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Some Stability Theorems for Systems of Volterra Integral Equations

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Many types of stability have been studied in the theory of ordinary differential equations. Our present purpose is to study perturbed Volterra integral equations (which are a generalization of the initial value problem for ordinary differential equations). As a natural generalization of these concepts for ordinary differential equations, we define stability, uniform stability, and asymptotic stability for integral equations and prove various theorems for linear and perturbed integral equations.

1. INTRODUCTION

In the theory of ordinary differential equations, many types of stability are studied. It is known that in general the usual stability (often called Liapunov stability) and asymptotic stability for linear equations are not preserved under perturbations of a simple and natural type, but that if one introduces a so-called uniform stabiliy then the properties of stability are preserved [2, p. 64]. Our purpose here is to study perturbed Volterra integral equations

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(a generalization of the initial value problem for ordinary differential equations) from this same point of view. Below, as a natural generalization of these concepts for ordinary differential equations, we define stability, uniform stability, and asymptotic stability for integral equations and prove various theorems for linear and perturbed linear equations. We will show that the standard results from the theory of ordinary differential equations generalize to Volterra equations for "initial functions" (i.e., the function φ in (1.1)) on arbitrary normed spaces provided a minimal amount of uniformity is present.

Consider the integral equation

$$u(x) = \varphi(x) + \int_a^x K(x, t, u(t))dt, \tag{1.1}$$

where K, u, and φ are n-vectors, x and t are real variables and $a \geqq x_0$, where x_0 is a fixed number. Let $\|u\|_{0,a} = \sup_{x \geqq a} |u(x)|$, where $|\cdot|$ is any vector norm, and suppose N is a normed space of functions defined on $x \geqq x_0$ with norm $\|\cdot\|$. If $\bar{u}(x)$ is a solution to (1.1) corresponding to $\bar{\varphi}(x) \in N$, then $\bar{u}(x)$ is said to be *stable on* N if to each $\varepsilon > 0$, there exists a $\delta = \delta(a, \varepsilon) > 0$ such that $\|\varphi - \bar{\varphi}\| \leqq \delta$, $\varphi \in N$, implies $\|u - \bar{u}\|_{0,a} \leqq \varepsilon$. If δ above can be found independent of $a \geqq x_0$, then $\bar{u}(x)$ is *uniformly stable* (*U.S.*) *on* N. Finally, if $\bar{u}(x)$ is stable on N and if, in addition, to each $a \geqq x_0$ there exists a $\delta = \delta(a) > 0$ such that $\|\varphi - \bar{\varphi}\| \leqq \delta$ implies $|u(x) - \bar{u}(x)| \to 0$ as $x \to +\infty$, then $\bar{u}(x)$ is said to be *asymptotically stable* (*A.S.*) *on* N.

We emphasize that by stability on a normed space N we always mean stability of the solution $u(x)$ in the sup norm $\|\cdot\|_{0,a}$ regardless of the norm on N. The space N is the space of allowable initial functions. The solution $u(x)$ need not and in general will not be in N.

Our concern here is with the stability of $\bar{u} \equiv 0$ (corresponding the $\bar{\varphi} \equiv 0$) for the systems

$$u(x) = \varphi(x) + \int_a^x A(x, t)u(t)\,dt, \tag{1.2}$$

$$u(x) = \varphi(x) + \int_a^x [A(x, t)u(t) + f(x, t, u(t))]\,dt, \tag{1.3}$$

where $A(x, t)$ is a continuous $n \times n$ matrix on $T = \{(x, t): x_0 \leqq t \leqq x < +\infty\}$, and $f(x, t, z)$ is a continuous n-vector on $T \times \{z: |z| < b\}$ for some $b > 0$ which satisfies $f(x, t, 0) \equiv 0$. Henceforth, we will merely refer to the stability (or U.S. or A.S.) of (1.2) and (1.3) by which we will mean the stability (or U.S. or A.S.) of $\bar{u} \equiv 0$ as defined above.

We restrict our attention here to continuous, but not necessarily differentiable solutions $u(x)$. As a result, although integro-differential equations of the form

$$du/dx = \psi(x) + B(x)u(x) + g(x, u(x)) + \int_a^x [C(x, t)u(t) + h(x, t, u(t))]\,dt \tag{1.4}$$

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can, by integration, be put into the form of (1.3), the converse is not necessarily true.

