Unit 3

10.1 Taylor Polynomials.
10.2 Taylor Series.
10.3 Finding and using Taylor Series.
11.1 What is a differential equation.
11.2 Slope fields.
11.4 Separation of Variables.
Exam 4 Thursday December 1.

Warm-up

- (Last year's final exam question)
  Suppose the power series $\sum C_n(x-3)^n$ converges for $x=-1$ and diverges for $x=10$. Determine if the power series converges or diverges for the following values.

  $x=\ 3$ converges diverges impossible to tell

  $x=\ 8$ converges diverges impossible to tell

  $x=-7$ converges diverges impossible to tell

Warm-up

- The Taylor polynomial of degree 7 of $g(x)$ is given by

$$P_7(x) = 1 - \frac{x}{3} + \frac{5s^2}{7} + 8x^3 - \frac{x^5}{11} + 8x^7$$

Find the Taylor Polynomial of degree 3 of $g(x)$.

Taylor Polynomial

- Taylor Polynomial of Degree n approximating $f(x)$ for x near a

$$f(x) \approx P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(}}{}$$

Find the third degree Taylor polynomial approximating e^x near 0.

- Find a simplified formula for $P_5(x)$, the fifth-degree Taylor polynomial approximating f near $x = 0$, using the values in the table:

$f(0)$	$f'(0)$	$f''(0)$	$f^{(3)}(0)$	$f^{(4)}(0)$	$f^{(5)}(0)$
-3	5	-2	0	-1	4

- The function $h(x)$ is approximated near $x = 0$ by the third-degree Taylor polynomial

$$P_3(x) = 2 - x - \frac{x^2}{3} + 2x^3.$$

Give the values of
(a) $h(0)$,
(b) $h'(0)$,
(c) $h''(0)$,
(d) $h'''(0)$.

Quiz for 4-7-16, - Section 10.1

- Find the second degree polynomial approximating $\frac{1}{1+x^2}$ near $x = 1$.

Discussion

- When we model the motion of a pendulum, we replace the differential equation

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin(\theta) \quad \text{by} \quad \frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$$

where θ is the angle between the pendulum and the vertical. Explain why, and under what circumstances, it is reasonable to make this replacement.

Section 10.2 - Taylor Series

- By recognizing the series as a Taylor series evaluated at a particular value of x, find the sum of the following convergent series.

$$1 - 0.1 + 0.1^2 - 0.1^3 + \dots$$

Power Series formulae.

- Show Webpage

Example 2

- Show that

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}.$$

Example 2

- Show that
$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}.$$

- Use this result to find the Talyor Series about $x = 0$ for $\dfrac{1}{(1-x)^2}$.

Like Examples 1 and 3

- Find the Taylor Series expansion of $\dfrac{1}{1+x^2}$ around $x = 0$.

- Use this result to find the Taylor Series expansion of $\arctan(x)$ around $x = 0$.

Using Products

- Find the Taylor Series expansion of: $\dfrac{s}{1-s}$ about $x = 0$

$\sin(\alpha)\cos(\alpha)$ about $\alpha = 0$

Warmup

- Using known Taylor series, find the first four nonzero terms of the Taylor series about 0 for the functions:

e^{-x}

$\sqrt{1+\sin(\theta)}$

$\cos(\theta^2)$

$e^t \cos(t)$

$t \sin(3t)$

$(1+x)^3$

Warmup - Similar to example 7

- Consider the functions $y = e^{-x^2}$ and $y = 1/(1+x^2)$.
- Write the Taylor Series expansions for the two functions about $x = 0$. What is similar, What is different?
- Looking at the series which function do you predict will be greater over the inteval $(-1, 1)$? Verify this.
- Are these functions even or odd? How can you tell by looking at the expansions?

Warmup - Similar to example 7

- Consider the functions $y = e^{-x^2}$ and $y = 1/(1+x^2)$.
- Write the Taylor Series expansions for the two functions about $x = 0$. What is similar, What is different?
- By looking at the coefficients, explain why it is reasonable how you can tell which series converges for all x and which series only converges on $(-1, 1)$.

Quiz for 11-16-16

- Find the first three non-zero terms of the Taylor series for $\ln(1-2t)$.
- For what values of t is your Taylor series a valid approximation.
- Find the first three non-zero terms of the Taylor series for $\int_0^x \ln(1-2t)dt$.

