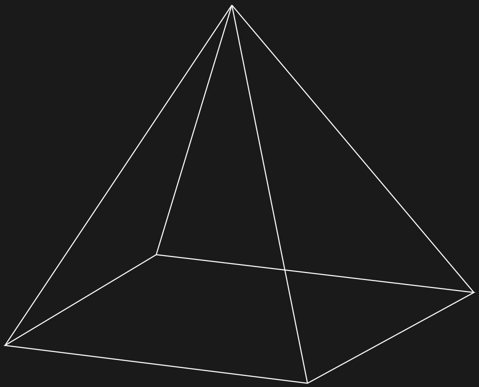
Unit 3

8.5	Physics.
9.1	Sequences.
9.2	Series.
9.3	Convergence.
9.4	Tests for Convergence
9.5	Power Series
Exam 3	Thursday November 3.

- Work in pairs to find the work done to build the Great Pyramid of Egypt. The pyramid is 410 feet high and has a square base of 755 feet by 755 feet. The stone making up the pyramid has a density of 200 pounds per cubic foot.
  (Hint: Think slices!)



- It is reported that the Great Pyramid of Egypt was built in 20 years. Make some reasonable assumptions on the amount of work that laborer can accomplish in a day to estimate the number of laborers needed to build the pyramid.

8.5 - A changing Mass

- A tank in the shape of an inverted cone has a height of 15 m and base radius of 4 m and is filled with water to a depth of 12 m. Determine the amount of work needed to pump all of the water to the top of the tank. Assume the density of water is 1000 kg/m^3.

8.5 - A changing Force

- Find the work done in compressing the spring from $x = 0$ to $x = 1$ and in compressing the spring from $x = 4$ to $x = 5$. Which of the two answers is larger? Why?

8.5 - WebAssign problem

- A flag in the shape of a right triangle is hung over the side of a building as shown below. The total weight of the flag is 220 pounds and it has uniform density. $a = 10$ and $b = 26$.
  - Find the density of the flag.
  - Find the approximate weight of the slice shown in the figure if it is located h feet below the roof of the building.
  - Find the approximate work needed to lift the slice onto the roof of the building.
  - Find the exact work needed to lift the entire flag onto the roof of the building.

8.5 - WebAssign Problem

- A 2000-lb cube of ice must be lifted at a rate of 1 ft / min a total distance of 100ft. It is melting at a rate of 4 lb per minute. Find the work required to raise the ice cube to the desired height?

Quiz - October 18

- A 28-meter uniform chain with a mass density of 2 kilograms per meter is dangling from the roof of a building. How much work is needed to pull the chain up onto the top of the building?

9.1 - Sequence Properties

- A sequence is *bounded* if there are numbers K and M such that $K \leq s_n \leq M$ for all n.
- A sequence is called *monotone* if it is either non-decreasing, $(s_n \leq s_{n+1})$ or if it is non-increasing $(s_n \geq s_{n+1})$.
- A sequence *converges* if there is a number L such that $\lim_{n \to \infty} s_n = L$.

Bounded, Monotone, Convergent?

- Which of the following sequences are (a) bounded, (b) monotone, and (c) convergent?
- $\left\{2^n\right\}$
- $\left\{3+e^{-n}\right\}$
- $\left\{\frac{n}{10}+\frac{10}{n}\right\}$
- $\left\{\cos(\pi n)\right\}$

9.1 - Sequences and Series

- Notation:

- What does it mean for a sequence to converge?
- What does it mean for a series to converge?

9.1 - Sequences and Series

- Are the following mathematical expressions examples of a *sequence* or a *series*?
  1. $2^2, \;\; 4^2, \;\; 6^2, \;\; 8^2, \ldots$
  2. $2^2 + 4^2 + 6^2 + 8^2 + \ldots$
  3. $1+2, \;\; 3+4, \;\; 5+6, \;\; 7+8, \ldots$

9.1 - Sequences and Series

- Are the following mathematical expressions examples of a *sequence* or a *series*?