Recent papers have been concerned with stability and asymptotic behavior of these equations on the particular normed space

$$N_0 = \{\varphi \in C^0[x_0, +\infty): \|\varphi\| = \|\varphi\|_{0,x_0}\};$$

see [3, 4, 7]. Our point of view here is different from that in these papers in that rather than placing sufficient assumptions on the resolvent kernel of (1.2) and on the perturbation term f in order to guarantee stability on N_0 (and, hence, implicitly requiring some "kind" of stability of (1.2)) we instead seek to make quite explicit the assumed stability of the unperturbed system (1.2). Our results will be of the nature of preservation of stability on N from (1.2) to (1.3). As noted above, this of necessity requires the uniform stability of (1.2). Unlike the results of the above mentioned literature our main result (Theorem 3.1) generalizes standard stability results for perturbed ordinary differential equations where A, f are independent of x and φ is a constant. Since in applications a variety of spaces may be appropriate, we give results for stability on N_0 as well as the other spaces listed below which suggest themselves quite naturally from our approach.

$$N_1 = \{\varphi \in C^1[x_0, +\infty): \|\varphi\| = \|\varphi\|_{0,x_0} + \|\varphi'\|_{0,x_0}\},$$

$$N_2 = \{\varphi \in C^1[x_0, +\infty): \|\varphi\| = \|\varphi\|_{0,x_0} + \textstyle\int_{x_0}^{+\infty} |\varphi'|\, ds\},$$

$$N_3 = \{\varphi: \varphi \equiv \text{constant}, \|\varphi\| = |\varphi|\}.$$

In Section 2, we give necessary and sufficient conditions for stability, U.S., and A.S. on N_3 for the linear system (1.2), and we give conditions under which these stabilities on any normed space N are preserved for (1.3) in Section 3. In Section 4 we offer other stability criteria for both (1.2) and (1.3) on the spaces N_0, N_1, and N_2. Finally, in Section 5 some applications are given. Also, it will be seen from the results of Section 2 and Section 4 that the concepts of U.S. on N_3 and of U.S. on N_2 are equivalent.

2. LINEAR SYSTEMS

Our approach will depend upon a "variation of constants" formula as given in the following lemma which can be proved by straightforward substitution.

LEMMA 2.1 *If $U(s, x)$ is an $n \times n$ matrix which solves the matrix equation*

$$U(s, x) = I + \textstyle\int_s^x A(x, t)U(s, t)\, dt$$

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for $a \leqq s \leqq x < +\infty$, *then*

$$u(x) = U(a, x)\varphi(a) + \int_a^x U(s, x)[\varphi'(s) + d/ds \int_a^s B(s, t)\, dt]\, ds$$

is the solution to

$$u(x) = \varphi(x) + \int_a^x [A(x, t)u(t) + B(x, t)]\, dt,$$

provided $B(x, t)$ *is a continuous n-vector possessing a continuous derivative in* x *on* $a \leqq t \leqq x < +\infty$ *and* $\varphi \in C^1(a, +\infty)$.

For a special case of this lemma see [1, p. 229]. The connection between this "variation of parameters" formula and the one used previously in the literature (see [3, 4, 7]) can be found by noticing that $\partial U(s, x)/\partial s = r(x, s)$ where $r(x, s)$ is the resolvent kernel associated with $A(x, t)$ (see [6, p. 10]); this relationship is derived by differentiating the integral equation for $U(s, x)$ after which one obtains the well known equation satisfied by $r(x, s)$. Using this relationship one can relate our results to those in [3, 4, 7]; although often similar they are independent. The advantage of the above formula is that it allows us to characterize stability on N_3 and, hence, to develop stability results for perturbed equations (1.3) in complete analogy with and as a generalization of standard stability results in the theory of ordinary differential equations. It is for these reasons that it seems natural to consider the "fundamental matrix" $U(s, x)$ and uniform stability on the normed space N_3 as defined above.

Throughout this paper we define the matrix norm of a matrix M to be $\|M\| = \sup_{|\xi|=1} |M\xi|$. The main results for the linear equation (1.2) are contained in the following theorem.