Determine which of the following functions is a solution to the differential equation. Select all that apply.

$$y\frac{dy}{dt} = 6t^2$$

1. $y = 4t^3$
2. $y = 2t^{3/2}$
3. $y = 6t^{3/2}$
4. $y = 2(t^3 + 1)^{1/2}$

The Family of solutions

- Family of Solutions: It can be shown that any solution to this differential equation is in the form $y = 2(t^3 + C)^{1/2}$, for some constant C.
- General Solution: We say that the family of functions $y = 2(t^3 + C)^{1/2}$ is the general solution to the differential equation.
- Initial Condition: Different members of the family can be distiguished by specifying an initial condition, that is, the value of y when $t = 0$.
- Particular Solution: The differential equation and the intial condition together are called the initial value problem.

Problem 22

1. Find the value of A so that the equation $y' - xy - x = 0$ has a solution of the form $y(x) = A + Be^{x^2/2}$ for any constant B.

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2. If $y(0) = 1$, find B.

Family of Solutions

1. Match the families of curves with the differential equations of which they are solutions.

$$\frac{dy}{dx} = ky$$

A. $y = x\,e^{kx}$
B. $y = x^k$
C. $y = e^{kx}$
D. $y = kx$

Family of Solutions

1. Match the families of curves with the differential equations of which they are solutions.

$$\frac{dy}{dx} = \frac{y}{x}$$

A. $y = x\,e^{kx}$
B. $y = x^k$
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Family of Solutions

1. Suggest a solution to the differential equation

$$\frac{d^2y}{dt^2} + y = 0$$

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Family of Solutions

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$$\frac{d^2y}{dt^2} + 9y = 0$$

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True or False

1. If $dx/dt = 1/x$ and $x = 3$ when $t = 0$, then x is a decreasing function of t.

2. If $y = f(t)$ is a particular solution to a first-order differential equation, then the general solution is $y = f(t) + C$, where C is an arbitrary constant.

3. Polynomials are never solutions to differential equations.

Not the quiz

1. Use the information below to find the second degree Taylor polynomial that can be used to approximate f near $x = 1$.

$$f(1) = 3 \quad f'(1) = 2 \quad f''(1) = -5$$

2. Approximate $f(1.1)$ using your answer in part (a).

Quiz for 11-22-16, - Section 11.2

1. Assume that a is a positive constant, and that r is a variable that is very small compared to a.
2. Show that

$$\frac{a}{(a+r)^2} = \frac{1}{a}\frac{1}{(1+\frac{r}{a})^2}$$

3. Find the third degree Taylor polynomial about $r = 0$ of

$$\frac{1}{a}\frac{1}{(1+\frac{r}{a})^2}$$

4. For what values of r does the Taylor Series converge.

Quiz for 11-22-16, - Section 11.5

- Find the third-order Taylor polynomial of $(1+x)\,e^x$ about the point $x=0$. Compute the Taylor polynomial directly from the definition - do not use the table of known Taylor series. [*10 points*]
- Use the result above to calculate the exact value of the sum

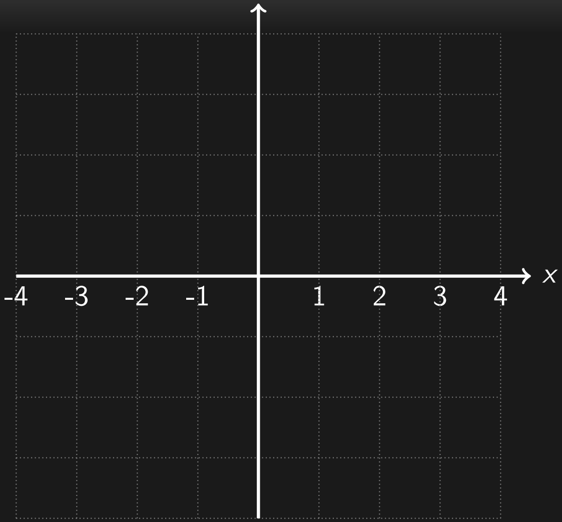
$$\sum_{n=0}^{\infty}(-1)^n \frac{n+1}{n!}$$

[*5 points*]

1. In the problem, we are going to plot the slope field for the differential equation $y' = x - y$.
At the point $(2,0)$, the differential equation shows that the derivative y' evaluates to $y' = 2 - 0$, indicating that the solution to the differential equation has positive slope. Use this fact to plot a short line segment of slope 2, "/", at the point $(2,0)$.

