho \left(\frac{d\mathbf{U}_I}{dt} \right)_I = \sum -F_{\text{forces}}$$

$$\rho \left[\left(\frac{d\mathbf{U}_R}{dt} \right) + 2\boldsymbol{\Omega} \times \mathbf{U}_R + \boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \mathbf{X} \right] = -F_{\text{gravity}} - F_{\text{pressure}} + F_{\text{friction}}$$

$$\rho \left[\left(\frac{d\mathbf{U}_R}{dt} \right) + 2\boldsymbol{\Omega} \times \mathbf{U}_R \right] = -\rho\mathbf{g} - \nabla p + F_{\text{friction}}$$

Navier-Stokes (ish)

$$\rho \left(\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} + 2\boldsymbol{\Omega} \times \mathbf{U} \right) = -\rho\mathbf{g} - \nabla p + F$$

Conservation of Mass

So far we have three equations and five unknowns (\mathbf{U}, ρ, p). Now use conservation of mass to get:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{U} + \mathbf{U} \cdot \nabla \rho = 0$$

For the ocean, assuming incompressibility $\nabla \cdot \mathbf{U} = 0$ simplifies the equation to:

$$\frac{\partial \rho}{\partial t} + \mathbf{U} \cdot \nabla \rho = 0$$

Equation of State

Ocean

$$p = \rho_0 [1 + \alpha(S - S_0) - \beta(T - T_0)]$$

where S is the salinity, T is temperature. This says pressure is assumed to be linearly dependent on temperature and salinity.

Atmosphere

$$p = \rho RT$$

This is the ideal gas law.

We're up to five equations, but we've now added a new variables S , T , so we still need more.

Conservation of Energy

Conservation of energy says change in internal energy equals rate of heat gain minus rate of work done:

$$\frac{de}{dt} = Q - W$$

$$e = C_v T \quad Q = \frac{k_T}{\rho} \Delta T \quad W = p \frac{dv}{dt}$$

where C_v is the heat capacity, k_T is thermal conductivity, $v = 1/\rho$ is volume per mass. Rearranging terms gives:

Conservation of Energy

$$\rho C_v \frac{dT}{dt} = k_T \Delta T - \rho \nabla \cdot \mathbf{U}$$

Conservation of Salt/Moisture

Salt content is conserved, but may be redistributed through diffusion, so we have:

$$\frac{dS}{dt} = \kappa_S \Delta S$$

where κ_S is a coefficient of salt diffusion.

Analogously (*waving hands*) for (specific) humidity:

$$\frac{dq}{dt} = \kappa_q \Delta q$$

Boussinesq Approximation

- Assume fluid density $\rho(x, y, z, t)$ does not depart too far from some mean reference ρ_0 , i.e.

$$\rho(x, y, z, t) = \rho_0 + \rho'(x, y, z, t) \quad |p'| \ll \rho_0$$

- For the ocean, density variations typically less than 1%.
- For the atmosphere, density approaches 0 as altitude increases, but much is explained by hydrostatic balance:

$$\frac{\partial p}{\partial z} = -\rho g$$

- Moreover, weather activity confined to the troposphere (up to about 10km), where density variations are still less than 5%.

Boussinesq Approximation

- Set $\rho(x, y, z, t) = \rho_0$
- Conservation of mass becomes conservation of volume:

$$\frac{\partial \rho_0}{\partial t} + \nabla \cdot (\rho_0 \mathbf{U}) = 0 \quad \Rightarrow \quad \nabla \cdot \mathbf{U} = 0$$

- In last (vertical) conservation of momentum equation, need to keep $\rho(x, y, z, t) = \rho_0 + \rho'(x, y, z, t)$, and decompose pressure $p(x, y, z, t) = p_0(z) + p'(x, y, z, t)$, where $p_0(z) = C - \rho_0 g z$:

$$\rho_0 \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} + 2\omega \cos \phi u \right) = -\rho' g - \frac{\partial p'}{\partial z} + F_z$$

- The approximation also affects friction force term F which is explicitly handled in classical fluid mechanics, but largely ignored in GFD.

Summary

Governing Equations in GFD

- Conservation of Momentum:

$$\rho \left(\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} + 2\boldsymbol{\Omega} \times \mathbf{U} \right) = -\rho \mathbf{g} - \nabla p + \mathbf{F}$$

- Conservation of Mass: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0$
- Equation of State

$$p = \rho_0 [1 + \alpha(S - S_0) - \beta(T - T_0)] \quad p = \rho RT$$

- Conservation of Energy: $\rho C_v \frac{dT}{dt} = k_T \Delta T - \rho \nabla \cdot \mathbf{U}$
- Conservation of Salt/Moisture: $\frac{dS}{dt} = \kappa_S \Delta S \quad \frac{dq}{dt} = \kappa_q \Delta q$

References & Acknowledgments

- Cushman-Roisin, Benoit. *Introduction to Geophysical Fluid Dynamics: Physical & Numerical Aspects*
engineering.dartmouth.edu/~cushman/books/GFD.html
- Christopher Jones, University of North Carolina and Warwick Mathematics Institute

Applause

Hey thanks!

Encore

Questions?