THEOREM 2.1 *Let* $U(s, x)$ *be as in Lemma* 2.1

a) The linear system (1.2) *is stable on* N_3 *if and only if there exists a constant* $M = M(a) > 0$ *such that* $\|U(a, x)\| \leqq M$, $x \geqq a$.

b) The linear system (1.2) *is U.S. on* N_3 *if and only if there exists a constant* $M > 0$, *independent of* a, *such that* $\|U(a, x)\| \leqq M$, $x_0 \leqq a \leqq x$.

c) The linear system (1.2) *is A.S. on* N_3 *if and only if* $\|U(a, x)\| \to 0$ *as* $x \to +\infty$ *for all* $a \geqq x_0$.

Remarks 1) We point out that Theorem 2.1 is false if N_3 is replaced by N_0; this is evident from the scalar equation

$$u(x) = \varphi(x) + \int_0^x (t-x)u(t)\, dt$$

for which $U(s, x) = \cos(s-x)$ and for which $u(x) = \frac{1}{2}(\sin x + x\cos x)$ when $\varphi(x) = \sin x$. Notice that this equation is stable on N_3 by Theorem 2.1(a), however. (In fact, by Theorem 4.1 below, this equation is U.S. on N_2.)

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2) In the theory of linear ordinary differential equations the matrix $U(s, x)$ is simply $Y(x)Y^{-1}(s)$, where $Y(x)$ is the fundamental matrix of the associated homogeneous equation with $Y(a) = I$. All of the above results are generalizations of the standard stability characterizations in that theory [2, p. 54].

Proof of Theorem 2.1 a) According to Lemma 2.1, the solution of (1.2) for $\varphi \in N_3$ is given by $u(x) = U(a, x)\varphi(a)$. Clearly, if $\|U(a, x)\| \leqq M(a)$, $x \geqq a$, then $|u(x)| \leqq \|M(a)\| |\varphi(a)|$ and stability follows immediately. Conversely, if to each $\varepsilon > 0$ there exists a $\delta = \delta(a, \varepsilon) > 0$ such that $\|u\|_{0,a} \leqq \varepsilon$ for all $\varphi \in N_3$ with $\|\varphi\| \leqq \delta$, then $|U(a, x)\xi| \leqq \varepsilon$ for all vectors ξ with $|\xi| \leqq \delta$. Let $\xi = \delta\eta$, where η is a unit vector. Then, for all η, $|U(a, x)\eta| \leqq \varepsilon\delta^{-1}$, and hence $\|U(a, x)\| \leqq M(a)$ where $\varepsilon\delta^{-1} = M(a)$.

b) As in (a), the solution to (1.2) with $\varphi \in N_3$ is $u(x) = U(a, x)\varphi(a)$. Clearly, if M is independent of a then $|u(x)| \leqq M|\varphi|$ implies U.S. on N_3. Conversely, the U.S. of (1.2) on N_3 implies δ and, hence, $M = \varepsilon\delta^{-1}$ are independent of a.

c) If $\|U(a, x)\| \to 0$ as $x \to +\infty$ for each $a \geqq x_0$, then $\|U(a, x)\|$ is bounded in x for each $a \geqq x_0$; thus, by part (a), (1.2) is stable on N_3. In this case, since $u(x) = U(a, x)\varphi(a)$ we have $u(x) \to 0$ as $x \to +\infty$ and, hence, (1.2) is A.S. on N_3.

On the other hand, the assumption of A.S. on N_3 leads to the fact that given any positive integer n, there exists a positive number $N = N(n)$ such that $x \geqq N$ implies $|U(a, x)\eta| \leqq 1/n\delta$, provided that η is a unit vector as above. Then, since δ is independent of n, it follows that $\|U(a, x)\| \leqq 1/n\delta$ for $x \geqq N$; i.e., $\|U(a, x)\| \to 0$ as $x \to +\infty$.

3. PERTURBED SYSTEMS

We now consider the system (1.3). Under given hypotheses, we say that the perturbed system (1.3) *preserves* stability on a normed space N provided (1.3) is stable on N whenever (1.2) is stable on N. The following hypotheses will be needed concerning the perturbation term f:

*H*1: $|f(x, x, z)| \leqq \gamma_1(x)|z|$, $x \geqq x_0$, $|z| < b$, where $\gamma_1(x)$ is defined for $x \geqq x_0$ and $\int_{x_0}^{+\infty} \gamma_1(x)\, dx < +\infty$;

*H*2: $f(x, t, z)$ is continuously differentiable in x and $|f_x(x, t, z)| \leqq \gamma_2(x, t)|z|$ on $T \times \{|z| < b\}$, where $\gamma_2(x, t)$ is defined on T and $\int_{x_0}^{+\infty} \int_{x_0}^{s} \gamma_2(s, t)\, dt\, ds < +\infty$.