At the point $(1,1)$, the differential equation shows that derivative y' evaluates to $y' = 1 - 1$, indicating that the solution to the differential equation has zero slope. Use this fact to plot a short, horizontal sloping line, "-", at the point $(1,1)$.

Repeat this for all integer pairs on the grid.
Use the slope field to sketch the approximate solution to the initial value problem $y' = x - y$, for $y(0) = 1$.

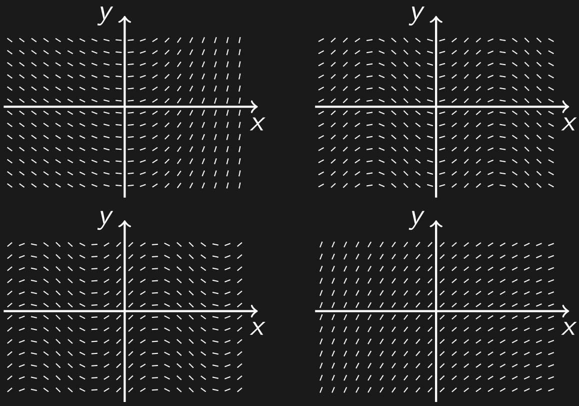


1.

Match the following slope fields with the appropriate differential equation

(a) $y' = \cos(x)$ (b) $y' = xe^x$

(c) $y' = \sin(x)$ (d) $y' = e^{-x}$



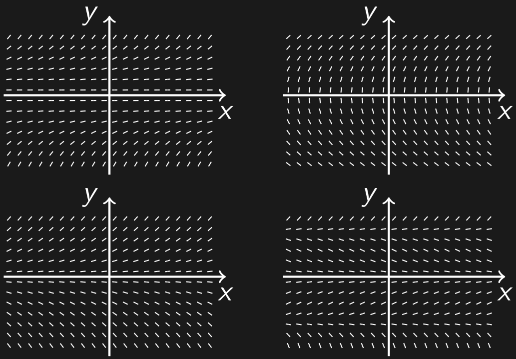
Match the following slope fields with the appropriate differential equation

(a) $y' = y^2$

(b) $y' = y$

(c) $y' = 1/y$

(d) $y' = y(y-1)(y+1)$



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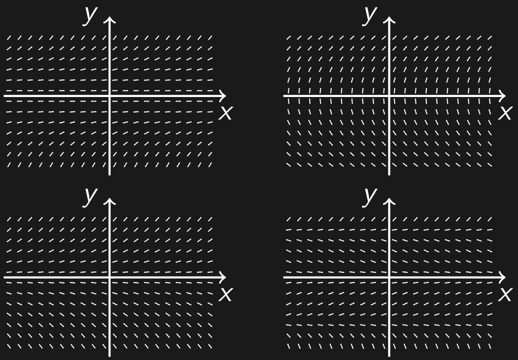
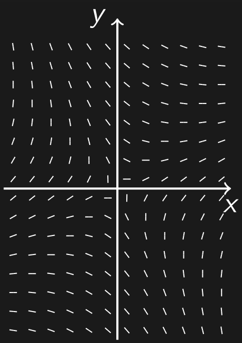
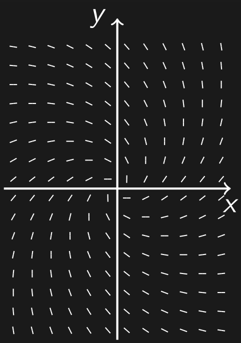
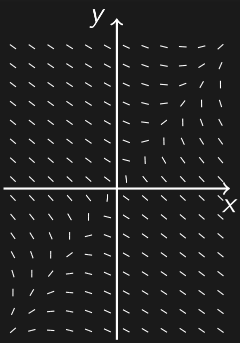


Which of the following is the slope field for $y' = (x+y)/(x-y)$?