4. $1, -2, \ \ 3, -4, \ \ 5, \ldots$
5. $1 - 2 + 3 - 4 + 5 - \cdots$
6. $1 + 2 + 3 + 4 + 5 + 6 + \ldots$
7. $-S_1 + S_2 - S_3 + S_4 - S_5 + \ldots$

9.2 - Geometric Series

- A finite geometric series has the form

$$a + ax + ax^2 + \cdots + ax^{n-1}$$

- An infinite geometric series has the form

$$a + ax + ax^2 + \ldots$$

9.2 - Geometric Series

- Which of the following are geometric series? For those which are, give the first term and the ratio between successive terms.
  - $5 - 10 + 20 - 40 + 80 - \ldots$
  - $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$
  - $2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

9.2 - Geometric Series

- Which of the following are geometric series?
  - $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$
  - $1 + 2z + (2z)^2 + (2z)^3 + (2z)^4 + \cdots$
  - $y^2 + y^3 + y^4 + y^5 + y^6 + \cdots$
  - $3 + 3z + 6z^2 + 9z^3 + 12z^4 + \cdots$

- Find the sum of the series. For what values of the variable does the series converge to this sum?

    - $1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\ldots$

    - $3+x+x^2+x^3+\ldots$

    - $80+\frac{80}{\sqrt{2}}+40+\frac{40}{\sqrt{2}}+20+\frac{20}{\sqrt{2}}+\cdots$

Not the Quiz

- Discuss what is wrong with the two statements.
- "The sequence below converges to $\dfrac{4}{1-1/4} = \dfrac{16}{3}$."

$$4, 1, \frac{1}{4}, \frac{1}{16}, \ldots$$

- "The sum of the infinite geometric series below is $\dfrac{1}{1+3/2} = \dfrac{2}{5}$."

$$1 - \frac{3}{2} + \frac{9}{4} - \frac{27}{8} + \cdots$$

Applicatons

- This problem shows how to estimate the cumulative effect of a tax cut on a country's economy. Suppose the government proposes a tax cut totaling \$100 million. We assume that all the people who have extra money spend 80% of it and save 20%. Thus, of the extra income generated by the tax cut, \$100(0.8) million = \$80 million is spent and becomes extra income to someone else. These people also spend 80% of their additional income, or \$80(0.8) million, and so on. Calculate the additional spending created by such a tax cut.

“Almost“ Geometric Series

- Explain why the series is not geometric.

$$\frac{2}{2}+\frac{3}{4}+\frac{4}{8}+\frac{5}{16}+\frac{6}{32}+\cdots$$

  Find an expression for the n^{th} term of the series.

  Describe what happens to the limit
  $\lim_{n\to\infty} |a_{n+1}/a_n|$.
  This suggests that the series *almost* a geometric series. Do you think it will converge or diverge?

Ratio Test

- Consider the series.

$$\sum_{n=0}^{\infty} \frac{(n!)(n!)}{(2n)!}$$

Describe what happens to the limit $\lim_{n\to\infty} |a_{n+1}/a_n|$.

This suggests that the series *almost* a geometric series. Do you think it will converge or diverge?

Ratio Test & Power Series

- Explain why the series is not geometric.

$$\frac{2}{2}+\frac{3}{4}+\frac{4}{8}+\frac{5}{16}+\frac{6}{32}+\cdots$$

Find an expression for the n^{th} term of the series.

Describe what happens to the limit
$\lim_{n\to\infty} |a_n/a_{n+1}|$.
This suggests that the series *almost* a geometric series. Do you think it will converge or diverge?

Quiz for 3-24-16, section 9.3

- Explain what is wrong with each of these statements.
  1. "The sequence
  $$4, 1, \frac{1}{4}, \frac{1}{16}, \ldots$$
  converges to $\dfrac{4}{1-1/4} = \dfrac{16}{3}$."
  2. "The sum of the infinite geometric series
  $$1 - \frac{3}{2} + \frac{9}{4} - \frac{27}{8} + \cdots$$
  is $\dfrac{1}{1+3/2} = \dfrac{2}{5}$."

The Integral Test

- Suppose $a_n = f(n)$, where $f(x)$ is positive and decreasing.
  - If $\int_1^\infty f(x)dx$ converges, then $\sum a_n$ converges.
  - If $\int_1^\infty f(x)dx$ diverges, then $\sum a_n$ diverges.

Does each series converge or diverge?