Under these assumptions, we can state the following general stability theorem for the perturbed system (1.3).

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THEOREM 3.1 *Suppose that the linear system* (1.2) *is U.S. on* N_3 *and that* f *satisfies* $H1$ *and* $H2$.

a) *Then the perturbed system* (1.3) *preserves stability and uniform stability on any normed space* N.

b) *Then the perturbed system* (1.3) *preserves asymptotic stability on any space* N *for which* $N_3 \subseteq N$.

The proof of (a) is an obvious application of the following lemma once it is noticed that Φ as defined there is nothing more than the solution to the linear equation (1.2).

LEMMA 3.1 *Let*

$$\Phi(a, x) \equiv U(a, x)\varphi(a) + \int_a^x U(s, x)\varphi'(s)\, ds.$$

If $f(x, t, z)$ *satisfies* $H1$, $H2$ *and if* (1.2) *is U.S. on* N_3, *then any solution of* (1.3) *for which* $\|\Phi\|_{0,a}$ *is sufficiently small exists for all* $x \geqq a$ *and satisfies* $\|u\|_{0,a} \leqq L\|\Phi\|_{0,a}$, *where* L *is a positive constant independent of* a.

Proof of Lemma 3.1 From our representation formula of Lemma 2.1, we have

$$u(x) = \Phi(a, x) + \int_a^x U(s, x)(d/ds) \int_a^s f(s, t, u(t))\, dt\, ds. \tag{3.1}$$

Then

$$|u(x)| \leqq |\Phi(a, x)| + \int_a^x \|U(s, x)\| \{|f(s, s, u(s))| + \int_a^s |f_s(s, t, u(t))| dt\}\, ds, \tag{3.2}$$

or

$$\begin{aligned} |u(x)| &\leqq |\Phi(a, x)| + M \int_a^x \{\gamma_1(s)|u(s)| + \int_a^s \gamma_2(s, t)|u(t)|\, dt\}\, ds \\ &\leqq |\Phi(a, x)| + M \int_a^x v(s)\gamma(s)\, ds, \end{aligned}$$

where $v(s) = \max_{[a,s]}|u(t)|$ and $\gamma(s) = \gamma_1(s) + \int_a^s \gamma_2(s, t)\, dt$. Therefore, since the integrand on the right hand side of the inequality above is non-negative, we obtain

$$v(x) \leqq |\Phi(a, x)| + M \int_a^x v(s)\gamma(s)\, ds.$$

Thus, using the standard inequality of Gronwall, we obtain for $x \geqq a$

$$|v(x)| \leqq |\Phi(a, x)| + M \int_a^x \gamma(s)|\Phi(a, s)| \exp\left(\int_s^x M\gamma(\tau)\, d\tau\right) ds,$$

or

$$v(x) \leqq \|\Phi\|_{0,a}\left[1 + \int_a^x M\gamma(s) \exp\left(\int_s^x M\gamma(\tau)\, d\tau\right) ds\right].$$

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Now, the bracketed expression on the right is actually the same as $\exp(M\int_a^x \gamma(s)\,ds)$. Hence, it follows that $\|u\|_{0,a} \leqq L\|\Phi\|_{0,a}$ where, using $H1$ and $H2$, we have set

$$L = \exp\left(M \int_{x_0}^{+\infty} \gamma(s)\,ds\right) < +\infty .$$

Also, by the continuation property of solutions to (1.3) [5], it follows that for sufficiently small $\|\Phi\|_{0,a}$ the solution exists on $[a, +\infty)$.