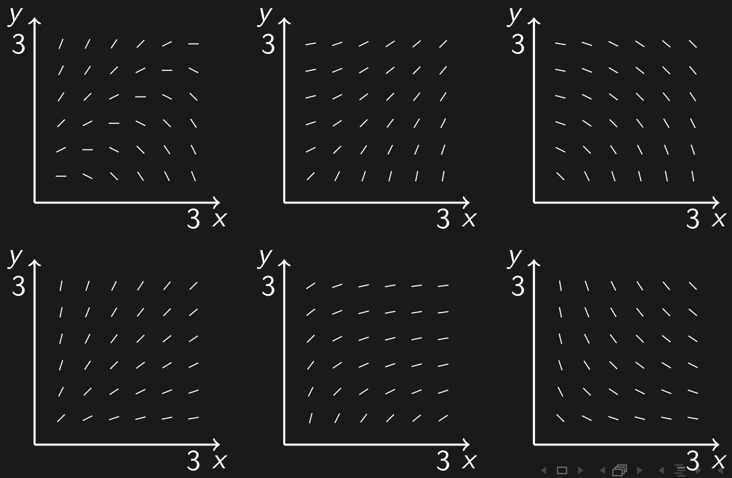




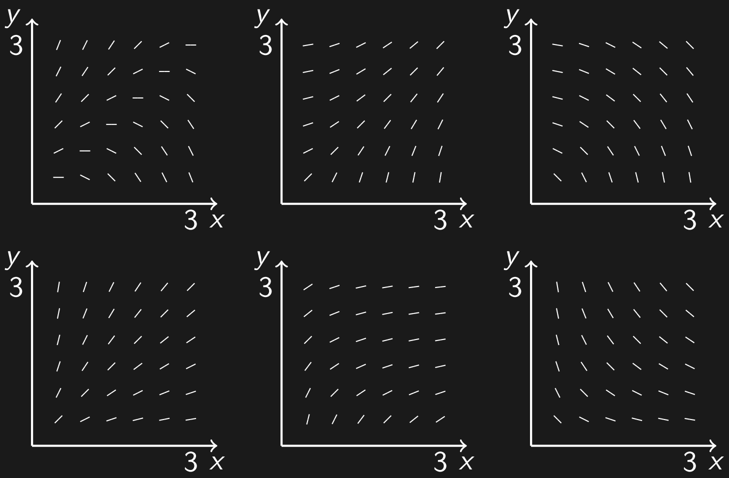
Match the following slope fields with the appropriate differential equation

(a) $y' = -x/y$ (b) $y' = x/y$ (c) $y' = 1/(xy)$

(d) $y' = -y/x$ (e) $y' = y/x$ (f) $y' = y - x$

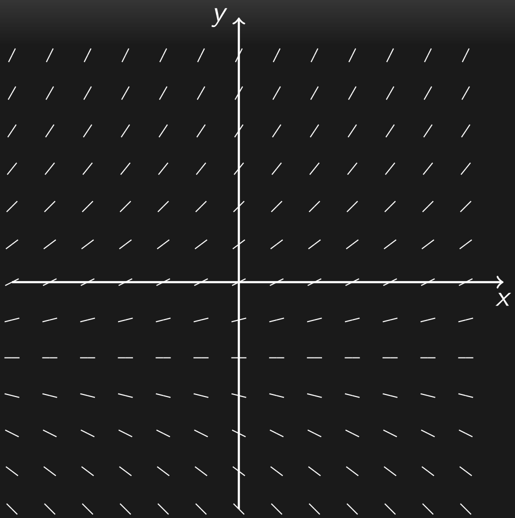


(f) $y' = y - x$
(e) $y' = y/x$
(b) $y' = x/y$
(c) $y' = 1/(xy)$
(a) $y' = -x/y$
(d) $y' = -y/x$



True or False:
All solutions to the differential equation whose slope field is show below have

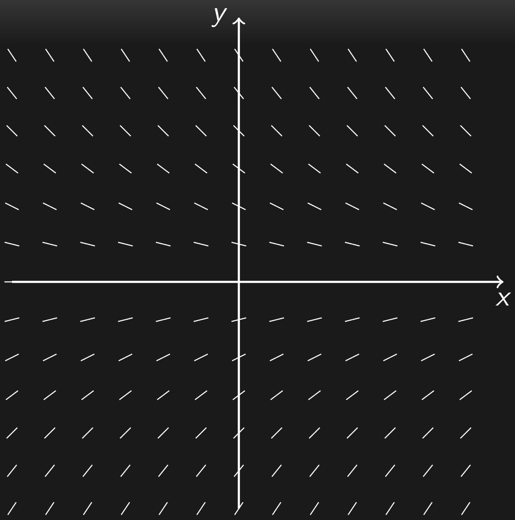
$$\lim_{x\to\infty} y = \infty$$



True or False:
All solutions to the differential equation whose slope field is show below have