-
  - $\sum_{n=0}^{\infty} \frac{4}{2n+1}$
  - $\sum_{n=1}^{\infty} \frac{3}{(2n-1)^2}$
  - $\sum_{n=1}^{\infty} \left(\left(\frac{3}{4} \right)^n + \frac{1}{n} \right)$
  - $\sum_{n=1}^{\infty} \frac{1}{n(1+\ln(n))}$

Properties of Convergence

- Consider the series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots$.
  1. Show that $\frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)}$.
  2. Use part (a) to find the partial sums S_3, S_{10}, and S_n.
  3. Use part (b) to show that the sequence of partial sums S_n converges to 1.
  4. Find the infinite sum $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$.

Euler's Result

- The book gives Euler's result
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

1. Find the sum of the first 20 terms in this series. Give your answer to three decimal places.
2. Use your answer to estimate π. Give your answer to two decimal places.
3. Repeat parts (a) and (b) with 100 terms.
4. Use a right sum approximation to bound the error in approximating $\pi^2/6$ by $\sum_{n=1}^{20}(1/n^2)$ and by $\sum_{n=1}^{100}(1/n^2)$.

Integral Tests

- Show that $\sum_{n=1}^{\infty} \frac{1}{ln(2^n)}$ diverges.
- Show that $\sum_{n=1}^{\infty} \frac{1}{(ln(2^n))^2}$ converges.

Absolute vs conditional convergence

- Given a series $\sum a_n$. What vocabulary might you use to differentiate between the following two situations:
  - When both $\sum a_n$ and $\sum |a_n|$ converge...
  - When $\sum a_n$ converges but $\sum |a_n|$ diverges...

True or False? Explain

- If $\sum |a_n|$ converges, then $\sum (-1)^n |a_n|$ converges.
- If $\sum a_n$ is absolutely convergent, then it is convergent.
- If $\sum a_n$ is conditionally convergent, then it is absolutely convergent.

Quiz November 1

1. The Alternating Series Test can be used to decide whether $\sum_{n=1}^{\infty}(-1)^{n-1}\left(2-\frac{1}{n}\right)$ converges or diverges.
2. Does the series $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{\sqrt{n}}$ converge? If so, does the series converge absolutely or does it converge conditionally?

True or false? Explain

- We can use the Comparison Test to prove that $\sum_{n=1}^{\infty} \frac{e^{-n}}{n^2}$ converges, since we already know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Determine whether the series converge

1. $\displaystyle\sum_{n=0}^{\infty} \frac{3}{n^2+4}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+1}{n^4+2n^3+2n}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4+e^n}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n)!}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{5n+1}{3n^2}$

Determine whether the series converge

1. $\sum_{n=1}^{\infty} \frac{1}{r^n n!}, r > 0$
2. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$
3. $\sum \frac{4\sin(n) + n}{n^2}$
4. $\sum_{n=1}^{\infty} \frac{n-4}{\sqrt{n^3 + n^2 + 8}}$
5. $\sum_{n=1}^{\infty} \frac{2^n + 1}{n2^n - 1}$
6. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{e^n}$

True or False? Explain

- The series $\sum \dfrac{1}{n^2+1}$ converges by the Ratio Test.

Power Series

- A power series about $x = a$ is a sum of constants times powers of $(x - a)$

$$\sum_{n=0}^{\infty} C_n (x-a)^n = C_0 + C_1(x-a) + C_2(x-a)^2 + \dots$$

Power Series

- A power series about $x = a$ is a sum of constants times powers of $(x - a)$

  $$\sum_{n=0}^{\infty} C_n(x-a)^n = C_0 + C_1(x-a) + C_2(x-a)^2 + \ldots$$

  The interval of convergence of a power series is the set of all values of x such that the sum converges.

Find the radius of convergence

- $\sum_{n=0}^{\infty} 5^n x^n$

$$\sum_{n=0}^{\infty} (2^n + n^2) x^n$$

$$1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \frac{16x^4}{4!} + \dots$$

Find the radius of convergence

- $\sum_{n=0}^{\infty} \frac{(x-3)^n}{n}$

  $\sum_{n=0}^{\infty} n!x^n$

  $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$

True or False? Explain

- If the power series $\sum C_n x^n$ converges at $x = 10$, then it converges at $x = -9$.