This lemma proves the preservation of stability and U.S. on N, and hence proves part (a) of Theorem 3.1. To prove part (b) we consider inequality (3.2). If (1.2) is A.S. on N, then by definition it is stable on N and, consequently, by part (a) the perturbed system (1.3) is also stable on N. Thus, for $\|\varphi\|$ sufficiently small, u or equivalenty v, as defined in the proof of the lemma, is bounded; i.e., $\|v(x)\|_{0,a} \leqq c$. From (3.2) we find

$$|u(x)| \leqq |\Phi(a, x)| + c\int_a^x \|U(s, x)\|\,\gamma(s)\,ds .$$

Now since $N_3 \subseteq N$, it follows that A.S. on N implies A.S. on N_3 which in turn implies by Theorem 2.1 that $\|U(s, x)\| \to 0$ as $x \to +\infty$ for each $s \geqq x_0$. From U.S. on N_3 we also have $\|U(s, x)\| \leqq M, x_0 \leqq s \leqq x$. Let $x_1 > x$ be so large that

$$\int_{x_1}^{+\infty} \gamma(s)\,ds \leq \varepsilon(cM)^{-1}$$

where ε is any given positive constant. Then we have

$$|u(x)| < |\Phi(a, x)| + c\int_a^{x_1} \|U(s, x)\|\,\gamma(s)\,ds + \varepsilon$$

and Lebesgue's dominated convergence theorem implies

$$\limsup_{x\to+\infty} |u(x)| \leqq \varepsilon$$

inasmuch as the A.S. on N of (1.2) implies $|\Phi(a, x)| \to 0$ as $x \to +\infty$. But $\varepsilon > 0$ was arbitrary so we concluded that

$$\lim_{x\to+\infty} |u(x)| = 0$$

and hence that (1.3) is A.S. on N. This completes the proof of Theorem 3.1.

In the above proof of part (b) the assumption $N_3 \subseteq N$ was used only in that it implies $\|U(a, x)\| \to 0$ as $x \to +\infty$ for $a \geqq x_0$. This condition can also be assured by the outright assumption that the unperturbed linear system (1.2) is A.S. on N_3. Thus, with this slight change in the proof above, the following modification of part (b) of Theorem 3.1 can be proved.

THEOREM 3.2 *Suppose that the linear system* (1.2) *is both U.S. and A.S. on* N_3 *and that* f *satisfies* $H1$ *and* $H2$. *Then the perturbed system* (1.3) *preserves A.S. on any normed space* N.

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As a final remark we note that if A, f are taken to be independent of x and if $N = N_3$, then Theorem 3.1 reduces to a well known stability result for ordinary differential equations [2, p. 64]. That the assumption of uniform stability on N_3 in the above results cannot be dropped is well known from examples in the theory of ordinary differential equations; nor can the hypotheses H1 be weakened to $\gamma_1 \equiv$ constant or even $\gamma_1 \to 0$ as $x \to +\infty$ [2].

4. OTHER STABILITY RESULTS

By reconsidering Lemma 2.1 we may easily obtain stability criteria for (1.2) on other normed spaces in terms of the fundamental matrix $U(s, x)$. This is often advantageous in determining the stability of (1.2) on particular spaces as related to the kernel $A(x, t)$. A subsequent paper will develop this idea in more detail. The following three theorems give sufficient conditions under which the linear system (1.2) is stable on each of the spaces N_0, N_1, and N_2.

THEOREM 4.1 *Let $U(s, x)$ be as in Lemma* 2.1.

a) If there exists a constant $M = M(a) > 0$ such that $\|U(s, x)\| \leqq M$, $a \leqq s \leqq x$, then (1.2) *is stable on N_2.*

b) If there exists a constant $M > 0$, independent of a, such that $\|U(a, x)\| \leqq M$, $x \geqq a \geqq x_0$, then (1.2) *is U.S. on N_2.*

c) If (1.2) *is U.S. on N_3 and $\|U(s, x)\| \to 0$ as $x \to +\infty$ for each $s \geqq a$, then* (1.2) *is A.S. on N_2.*

THEOREM 4.2 *Let $U(s, x)$ be as in Lemma* 2.1.