$$\lim_{x\to\infty} y = 0$$

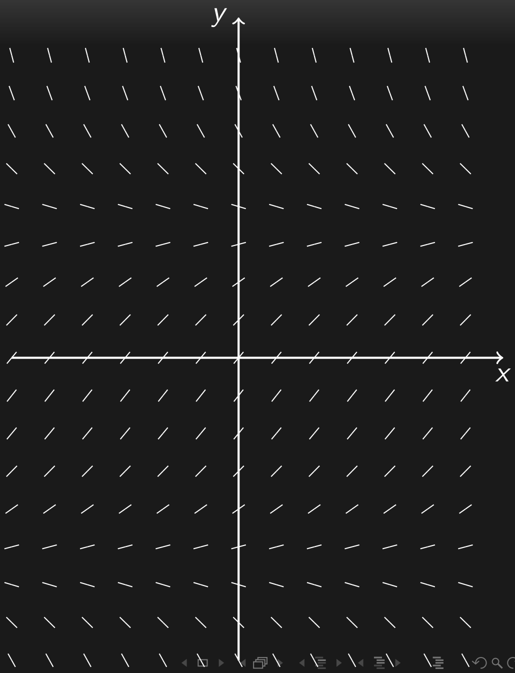
.



True or False:
All solutions to the differential equation whose slope field is show below have the same limiting values as

$$x \to \infty$$

.



Seperable Differential Equations

Which of the following are seperable

a) $y' = y$

b) $y' = x + y$

c) $y' = xy$

d) $y' = \sin(x + y)$

e) $y' - xy = 0$

f) $y' = y/x$

g) $y' = \ln(xy)$

h) $y' = \ln(x^y)$

i) $y' = (\sin(x))(\cos(y))$

j) $y' = (\sin(x))(\cos(xy))$

k) $y' = x/y$

l) $y' = 2x$

m) $y' = (x + y)/(x + 2y)$

Which of the following are seperable

a) $y' = y$ Yes

b) $y' = x + y$ No

c) $y' = xy$ Yes

d) $y' = \sin(x + y)$ No

e) $y' - xy = 0$ Yes

f) $y' = y/x$ Yes

g) $y' = \ln(xy)$ No

h) $y' = \ln(x^y)$ Yes

i) $y' = (\sin(x))(\cos(y))$ Yes

j) $y' = (\sin(x))(\cos(xy))$ No

k) $y' = x/y$ Yes

l) $y' = 2x$ Yes

m) $y' = (x + y)/(x + 2y)$ No

Separation of variables

- Use separation of variables to find the solution to the differential equation.

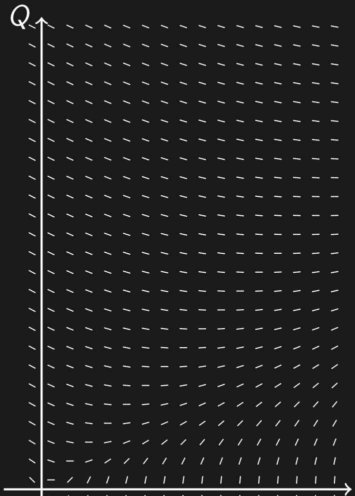
$$\frac{dy}{dx} = -\frac{x}{y} \quad y = 1 \text{ when } x = 0.$$

Separation of variables

- Use separation of variables to find the solution to the differential equation

$$\frac{dz}{dy} = zy \quad z = 1 \text{ when } y = 0.$$

$$Q' = t/Q - 0.5$$



11.1 - Verifying Solutions to Differential Equations

- Is the differential equation below separable?

$$xy' = y - x$$

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- Is the differential equation below separable?

$$xy' = y - x$$

- Show that $y = c_1 x - x\ln(x)$ is a solution to the differential equation.

$$xy' = y - x$$

Quiz for 11-22-16, - Section 11.4

- Determine whether the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n \ln(n)}$$

is (a) Conditionally convergent (b) Absolutely convergent. Clearly state which tests you use and clearly show that all requirements of the test are met.

