Unit 2: Sections 7.6 - 8.4

In Section 7.6, we are going to make sense of the expression

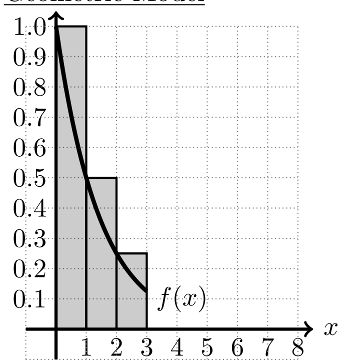
$$\int_0^\infty f(s)ds$$

Recall that the area under a function can be approximated by a left hand sum:

$$\int_a^b f(t)\,dt \approx \sum_{i=0}^{N-1} f(x_i)\Delta x$$

We could use this geometric model with $n = 3$, and $\Delta x = 1$ to approximate the area under the curve $f(x) = (1/2)^x$: Complete the calculation. (Show your work.)

Geometric Model



Calculation:

$$\int_0^3 \left(\frac{1}{2}\right)^x dx$$

$$\approx \sum_{i=0}^{2} \left(\frac{1}{2}\right)^i \cdot 1$$

$$=$$

$$=$$

Obviously, *if* we wanted a a better approximation, we *could* use more subdivisions, but we're not going to do that *yet*. Sections 7.6 and 7.7 involve a different kind of limit. We want to understand what happens to the area under the curve in the limit as the upper bound tends to infinity.

This algorithm that will help us make sense of this limit.

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Find and shade a shape of area 1.  (done)
Find and shade a shape of area 1/2.
Find and shade a shape of area 1/4.
Find and shade a shape of area 1/8.
Proceed infinitely many times.
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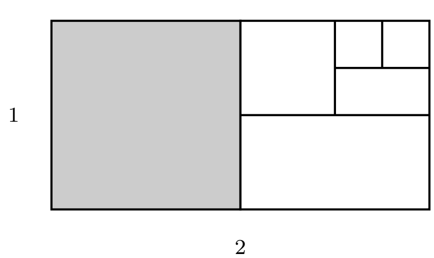
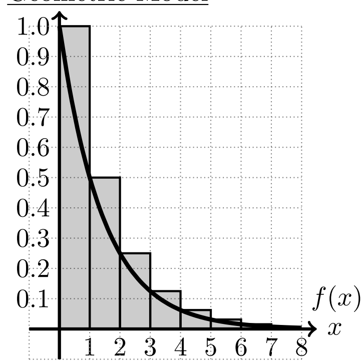


After infinitely many iterations of the algorithm, the shading stays within the rectangle of area 2. We say that as N tends to infinity, the total area shaded tends to 2.

$$\lim_{N\to\infty} \text{(Shaded Area)} = 2$$

So, keep $\Delta x = 1$, pick a large n, (say, 10, or 10,000,000) and complete the calculation on the right .

Geometric Model



Calculation:

$$\int_0^a \left(\frac{1}{2}\right)^x dx$$

$$\approx \sum_{i=0}^{N-1} \left(\frac{1}{2}\right)^i \cdot 1$$

$$=$$

$$=$$

For any finite number of terms (no matter how large), the sum adds to a number less than (but approaching) 2.

Therefore, for any finite upper bound of the integral, the true value of the integral is less than the approximating finite sum, and the approximation is less than (but approaching) 2.

$$\int_0^\infty \frac{1}{2^x}\,dx = \lim_{a\to\infty} \int_0^a \frac{1}{2^x}\,dx < \lim_{N\to\infty} \sum_{i=0}^{N-1} \frac{1}{2^i} = \sum_{i=0}^{\infty} \frac{1}{2^i} = 2$$

The equation above is a very crude approximation of the integral. (Notice all the extra area in the figure.) However, improving our method by using using finer subdivisions is a tricky process and will be demonstrated on the next page.

Improper Integrals

Section 7.6
September 13, 2016

Name:
Math 129 - 01

Understand	Use the terminology "improper integral" for $\int_a^\infty f(x)\,dx$.
Understand	Know that $\int_a^\infty f(x)\,dx$ with positive integrand $f(x)$ means $\lim_{b\to\infty} \int_a^b f(x)\,dx$.
Understand	Use known integration techniques to find $\int_a^b f(x)\,dx$ and then take the limit as $b \to \infty$.
Understand	Use correct vocabulary (*converges* or *diverges*) to describe the result of an improper integral calculation.

1. To calculate the exact value of the improper integral

$$\int_0^\infty \left(\frac{1}{2}\right)^x dx,$$

first write the definite integral from 0 to b of the integrand above, and calculate it using whatever methods you know from previous sections (your answer should have b appearing in it).

$\int_0^b \left(\frac{1}{2}\right)^x dx =$

Now, take your answer from the part above (it should simplify to $\big((1/2)^b - 1\big) / \ln(2))$, and write "$\lim_{b\to\infty}$ and then your answer from above". Calculate the result of this limit.

$\lim_{b\to\infty} \int_0^b \left(\frac{1}{2}\right)^x dx =$

What was the result of your limit?

2. Repeat the same process for the improper integral

$$\int_0^\infty \frac{1}{x+1}\,dx$$

Frst calculate the definite integral from 0 to b of the integrand above .

$\int_0^b \frac{1}{x+1}\,dx =$

Now take the limit of your expression above.

If the result of your limit is a finite number, write $\int_0^\infty f(x)\,dx = \lim_{b\to\infty} \int_0^b f(x)dx =$"your limit"If the result is infinite, write "$\int_0^\infty f(x)dx$ diverges".

Quiz (Leave this space blank)