a) If there exists a constant $K = K(a)$ such that

$$\|U(a, x)\| + \int_a^x \|U(s, x)\| \, ds \leqq K$$

for all $x \geqq a$, then (1.2) *is stable on N_1.*

b) If K is independent of a in part (a), *then* (1.2) *is U.S. on N_1.*

c) If

$$\int_a^x \|U(s, x)\| \, ds + \|U(a, x)\| \to 0 \text{ as } x \to +\infty,$$

then (1.2) *is A.S. on N_1.*

THEOREM 4.3 *Let $U(s, x)$ be as in Lemma* 2.1 *and suppose $U(s, x)$ possesses a continuous partial derivative in s for $a \leqq s \leqq x$.*

a) If there exists a constant K such that

$$\int_{x_0}^x \|(\partial U/\partial s)(s, x)\| \, ds \leqq K$$

for all $x \geqq x_0$, then (1.2) *is U.S. on N_0.*

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b) If

$$\int_a^x \|(\partial U/\partial s)(s, x)\| ds \to 0 \quad \text{as} \quad x \to +\infty,$$

then (1.2) *is A.S. on* $N_0 \cap \{\varphi: |\varphi| \to 0 \text{ as } x \to +\infty\}$.

Proof of Theorem 4.1 (a) For $\varphi \in N_3$, the solution to (1.2) is given by

$$u(x) = U(a, x)\varphi(a) + \int_a^x U(s, x)\varphi'(s)\, ds,$$

from which we have

$$\begin{aligned} |u(x)| &\leqq M(a)[\|\varphi\|_{0,a} + \textstyle\int_a^x |\varphi'(s)| ds] \\ &\leqq M(a)\|\varphi\|. \end{aligned}$$

Therefore, $\|u\|_{0,a} \leqq M(a)\|\varphi\|$, and stability follows.

b) If M is independent of a in the above proof, then uniform stability follows from the same argument.

c) Let M be the uniform bound on $\|U(a, x)\|$ as in Theorem 2.1(b). Then, given $\varepsilon > 0$ there exists a real $x_1 > a$ such that for $\varphi \in Nv$

$$M \int_{x_1}^{+\infty} |\varphi'(s)| ds < \varepsilon.$$

Again using the representation for the solution $u(x)$, we obtain for $x \geqq x_1$

$$|u(x)| \leqq \|U(a, x)\| |\varphi(a)| + \int_a^{x_1} \|U(s, x)\| |\varphi'(s)| ds + \varepsilon.$$

Now, using the hypothesis that $\|U(s, x)\| \to 0$ together with an application of the Lebesgue dominated convergence theorem, we conclude from this inequality that

$$\limsup_{x \to +\infty} |u(x)| \leqq \varepsilon,$$

which, since ε is arbitrary, implies the desired result that $|u(x)| \to 0$ as $x \to +\infty$. This proves Theorem 4.1.

A proof of Theorem 4.2 can be constructed in a very similar manner using the representation formula in Lemma 2.1 and will not be reproduced here.

Proof of Theorem 4.3 An integration by parts in the representation formula for the solution to (1.2) yields

$$u(x) = \varphi(x) - \int_a^x (\partial U/\partial s)(s, x)\varphi(s)\, ds.$$

(To be precise, this formula for the solution $u(x)$ can be directly verified and hence is valid for $\varphi \in N_0$ without any differentiability assumption on φ.) It then easily follows that

$$\|u\|_{0,a} \leqq (1+K)\|\varphi\|_{0,a} \leqq (1+K)\|\varphi\|,$$

from which we conclude (a).

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From the above representation for the solution $u(x)$ we have

$$|u(x)| \leqq |\varphi(x)| + \|\varphi\| \int_a^x \|(\partial U/\partial s)(s, x)\| ds,$$

and hence $|u(x)| \to 0$ as $x \to +\infty$.

Using these criteria for stability we can obtain corollaries of Theorem 3.1 with respect to the stability of the perturbed system (1.3) on N_0, N_1, and N_2.

COROLLARY 4.1 *Suppose* (1.2) *is U.S. on* N_3 *and f satisfies H*1 *and H*2. *Then the perturbed system* (1.3) *is U.S. on* N_2.

COROLLARY 4.2 *Suppose* (1.2) *is both U.S. and A.S. on* N_3 *and that f satisfies H*1 *and H*2. *Then the perturbed system* (1.3) *is both U.S. and A.S. on* N_2.

Since the U.S. on N_3 of (1.2) implies the boundedness in x of $\|U(s, x)\|$ by Theorem 2.1(b) which in turn, by Theorem 4.1, implies the U.S. on N_2 of (1.2), these two corollaries follow from a direct application of Theorem 3.1.