11.4 Separation of Variables

- For $k > 0$, find and graph solutions of

$$\frac{dH}{dt} = -k(H - 20)$$

11.4 Separation of Variables

- Find and sketch the solution to the initial value problem

$$\frac{dP}{dt} = 2P - 2Pt \quad P(0) = 5$$

Growth and Decay

- Nicotine leaves the body at a rate proportional to the amount present, with constant of proportionality 0.347 if the amount of nicotine is in mg and time is in hours. The amount of nicotine in the body immediately after smoking a cigarette is 0.4 mg.

Growth and Decay

- Warfarin is a drug used as an anticoagulant. After administration of the drug is stopped, the quantity remaining in a patient's body decreases at a rate proportional to the quantity remaining. The half-life of warfarin in the body is 37 hours.

Growth and Decay

- Before Galileo discovered that the speed of a falling body with no air resistance is proportional to the time since it was dropped, he mistakenly conjectured that the speed was proportional to the distance it had fallen.
  1. Assume the mistaken conjecture to be true and write an equation relating the distance fallen, $D(t)$ aat time t and the derivative.
  2. Show that the only mathematically correct solution to this equaiton is $D = 0$ for all time.

- A stream flowing into a lake brings with it a pollutant at a rate of 8 metric tons per year. The river leaving the lake removes the pollutant at a rate prooportional to the quantitiy in the lake , with constatn of proportionality $k = -0.16/yr$.
  1. Is the quantity of pollutant in the lake increasing or decreasing when there is 45 metic tons of pollutant in the lake? ...55 metric tons?
  2. What is the quantity of pollutant in the lake after a long time?

- A spherical snowball melts at a rate proportional to its surface area.
- Water leaks out of a barrel at a rate proportional to the square root of the depth of the water.

- The balance in an investment fund is gaining value at a continuous rate of 8 % per year and money is being withdrawn from the fund at a continous rate of \$20,000 per year.

A detective finds a murder victim at 9 am. The temperature of the body is measured at 90.3 degrees F. One hour later, the temperature of the body is 89.0 degrees F. The temperature of the room has been maintained at a constant 68 degrees F.

1. Assuming the temperature, T, of the body obeys Newton's Law of Cooling, write a differential equation for T.
2. Solve the differential equation to estimate the time the murder occurred.

A water reservoir holds 100 million gallons of water and supplies a city with 1 million gallons a day. The reservoir is partly refilled by a spring that provides 0.9 million gallons a day, and the rest of the water, 0.1 million gallons a day, comes from run-off from the surrounding land. The spring is clean, but the run-off contains salt with a concentration of 0.0001 pound per gallon. There was no salt in the reservoir initially, and the water is well mixed (that is, the outflow contains the concentration of salt in the tank at that instant). We find the concentration of salt in the reservoir as a function of time.

At 1:00 pm one winter afternoon, there is a power failure at your house in Wisconsin, and your head does not work without electricity. When the power goes out, it is 68 degrees F in your house. At 10:00pm, it is 57 degrees F in the house, and you notice that it is 10 degrees F outside.

1. Assuming that the temperature, T, in your home obeys Newton's Law of Cooling, write the differential equation satisfied by T.
2. Solve the differential equation to estimate the temperature in the house when you get up at 7:00 am the next morning. Should you worry about your water pipes freezing?
3. What assumption did you make in part (a) about the temperature outside? Given this (probably incorrect) assumption, would you revise your estimate up or down? Why?

Quiz for 4-7-16, - Section 11.4

- Padé approximants are rational functions used to approximate more complicated functions. In this problem, you will derive the Padé approximate to the exponential function. [*20 points*]
  1. Let $f(x) = (1 + ax)/(1 + bx)$, where a and b are constants. Use any method you choose to write down the first four terms in the Taylor series for $f(x)$ about $x = 0$.
  2. What is the radius of convergence of the Taylor Series in (a)?
  3. Write down the first four terms in the Taylor series of $g(x) = e^x$ about $x = 0$.
  4. Use your results from (a) and (c) to explain why selecting $a = 1/2$ and $b = -1/2$ makes $f(x)$ the best possible Padé approximate to $g(x)$ near $x = 0$.

- Write out the first four nonzero terms for the Taylor series of the function for x near 0. You may use the table provided.

$$g(x) = \frac{1}{1 - \ln(x+1)}$$