COROLLARY 4.3 *Assume that f satisfies H*1 *and H*2.

a) *If the linear system* (1.2) *is U.S. on* N_3 *and if* $U(s, x)$ *satisfies the hypothesis of Theorem* 4.2(*a*), *then* (1.3) *is stable on* N_1.

b) *If* $U(s, x)$ *satisfies the hypothesis of Theorem* 4.2(*b*), *then* (1.3) *is U.S. on* N_1.

c) *If* $U(s, x)$ *satisfies the hypothesis of Theorem* 4.2(*c*), *then* (1.3) *is A.S. on* N_1.

This corollary also follows immediately from Theorem 3.1 and from Theorem 4.2 once it is noticed that both the hypotheses of Theorem 4.2, parts (b) and (c), individually imply the U.S. of (1.2) on N_3 by virtue of Theorem 2.1.

COROLLARY 4.4 *Assume that f satisfies H*1 *and H*2.

a) *If* $U(s, x)$ *satisfies the hypothesis of Theorem* 4.3(*a*), *then the perturbed system* (1.3) *is U.S. on* N_0.

b) *If* $U(s, x)$ *satisfies the hypothesis of Theorem* 4.3(*c*), *then* (1.3) *is A.S. on* $N_0 \cap \{\varphi : |\varphi| \to 0 \text{ as } x \to +\infty\}$.

Part (a) is immediate from Theorem 3.1 and Theorem 4.3(a). To prove part (b) we need only note that the hypotheses of Theorem 4.3(a) and (b) both individually imply that $\|U(s, x)\|$ is uniformly bounded for $x_0 \leqq s \leqq x$ and consequently, by Theorem 2.1, that (1.2) is U.S. on N_3; indeed,

$$\|U(s, x)\| = \|I - \int_x^s (\partial U/\partial t)(t, x)\, dt\| \leqq 1 + K,$$

for $x_0 \leqq s \leqq x$. Part (b) is now seen to follow from Theorems 3.1 and 4.3(b).

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5. SOME APPLICATIONS

In this section we briefly point out some applications of the above results to some problems which have been of interest in recent literature.

First, equations (1.2) in which $A(x, t) \equiv A(x-t)$ (so called "renewal" equations) are of interest in a great diversity of applications. For such problems it is easily seen that $U(s, x) \equiv U(x-s)$. One way of seeing this is to recall the well known fact that for such kernels the resolvent kernel $r(s, x)$ is a function of the difference $x-s$ [6]; inasmuch as $\partial U/\partial s \equiv r$, as pointed out above, it follows that U is also a function of the difference $x-s$. Consequently stability and uniform stability on N_3 are equivalent for such equations, since by Theorem 2.1 both are equivalent to the boundedness of $U(z)$, $z \geqq 0$. The perturbation theorems of Section 3 then apply to any stable system of this type. For example, from Theorem 3.1 we have the following variant of a theorem of Miller (Theorem 4 in [7], which itself is a generalization of a theorem of Paley and Wiener).

THEOREM 5.1 *Let $A(s, x) \equiv A(x-t)$. Suppose that the linear equation* (1.2) *is stable on N_3 and that f satisfies $H1$ and $H2$. Then the perturbed equation* (1.3) *preserves stability on any normed space N and asymptotic stability on any normed space N for which $N_3 \subseteq N$.*

A theorem of Paley and Wiener (see [8]) states that if $\|A(\cdot)\|$ is integrable on $[0, +\infty)$ (i.e., $\|A\| \in L^1[0, +\infty)$), then $\|r\| \in L^1[0, +\infty)$ if and only if

$$\det\,(I-\textstyle\int_0^{+\infty} \exp\,(-st)A(t)\,dt) \neq 0 \tag{5.1}$$

for all complex s in the right half plane $\operatorname{Re} s \geqq 0$. This theorem is used in [7] to relate the assumption $\|r\| \in L^1[0, +\infty)$ back to the kernel of (1.2). In our case the assumption $\|r\| \in L^1[0, +\infty)$ is, in fact, the assumption that $\|\partial U/\partial s\| \in L^1[0, +\infty)$ which, by Theorem 4.3(a), implies stability on N_0 and hence on N_3. We have then the following result.

COROLLARY 5.1 *Let $A(x, t) \equiv A(x-t)$ and suppose $\|A\| \in L^1[0, +\infty)$. Also assume that f satisfies $H1$ and $H2$. If the Paley–Wiener condition* (5.1) *holds, then the perturbed system* (1.3) *is stable on N_0. If, in addition to $\varphi \in N_0$, we have that $|\varphi(x)| \to 0$ as $x \to +\infty$, then $|u(x)| \to 0$ as $x \to +\infty$ where $u(x)$ is any solution to* (1.3).

We have only to prove the last statement. Inasmuch as $\partial U(s, x)/\partial s \equiv r(x-s)$ we have from Lemma 2.1 (after an integration by parts)

$$\begin{aligned} u(x) &= \varphi(x)-\textstyle\int_a^x r(x-s)\varphi(s)\,ds \\ &= \varphi(x)-\textstyle\int_0^{x-a} r(s)\varphi(x-s)\,ds. \end{aligned}$$

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Condition (5.1) and $\varphi \in N_0$ imply that the integrand above is in $L^1[0, +\infty)$. Hence, there exists, for each $\varepsilon > 0$, a real $x_1 > a$ such that

$$\int_{x_1}^{+\infty} \|r(s)\| ds \leqq \varepsilon c^{-1}$$

where $c = \|\varphi\|_{0,x_0}$. Writing, for large x

$$u(x) = \varphi(x) - \int_0^{x_1-a} r(s)\varphi(x-s)\, ds - \int_{x_1-a}^{x-a} r(s)\varphi(x-s)\, ds$$

we find, upon applying Lebesgue's dominated convergence theorem to the middle term, that

$$\limsup_{x\to+\infty} |u(x)| \leqq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $|u(x)| \to 0$ as $x \to +\infty$. This proves the corollary.

Our final remarks concern the integro-differential equation (1.4). If this equation is integrated we obtain an equation of the form (1.3) with

$$\varphi(x) \equiv u_0 + \int_a^x \psi(s)\, ds,\ u(a) = u_0$$

$$A(x, t) \equiv B(t) + \int_t^x C(s, t)\, ds$$

$$f(x, t, u(t)) \equiv g(t, u(t)) + \int_t^x h(s, t, u(t))\, ds.$$

Note that $\varphi \in N_2$ if and only if $\psi \in L^1[0, +\infty)$. Also, $\varphi \in N_3$ if and only if $\psi \equiv 0$. Thus, stability of (1.4) on N_3, as defined above for the equivalent system (1.3), becomes equivalent to the usual definitions for ordinary differential equations. Uniform stability on N_2 for (1.4) means that to any $\varepsilon > 0$ there exists a $\delta > 0$, independent of a, such that $|u_0| + \int_{x_0}^{+\infty} |\psi| ds \leqq \delta$ implies $\|u\|_{0,a} \leqq \varepsilon$; if, in addition, $|u(x)| \to 0$ as $x \to +\infty$, then (1.4) is asymptotically stable on N_2. Corollaries 3.7 and 3.2 now yield the following result.

THEOREM 5.2 *Suppose g and h satisfy the conditions*

$$|g(x, z)| \leqq \gamma_1(x)|z|,\ x \geqq x_0,\ |z| < b,$$

$$|h(x, t, z)| \leqq \gamma_3(x, t)|z| \text{ on } T \times \{|z| < b\}$$

where

$$\int_{x_0}^{+\infty} \gamma_1(x)\, dx < +\infty,\ \int_{x_0}^{+\infty} \int_{x_0}^{s} \gamma_2(s, t)\, dt\, ds < +\infty.$$

Then if the linear system

$$du/dx = B(x)u(x) + \int_a^x C(x, t)u(t)\, dt,\ u(a) = u_0, \tag{5.2}$$

is U.S., then (1.4) *is U.S. on* N_2. *If, in addition,* (5.2) *is A.S., then* (1.4) *is also A.S. on* N_2.

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In particular note that if $C(x, t) \equiv 0$, then (5.2) is an initial value problem for an ordinary differential equation for which all the many well known techniques are available in deciding its stability properties. For example, if $C(x, t) \equiv 0$ and $B(x) \equiv$ constant, then, under the hypotheses of Theorem 5.2 concerning the perturbation terms h and g, the U.S. and/or A.S. of (1.4) on N_2 can be decided on the basis of the well-known eigenvalue criteria for the matrix B.

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